Sultan Qaboos University-College of Science
Department of Mathematics and Statistics
Final Exam
MATH 2202 Linear Algebra I Fall 2010
January 11, 2011 Total Marks: 100 TIME Allowed: THREE hours

Attempt all the **EIGHT** questions and show all details of your work.

Q1. (a) [7 marks] Consider the linear system

$$\begin{array}{llll} x_1 & +2x_2 & +\alpha x_3 & =1 \\ \alpha x_1 & +\alpha x_2 & & =1 \\ \alpha x_1 & +3\alpha x_2 & +8x_3 & =3, \end{array}$$

where α is a real number. For what value(s) of α does the system have
(i) A unique solution (ii) Infinitely many solutions.

(b) [4 marks] Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be a linear transformation that maps $v_1=\begin{bmatrix}1\\0\end{bmatrix}$ into $\begin{bmatrix}3\\0\\1\end{bmatrix}$ and maps $v_2=\begin{bmatrix}0\\-1\end{bmatrix}$ into $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. Find the image under T of $u=\begin{bmatrix}2\\2\end{bmatrix}$.

Q2. Consider the matrices

$$A=\begin{bmatrix}4&1&6\\-7&5&3\\9&-3&3\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix}$$

and the vectors

$$b=\begin{bmatrix}2\\10\\-6\end{bmatrix} \quad\text{and}\quad p=\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

(a) [4 marks] Show that A is row equivalent to B.
(b) [2 marks] Do the columns of A span $\mathbb{R}^3$? Justify your answer.
(c) [2 marks] Without calculations, list rank A and dim Nul A. Justify your answer.
(d) [2 marks] Find the bases for Col A and row A.
(e) [3 marks] Write the third column of A as a linear combination of the other two columns.
(f) [4 marks] Use part (e) to find the solution of $Ax=0$, then give a basis for Nul A.
(g) [3 marks] Verify that p is a particular solution of $Ax=b$, then find all solutions of $Ax=b$.

Q3.

(a) [3 marks] Explain why the set $V=\left\{\begin{bmatrix}x\\y\end{bmatrix}: xy\le 0\right\}$ is not a subspace of $\mathbb{R}^2$.

(b) [5 marks] Show that $W=\left\{\begin{bmatrix}a-b+3c\\b-c\\2a+3b+c\end{bmatrix}: a,b,c\in\mathbb{R}\right\}$ is a subspace of $\mathbb{R}^3$. Find a basis for W.

(c) [4 marks] Show that $H=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}: \begin{array}{c} a-3b+2c=0\\ 3a=4c\end{array}\right\}$ is a subspace of $\mathbb{R}^4$, and determine the dimension of H.

Please turn over for the remaining questions.

Q 4. Let P_3 be the vector space of the polynomials of degree at most 3 and define $T : P_3 \to \mathbb{R}^3$ by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

(a) [2 marks] Find the image under T of $p(t) = 1 + t + t^2 + t^3$.
(b) [4 marks] Show that T is a linear transformation.
(c) [3 marks] Find a non-zero polynomial $p \in P_3$ which belongs to Ker T.

Q5. Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
(a) [2 marks] Show that v_1 and v_2 are eigenvectors of A.
(b) [3 marks] Use part (a) to diagonalize A.
(c) [4 marks] Compute A^{2011}.

Q6. Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 5 & 5 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.
(a) [5 marks] Show that the characteristic equation of A is $(\lambda - 4)(\lambda + 1)^3 = 0$.
(b) [8 marks] Find the eigenvalues of A and then the basis for the eigenspace corresponding to each eigenvalue.
(c) [4 marks] Diagonalize A.

Q7. Prove the following statements.
(a) [4 marks] If A and B are $n \times n$ matrices such that A is similar to B, then A and B have the same eigenvalues.
(b) [4 marks] For any $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.
(c) [2 marks] For any $m \times n$ matrix A, rank A+dim Nul$A = n$.
(d) [3 marks] For all real numbers a, b and c, the matrix

$$\begin{bmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{bmatrix}$$

is not invertible.

Q8. Give an example for each of the following statements. Justify your example.
(a) [3 marks] A diagonalizable matrix that is not invertible.
(b) [3 marks] An invertible matrix that is not diagonalizable.
(c) [3 marks] 2×2 non-zero matrices A and B such that $\det(A + B) = \det A + \det B$.

—————— // ——————