MATH 2202 - FALL 2009
FINAL EXAM SOLUTIONS

1. (a)

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ -4 & -4 & 2 & 0 & -2 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & -6 & 12 & -6 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & -6 & 12 & -6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(b) The columns of A do not span $\mathbb{R}^4$ because we don't have a pivot position in each row.

(c) Observe that B has pivots in 1, 3, and 4. Hence, columns 1, 3, and 4 of A form a basis for $Col\ A$:

$$\text{Basis for } Col\ A : \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$A \sim B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the equation $A\mathbf{x} = \mathbf{0}$ is equivalent to $B\mathbf{x} = \mathbf{0}$, that is:

$$\begin{array}{ccccccc} x_1 & +x_2 & & & +x_5 & = & 0 \\ & & x_3 & & +x_5 & = & 0 \\ & & & x_4 & & = & 0 \end{array}$$

So, $x_1 = -x_2 - x_5$, $x_3 = -x_5$, $x_4 = 0$, with x_2 and x_5 free variables. Hence, the basis of $Nul\ A$ is given by:

$$\text{Basis for } Nul\ A : \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) There are two free variables x_2 and x_5. Hence $\dim(Nul\ A) = 2$. Also, $Rank\ A = 3$, because the number of pivot columns is 3.

2. (a) $B\left(A^TA+X\right)B=I \Rightarrow B^{-1}B\left(A^TA+X\right)B=B^{-1} \Rightarrow \left(A^TA+X\right)B=B^{-1}$
$\Rightarrow \left(A^TA+X\right)BB^{-1}=(B^{-1})^2 \Rightarrow A^TA+X=(B^{-1})^2$
$\Rightarrow X=(B^{-1})^2-A^TA.$

(b) $X^T=(B^{-1}B^{-1}-A^TA)^T=(B^{-1}B^{-1})^T-\left(A^TA\right)^T=(B^{-1})^T(B^{-1})^T-A^TA$
$=(B^T)^{-1}(B^T)^{-1}-A^TA=(B^{-1})(B^{-1})-A^TA$ (because $B^T=B$)
$\Rightarrow X^T=(B^{-1})^2-A^TA=X.$

(c) Since $\det A^T=\det A,\ \det(A^3)=(\det A)^3,\ \det B^2=(\det B)^2$,
and $\det B^{-1}=1/\det B$, then
$\det(A^3B^{-1}A^TB^2)=(\det A)^4\times\det B=16\times -3=-48.$

3. (a) W_1 is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector.

(b) W_2 is not a subspace of $\mathbb{R}^3$ because it is not closed under scalar multiplication.
For example: the vector $u=\begin{bmatrix}2\\-1\\1\end{bmatrix}\in W_2$,but the vector $-u=\begin{bmatrix}-2\\1\\-1\end{bmatrix}\notin W_2$.

(c) Clearly W_3 contains the zero vector. Moreover, if $\mathbf{u}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ are two vectors belonging to W_2 and $c\in\mathbb{R}$, then:
$x_3=2x_2,\ x_1+x_2=0$ and $y_3=2y_2,\ y_1+y_2=0 \Rightarrow x_3+y_3=2(x_2+y_2)$ and $x_1+y_1+x_2+y_2=0 \Rightarrow \mathbf{u}+\mathbf{v}\in W_3$. Also, we have $cx_3=2cx_2,\ cx_1+cx_2=0$
$\Rightarrow c\mathbf{u}\in W_3$. Hence, W_3 is a subspace of $\mathbb{R}^3$.

4. (a) $W=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\in\mathbb{R}^4;\ \left\{\begin{array}{r}2a+3b=-c\\4b-c+2a=2d\end{array}\right.\right\}$

$=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\in\mathbb{R}^4;\ \begin{bmatrix}2&3&1&0\\2&4&-1&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\mathbf{0}\right\}$

$=\left\{\begin{bmatrix}-\frac{3}{2}b-\frac{1}{2}c\\b\\c\\-b-\frac{3}{2}c\end{bmatrix};\ b,c\in\mathbb{R}\right\}.$

Hence, $W=Nul\ A=Col\ B$,

where $A=\begin{bmatrix}2&3&1&0\\2&4&-1&-2\end{bmatrix}$ and $B=\begin{bmatrix}-\frac{3}{2}&-\frac{1}{2}\\1&0\\0&1\\-1&-\frac{3}{2}\end{bmatrix}$.

