MATH 2202 - FALL 2008
MIDTERM EXAM 1 SOLUTIONS

1. (a) $A = \begin{bmatrix} -1 & 3 & -3 & 2 \\ 2 & -5 & 2 & -2 \\ -3 & 7 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & -4 & 2 \\ 0 & -2 & 8 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -9 & 4 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 3R_2 \\ \\ \\ \end{matrix}$

(b) The columns of A do not span $\mathbb{R}^3$ because row 3 does have a pivot postion.

(c) $A\mathbf{x} = \mathbf{0} \iff [A\ \mathbf{0}] \sim [B\ \mathbf{0}] \Rightarrow x_1 = 9x_3 - 4x_4$; $x_2 = 4x_3 - 2x_4$; x_3 and x_4 are free variables.

$\Longrightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\mathbf{v}_1 + x_4\mathbf{v}_2$, where $\mathbf{v}_1 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the solution set is spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(d) $A\mathbf{p} = \begin{bmatrix} -1 & 3 & -3 & 2 \\ 2 & -5 & 2 & -2 \\ -3 & 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-3+3+2 \\ 2+5-2-2 \\ -3-7+1+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}$

i.e. $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$. Hence, from (c) the general solution of $A\mathbf{x} = \mathbf{b}$ is given by: $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}_1 + x_4\mathbf{v}_2$ which represents a plane through the vector $\mathbf{p}$.

2. *(i)* $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a linear independent set because the vectors $\mathbf{v}_1$, and $\mathbf{v}_3$are not multiples of each other.
(ii) $\{\mathbf{v}_2, \mathbf{v}_4, -3\mathbf{v}_2 + 4\mathbf{v}_4\}$ is linear dependent set because the vector $-3\mathbf{v}_2 + 4\mathbf{v}_4$ is a linear combination of the two first vectors $\mathbf{v}_2$ and $\mathbf{v}_4$.
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linear dependent because the numbers of vectors is greater than the number of entries in each vector.
(iv) $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4\}$ is linear dependent set because the vector $\mathbf{v}_2 - \mathbf{v}_4$ is a linear combination of the two first vectors $\mathbf{v}_2$ and $\mathbf{v}_4$.

(a) $T(\mathbf{e}_1) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \Longrightarrow A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$.

$A \sim \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ which has a pivot position in every row, then $A\mathbf{x} = \mathbf{0}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^2$. It follows that T is maps $\mathbb{R}^2$ onto $\mathbb{R}^2$. Further, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then T is a one to one mapping.

(b) Let $\mathbf{w} = \begin{bmatrix} 1 \\ -13 \\ 11 \end{bmatrix}$. First, we need to solve for x_1 and x_2 the vector equation

$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w} \Leftrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -13 \\ -1 & 2 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -15 \\ 0 & 4 & 12 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2/-5 \\ R_3/4 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2/-5 \\ R_3/4 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$

$\Longrightarrow x_2 = 3$ and $x_1 = 1 - 2x_2 = -5$.

Hence,

$$T(\mathbf{w}) = T(-5\mathbf{u} + 3\mathbf{v}) = -5T(\mathbf{u}) + 3T(\mathbf{v}) = -5\begin{bmatrix} -4 \\ 6 \end{bmatrix} + 3\begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -21 \end{bmatrix}.$$

(a) The number of pivot positions is at most $nm < n$.So, the system $A\mathbf{x} = \mathbf{0}$ has free variables and consequently non trivial solutions.

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly independent set $\Longrightarrow$ there exist x_1, x_2, and x_3 not zero such that:

$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 0 \Longrightarrow T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) = T(0) = 0$
$\Longrightarrow x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) + x_3T(\mathbf{v}_3) = 0 \Longrightarrow \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set.

(c) $T(c\mathbf{x}) = T(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}) = \begin{bmatrix} cx_1 + 4cx_2 \\ cx_2 - 3\,|cx_1| \end{bmatrix} = \begin{bmatrix} cx_1 + 4cx_2 \\ cx_2 - 3\,|c|\,|x_1| \end{bmatrix}$
$\neq T(c\mathbf{x}) = c\begin{bmatrix} x_1 + 4x_2 \\ x_2 - 3c\,|x_1| \end{bmatrix}$, for $c \in \mathbb{R}$. Hence, T is not a linear transformation.

