SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 2202- Linear Algebra I

Date: 20.10.2008	**MIDTERM EXAM 1 - FALL 2008**	Time: 60 minutes

Answer the following questions (show all work).

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1. [13 marks] Let $A = \begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 & -9 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) Show that A is row equivalent to B.

   (b) Do the columns of A span $\mathbb{R}^3$? Justify your answer.

   (c) Are the columns of A linearly independent? Justify your answer.

   (d) Solve the linear system $A\mathbf{x} = \mathbf{0}$. Write the solution in parametric vector form, and give a geometric description of the solution set.

2. [8 marks] Determine the value(s) of the parameters a and b so that the system

$$\begin{cases} x_1 & - & x_2 & - & x_3 & = & 1 \\ -x_1 & + & x_2 & + & ax_3 & = & 2b \\ ax_1 & + & 3ax_2 & + & x_3 & = & 3 \end{cases} ; \text{ has}$$

   (a) no solution ; (b) a unique solution ; (c) infinity of solutions

3. [11 marks]

   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the line $x_2 = -x_1$, and then reflects the resulting points through the x_1-axis. Find the standard matrix for T. Is T a one-to-one mapping? Justify your answer.

   (b) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$. Find the image under T of $\begin{bmatrix} -5 \\ -6 \end{bmatrix}$.

4. [8 marks] The following statements are **true**. Give a justification for each one.

   (a) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then the set $\{2\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, -\mathbf{v}_1 + \mathbf{v}_3\}$ is also linearly independent.

   (b) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by: $T(x_1, x_2) = (1 + x_1^2, x_1 + x_2)$ is not linear.

   (c) If $A\mathbf{x} = \mathbf{b}$ has a solution and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the solution of $A\mathbf{x} = \mathbf{b}$ is unique.

Total: 40 marks

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DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 2202- Linear Algebra I

Date: 30.11.2008	**MIDTERM EXAM 2 - FALL 2008**	Time: 60 minutes

Answer the following questions (show all work).

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1. [8 marks] Assume that $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \\ -1 & -2 & 1 & 0 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find an explicit description of **Nul** A, by listing vectors that span the null space.

   (b) Find a nonzero vector in **Col** A.

2. [7 marks] Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4; \quad \begin{array}{r} a - c = -3b \\ b + c - d = -a \end{array} \right\}$

   (a) Find a matrix A such that $W = \textbf{Nul}\ A$.

   (b) Use the answer of part (a) to show that W is a subspace of $\mathbb{R}^4$.

3. [9 marks] Explain why the following sets are not subspaces of $\mathbb{R}^2$.
   (i)$\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2;\ ab \geqslant 0 \right\}$, (ii)$\left\{ \begin{bmatrix} x \\ 2x+3 \end{bmatrix};\ x \in \mathbb{R} \right\}$, and (iii)$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2;\ 2x + 5y < 0 \right\}$.

4. [8 marks] Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most two, and define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(0) \\ p(-1) \\ p(0) \end{bmatrix}$

   (a) Show that T is a linear transformation.

   (b) Find a polynomial $p \in \mathbb{P}_2$ that spans the kernel of T.

5. [8 marks] The following statements are **true**. Give a justification for each one.

   (a) If $A,\ B$ are two matrices such that $(A+B)^2 = A^2 + B^2$, then $BA = -AB$.

   (b) If A is an $n \times n$ matrix such that $A^T A = I$, then $det\ A = \pm 1$.

   (c) If $A,\ B,\ X$ are $n \times n$ matrices with $X,\ B + BX$ invertible, then $(B + BX)^{-1} = X^{-1}A$ implies that A is also invertible.

$\boxed{\text{Total: 40 marks}}$

SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 2202 - Linear Algebra I

Date: 10.1.2009	**FINAL EXAM - FALL 2008**	Time: 150 minutes

Answer the following questions (show all work).

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1. [20 marks] Let $A = \begin{bmatrix} -1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Show that A is row equivalent to B.

   (b) Do the columns of A span $\mathbb{R}^4$? Justify your answer.

   (c) Find bases for $Col\ A$, $Row\ A$, and $Nul\ A$.

   (d) Find $\dim(Nul\ A)$ and $Rank\ A$.

2. [12 marks] Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

   (a) Use cofactor expansion to show that the characteristic polynomial of A is $(-\lambda - 1)(2 - \lambda)^2$.

   (b) Find the eigenvalues and the corresponding eigenvectors of A.

3. [10 marks] Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$T(x_1, x_2, x_3) = (x_1 + 2x_3, x_3 - x_2, -x_1 + x_2)$$

   (a) Show that T has standard matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

   (b) Use Cramer's Rule to find the second component of $\mathbf{x}$ if $T(\mathbf{x}) = (-1, -3, 1)$.

   (c) Is $T \circ T$ a one-to-one linear transformation? Explain.

4. [9 marks] Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4;\ \left\{ \begin{array}{r} a + 3b = c \\ b + c + a = d \end{array} \right. \right\}$.

   (a) Find two matrices A and B such that $W = Nul\ A$ and $W = Col\ B$.

   (b) Use the answer of part (a) to show that W is a subspace of $\mathbb{R}^4$.

PLEASE TURN OVER FOR THE REMAINING QUESTIONS $\ggg$

5. [12 marks] Given u in $\mathbb{R}^3$, with $u^T u = 1$, Let $P = u\, u^T$ and $Q = I - 2P$.

   (a) Verify that the following statements are true.
   (i) $P^2 = P$ (ii) $P^T = P$ (iii) $Q^2 = I$

   (b) If $u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find P and Q. Then, determine the geometric transformation defined, for $\mathbf{x} \in \mathbb{R}^3$, by $S(\mathbf{x}) = Q\mathbf{x}$.

6. [14 marks] Let $P = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$, and $\mathfrak{B} = \{1-t, t-t^2, 2t^2-1\}$.

   (a) Find P^{-1}.

   (b) Show that $\mathfrak{B}$ is a basis for $\mathbb{P}_2$ (the set of real polynomials of degree at most 2).

   (c) Use P^{-1} to find the coordinate vector $\mathbf{x}_{\mathsf{B}}$ of the polynomial $p(t) = 1 - 2t + 3t^2$ with respect to $\mathfrak{B}$.

7. [14 marks] The following statements are true. Explain why?

   (a) If A is a 3×3 matrix, then $\det 5A = 5^3 \det A$.

   (b) Let A be an $m \times n$ matrix. If the sum of the column vectors of A is equal to zero, then $Rank\ A < n$.

   (c) If A is an $n \times n$ matrix having an eigenvalue equal to zero, then A is not invertible.

   (d) If A is an invertible $n \times n$ matrix, then $Adj\ A$ is invertible and $(Adj\ A)^{-1} = \frac{1}{\det A} A$.

8. [9 marks]

   (a) Use row elimination to show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

   (b) If A is a 3×3 matrix satisfying $A^2 - 2A + I = 0$, show that $A^3 = 3A - 2I$.