(b) Since $W=Nul\ A$, and $Nul\ A$ is a subspace of $\mathbb{R}^4$, so is W.

5. (a) $\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & -2-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{vmatrix}=(1-\lambda)\left[(-2-\lambda)^2-0\right]-1(0-0)$
$=(1-\lambda)\left(-2-\lambda\right)^2=(1-\lambda)(2+\lambda)^2.$

(b) The eigenvalues of $A:\;\det(A-\lambda I)=0\Rightarrow(1-\lambda)(2+\lambda)^2=0\Rightarrow\lambda=1$ or $\lambda=-2.$

<u>Eigenvector for $\lambda=1$:</u> $(A-\lambda I)\,x=0\sim\begin{bmatrix} 0 & 0 & -1 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}\sim\begin{bmatrix} 3 & -3 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\sim\begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x_1=x_2,\;x_3=0$, and x_2 a free variable. Hence the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvetor of A.

<u>Eigenvector for $\lambda=-2$:</u> $(A-\lambda I)\,x=0\sim\begin{bmatrix} 3 & 0 & -1 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim\begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\Rightarrow x_1=x_3=0$, and x_2 is a free variable.

Hence the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvetor of A.

6. (a) Let $\det P=1\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix}-1\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}=1$. Let C be the cofactors matrix, so we have:
$C_{11}=-1,\;C_{12}=2,\;C_{13}=1,$
$C_{21}=-2,\;C_{22}=3,\;C_{23}=1,$
$C_{31}=-2,\;C_{32}=2,$ and $C_{33}=1.$
Hence,

$$P^{-1}=\tfrac{1}{\det A}Adj\;A=\tfrac{1}{\det A}C^T=\begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) $c_1(1-t^2)+c_2(t-t^2)+c_3(2-2t+t^2)=0\Rightarrow c_1\left[1-t^2\right]_{\mathsf{B}}+c_2\left[t-t^2\right]_{\mathsf{B}}$
$+\,c_3\left[2-2t+t^2\right]_{\mathsf{B}}=[0]_{\mathsf{B}}$
$\sim\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\Rightarrow c_1=c_2=c_3=0\Rightarrow$ *the* polynomials of $\mathfrak{B}$ are linearly independent.
Since the $\dim(\mathbb{P}_2)=3$, then $\mathfrak{B}$ is a basis for $\mathbb{P}_2$.

(c) $\mathbf{x}_{\mathsf{B}}=P^{-1}\mathbf{x}=\begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}=\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$

7. (a) $T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(\frac{1}{2}) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(\frac{1}{2}) \end{bmatrix} = T(p) + T(q),$
and $T(cp) = \begin{bmatrix} (cp)(0) \\ (cp)(\frac{1}{2}) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p(\frac{1}{2}) \end{bmatrix} = cT(p),$ for $c \in \mathbb{R}$.
Hence, T is a linear transformation.

(b) $p \in Ker\ T \Rightarrow p(t) = a_0 + a_1 t + a_2 t^2 = 0$ with $p(0) = p(\frac{1}{2}) = 0$
$\Rightarrow \begin{cases} a_0 = 0 \\ \frac{1}{2}a_1 + \frac{1}{4}a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_2 = -2a_1 \end{cases} \Rightarrow p(t) = a_0 + a_1 t + a_2 t^2 = a_1 t(1-2t).$
Hence, $t(1-2t)$ spans $Ker\ T$.

8. (a) $B = \{b_1, b_2, ..., b_n\} \Rightarrow AB = A\,[b_1 b_2 ... b_n] = [Ab_1 Ab_2 ... Ab_n]$. Hence, the set $\{Ab_1 Ab_2 ... Ab_n\}$ spans $Col\ AB$.

(b) A is invertible $\Rightarrow$ the columns of A are linearly independent, which by the rank Th. implies that $\dim(Nul\ A) = 0$.

(c) Let λ be an eigenvalue of A, then linearity of multiplication gives
$A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda(\lambda x) = \lambda^2 x.$

(d) $(I+A)(I-A+A^2) = I - A + A^2 + A - A^2 + A^3 = I$ (because $A^3 = 0$).
Hence, by the inverse matrix Th. $(I+A)$ is invertibe and $(I+A)^{-1} = I - A + A^2$.