SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 2202- Linear Algebra I

Date: 9.3.2009	**MIDTERM EXAM 1 - SPRING 2009**	Time: 60 minutes

Answer the following questions (show all work).

---

1. [13 marks] Let $A = \begin{bmatrix} -1 & 3 & -3 & 2 \\ 2 & -5 & 2 & -2 \\ -3 & 7 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -9 & 4 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

   (a) Show that A is row equivalent to B.

   (b) Do the columns of A span $\mathbb{R}^3$? Justify.

   (c) Solve the linear system $A\mathbf{x} = \mathbf{0}$. Write the solution in parametric vector form, and give the vectors that span the solution set.

   (d) Verify that $A\mathbf{p} = \mathbf{b}$. Then, obtain the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and give a geometric description of the solution.

2. [8 marks] Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Determine whether the following sets are linearly independent. Use as few calculations as possible. Justify your answer.
   (i) $\{\mathbf{v}_1, \mathbf{v}_3\}$ *(ii)* $\{\mathbf{v}_2, \mathbf{v}_4, -3\mathbf{v}_2 + 4\mathbf{v}_4\}$ *(iii)* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ *(iv)* $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4\}$

3. [10 marks]

   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points counterclockwise through $\frac{3\pi}{4}$ and then reflects the resulting points through the x_1-axis. Find the standard matrix for T. Is T a one to one mapping? Does T map $\mathbb{R}^2$ onto $\mathbb{R}^2$? Justify.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} -6 \\ 3 \end{bmatrix}$. Find the image under T of $\begin{bmatrix} 1 \\ -13 \\ 11 \end{bmatrix}$.

4. [9 marks] The following statements are **true**. Give a justification for each one.

   (a) Let A be an $m \times n$ matrix, where $m < n$. Then, the linear system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution.

   (b) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^4$, and $T : \mathbb{R}^4 \to \mathbb{R}^3$ a linear transformation. Then, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is also a linearly dependent set.

   (c) The transformation defined by $T(x_1, x_2) = (x_1 + 4x_2, x_2 - 3\,|x_1|)$ is not linear.

Total: 40 marks

SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 2202- Linear Algebra I

Date: 20.4.2009 | **MIDTERM EXAM 2 - SPRING 2009** | Time: 60 minutes

Answer the following questions (show all work).

---

1. [8 marks]

   (a) Determine by inspection (no calculation) which of the following matrices have inverses. Explain.
   $$X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ Y = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 & 1 & 4 \end{bmatrix}, \ Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

   (b) If $A,\ B,\ C$ are invertible matrices such that $A = B(2C - 3I)B^T$, show that $C = \frac{1}{2}[B^{-1}A(B^{-1})^T + 3I]$.

2. [12 marks] Let $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 3 & -4 & 7 & -4 \\ -1 & -4 & -5 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

   (a) Find an LU factorization of the matrix A. Check your answer.

   (b) Use the LU factorization found in (a) to solve the equation $A\mathbf{x} = \mathbf{b}$.

3. [12 marks] The following statements are true. Explain why?

   (a) Suppose that A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$. Then, each equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

   (b) If an $n \times n$ matrix A satisfies $A^2 - 2A + I = 0$, then $A^3 = 3A - 2I$.

   (c) If A is an $n \times n$ matrix, then $\det(-A) = (-1)^n \det A$.

   (d) If a, b, c, d, and e are real numbers, then
   $$\begin{vmatrix} a & b & c \\ a+d & b+d & c+d \\ a+e & b+e & c+e \end{vmatrix} = 0.$$

4. [8 marks]

   (a) Suppose A_{11} is invertible. Find X, and Y such that
   $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix},$$
   where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

   (b) Suppose that A and B are two $n \times n$ invertible matrices such that $\det A = -2$ and $\det B = 4$. Compute $\det(A^2(B^{-1})^2A^TB^3)$.

Total: 40 marks

SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 2202 - Linear Algebra I

Date: 18.5.2009	**FINAL EXAM - SPRING 2009**	Time: 150 minutes

Answer the following questions (show all work).

---

1. [18 marks] Let $A = \begin{bmatrix} 1 & 1 & -1 & -4 & 1 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & -2 & 6 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,
and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Show that A is row equivalent to B.
   (b) Determine if $\mathbf{v}$ is in $Nul\ A$.
   (c) Find bases for $Col\ A$ and $Nul\ A$.
   (d) Find $\dim(Nul\ A)$ and $Rank\ A$.

2. [11 marks] Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

   (a) Show that $\mathbf{v}$ is an eigenvector of A, and determine its eigenvalue.
   (b) Find the eigenvalues of A, and give the corresponding eigenvectors.

3. [17 marks] Let $A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$.

   (a) Find an LU factorization for A, and show that A is invertible.
   (b) Use the factorization found in (a) to compute A^{-1}.

4. [9 marks] Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4;\ \begin{cases} a - 2b = 4c \\ 2a = c + 3d \end{cases} \right\}$.

   (a) Find two matrices A and B such that $W = Nul\ A$ and $W = Col\ B$.
   (b) Use the answer of part (a) to show that W is a subspace of $\mathbb{R}^4$.

PLEASE TURN OVER FOR THE REMAINING QUESTIONS $\gg\gg$

5. [10 marks] Consider the set of polynomials $\mathfrak{B} = \{1 - t, t + t^2, 1 + 2t + t^2\}$.

(a) Show that $\mathfrak{B}$ is a basis for $\mathbb{P}_2$ (the set of real polynomials of degree at most 2).

(b) Assuming that a polynomial p in $\mathbb{P}_2$ has coordinate vector, relative to $\mathfrak{B}$, $[p]_{\mathsf{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, find the polynomial p.

6. [13 marks] The following statements are true. Explain why?

(a) Let A be an $m \times n$ matrix. If the sum of the column vectors of A is equal to zero, then $rank\ A < n$.

(b) If $B = [\mathbf{b}_1\ \mathbf{b}_2\ ...\ \mathbf{b}_p]$, then $\{A\mathbf{b}_1, A\mathbf{b}_2, ..., A\mathbf{b}_p\}$ spans $Col\ AB$.

(c) If λ is an eigenvalue of a square matrix A, then λ^2 is an eigenvalue of the matrix A^2.

(d) Let A be an $n \times n$ matrix. If $A^3 = 0$, then $(I - A)(I + A + A^2) = 0$, and $(I - A)^{-1} = I + A + A^2$.

(e) If A is an invertible $n \times n$ matrix, then $Adj\ A$ is invertible and $(Adj\ A)^{-1} = \frac{1}{\det A} A$.

7. [12 marks] Let $M_{2\times 2}$ be the vector space of all 2×2 real matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by: $T(A) = A + A^T$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in \mathbb{R}$.

(a) Show that T is a linear transformation.

(b) Find a nonzero 2×2 matrix that spans the kernel of T.