Q. No. 1 $\quad A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

(i) Since A has three rows, L should be 3×3.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & -5 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A = LU$$

(ii) $Ax = LU(x) = L(U(x)) = b$. Let $U(x) = y$. Then

$Ly = b$.

Solve $Ly = b$:

$$[L\ b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -3 & 1 & 0 & 6 \\ 1 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 12 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & -10 \end{array}\right] = [I\ y]$$

$$\therefore y = \begin{bmatrix} 2 \\ 12 \\ -10 \end{bmatrix}$$

Now we solve $Ux = y$

$$[U\ y] = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 1 & 7/2 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 4 \\ 0 & 1 & 7/2 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Q. No.2

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

(i) $\det A = (4+12) - 2(9-2) = 16 - 14 = 2$

(ii) $C_{11} = (4+12) = 16 \quad C_{21} = -(0-6) = 6 \quad C_{31} = (0+2) = 2$

$C_{12} = -(-12-8) = 20 \quad C_{22} = (4+4) = 8 \quad C_{32} = -(4-6) = 2$

$C_{13} = (9-2) = 7 \quad C_{23} = -(-3-0) = 3 \quad C_{33} = (1-0) = 1$

$$\therefore \operatorname{Adj} A = \begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix}$$

(iii) $$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{2}\begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Q. No.3 (i) $$\begin{bmatrix} a+3b \\ b-c \\ 2c-a \\ 4b \end{bmatrix} = a\underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} + b\underset{v_2}{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \end{bmatrix}} + c\underset{v_3}{\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}}$$

Since $H = \operatorname{Span}\{v_1, v_2, v_3\}$, therefore H is a subspace of $\mathbb{R}^4$

(ii) $$A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 3 & -6 & 1 & -1 \\ 2 & -4 & 5 & 8 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$$

$$[A\ w] = \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 3 & -6 & 1 & -1 & 7 \\ 2 & -4 & 5 & 8 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & -5 & -10 & 10 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & -5 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It is clear that the equation $Ax = w$ is consistent, so w is in $\operatorname{Col} A$. W is not in $\operatorname{Nul} A$, since W has only three entries and $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^4$.

Q. No. 4 The coordinate ~~vectors~~ mapping produces the coordinate vectors:
$(1,-3,3,-1)$, $(4,-12,9,0)$, $(0,0,3,-4)$, respectively. We shall see linear independence of these vectors by writing them as columns of a matrix.

$$\begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 0 & 4 & -4 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the matrix does not have a pivot in each column, its columns are linear dependent and so the corresponding polynomials are linear dependent.

Q. No. 5 (i) $A = \begin{bmatrix} 1 & 4 & 5 & -9 \\ -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & -3 & -6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 0 & -5 \end{bmatrix} = B$

Basis for Row $A = \{(1,4,5,-9), (0,2,4,-6), (0,0,0,-5)\}$

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 4 \end{bmatrix} \right\}$

To find basis for Nul A, we need reduced echelon form of A.

$$A \sim B = \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Basis for Nul $A = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(ii) Given A is 9×14 matrix and dim Nul $A = 8$.
By the Rank Theorem, Rank $A = 14 -$ dim Nul $A = 6$.
$\therefore$ dim Row A = Rank $A = 6$.