MATH 2202 - SPRING 2009
MIDTERM EXAM 2 SOLUTIONS

1. (a) (i) X is not invertible because it is not a square matrix.

   (ii) The third column of the matrix Y is equal to the sum of columns 1 and 2 i.e. the column of Y are linearly dependent. It follows by the IMT that the matrix Y is not invertible.

   (iii) Since Z is the product of two triangular matrices, $\det Z = 2 \cdot 1 = 2 \neq 0 \Rightarrow Z$ is invertible.

   (b) $A = B(2C - 3I)B^T \Rightarrow B^{-1}A = B^{-1}B(2C - 3I)B^T = (B^{-1}B)(2C - 3I)B^T$
   $= (2C - 3I)B^T$
   $\Rightarrow (B^{-1}A)(B^T)^{-1} = (2C - 3I)(B^T(B^T)^{-1}) = 2C - 3I \Rightarrow B^{-1}A(B^{-1})^T = 2C - 3I$
   $\Rightarrow 2C = B^{-1}A(B^{-1})^T + 3I. \Rightarrow C = \frac{1}{2}\left[B^{-1}A(B^{-1})^T + 3I\right].$

(a) $$A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 3 & -4 & 7 & -4 \\ -1 & -4 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & -6 & -3 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + R_1 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} .$$
$$i.e\ L = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -1 & -3 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) (i) $$L\mathbf{y} = \mathbf{b} \Leftrightarrow \begin{cases} y_1 = 1 \\ 3y_1 + y_2 = 4 \\ -y_1 - 3y_2 + y_3 = -2 \end{cases} \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 4 - 3y_1 = 4 - 3 = 1 \\ y_3 = -2 + y_1 + 3y_2 = -2 + 1 + 3 = 2 \end{cases}$$
$$i.e\ \ \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(ii) $$U\mathbf{x} = \mathbf{y} \Leftrightarrow \begin{bmatrix} 1 & -2 & 2 & -1 & 1 \\ 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 6 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & 2 & 0 & 3 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\Rightarrow x_1 = 6 - 3x_3,\ x_2 = \frac{3}{2} - \frac{x_3}{2},\ x_4 = 2, \text{ and } x_3 \text{ a free variable.}$$

(a) $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \Rightarrow$ the matrix A has a pivot in each row

$\Rightarrow$ the matrix A has a pivot in each column $\Rightarrow A\mathbf{x} = \mathbf{b}$ has no free variable
$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution.

(b) $A^2 - 2A + I = 0 \Rightarrow A^2 = 2A - I \Rightarrow A^3 = 2A^2 - A \Rightarrow A^3 = 2(2A - I) - A = 3A - 2I.$

(c) $-A$ is obtained by multiplying each row of A by -1. Hence, by the properties of determinants
$$\det(-A) = \underbrace{(-1)(-1)...(-1)}_{n\ times} \det A = (-1)^n \det A.$$

(d) $\begin{vmatrix} a & b & c \\ a+d & b+d & c+d \\ a+e & b+e & c+e \end{vmatrix} = \begin{vmatrix} a & b & c \\ a+d-a & b+d-b & c+d-c \\ a+e-a & b+e-b & c+e-c \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & d & d \\ e & e & e \end{vmatrix}$
$= de \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$ (determinant with two rows that are equal).

(a) Since $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ XA_{11} & S \end{bmatrix}$,
then $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ XA_{11} & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{bmatrix}$.
Hence, $A_{11}Y = A_{12}$, $XA_{11} = A_{21}$, and $XA_{11}Y + S = A_{22}$.
Since A_{11} is invertible, it follows that $Y = A_{11}^{-1}A_{12}$, and $X = A_{21}A_{11}^{-1}$.
Substituting for X and Y in the partitioned matrix we get
$XA_{11}Y + SA_{21}A_{11}^{-1}A_{11}A_{11}^{-1}A_{12} + S = A_{22} \Rightarrow S = A_{22} - A_{21}A_{11}^{-1}A_{12}$
which is the given value for S.

(b) Since $\det A^T = \det A = -2$, and $\det B = 1/\det B = 1/4$, then,
$\det(A^2(B^{-1})^2A^TB^3) = (\det A)^2(1/\det B)^2(\det A)(\det B)^3 = (\det A)^3 \det B$
$= (-8)(4) = -32$.