Answers to the Final Exam MATH 2202, Fall 2010

(1) (a) Denote by $\tilde{A}$ the augmented matrix of the given linear system, we have

$$\tilde{A} = \begin{bmatrix} 1 & 2 & \alpha & 1 \\ \alpha & \alpha & 0 & 1 \\ \alpha & 3\alpha & 8 & 3 \end{bmatrix} \underset{-\alpha R_1 + R_3}{\overset{-\alpha R_1 + R_2}{\sim}} \begin{bmatrix} 1 & 2 & \alpha & 1 \\ 0 & -\alpha & -\alpha^2 & 1-\alpha \\ 0 & \alpha & 8-\alpha^2 & 3-\alpha \end{bmatrix}$$

$$\overset{R_2 + R_3}{\sim} \begin{bmatrix} 1 & 2 & \alpha & 1 \\ 0 & -\alpha & -\alpha^2 & 1-\alpha \\ 0 & 0 & 8-2\alpha^2 & 4-2\alpha \end{bmatrix}.$$

Clearly the system is inconsistent for $\alpha = 0$.

(i) The system has a unique solution if $\alpha \neq 0$ and $8 - 2\alpha^2 \neq 0$. That is if $\alpha \neq 0$ and $\alpha \neq \mp 2$.

(ii) The system has infinitely many solutions if $\alpha \neq 0$, $8 - 2\alpha^2 = 0$ and $4 - 2\alpha = 0$. That is $\alpha \neq 0$, $\alpha = \mp 2$ and $\alpha = 2$. That is if $\alpha = 2$.

(b) We have $u = 2v_1 - 2v_2$. Since T is linear, we have

$$T(u) = T(2v_1 - 2v_2) = 2T(v_1) - 2T(v_2)$$

$$= 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 2 \end{bmatrix}.$$

(2)

(a) $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix} \underset{-9/4 R_1 + R_3}{\overset{7/4 R_1 + R_2}{\sim}} \begin{bmatrix} 4 & 1 & 6 \\ 0 & 27/4 & 54/4 \\ 0 & -21/4 & -42/4 \end{bmatrix}$

$\underset{-\frac{4}{21} R_3}{\overset{\frac{4}{27} R_2}{\sim}} \begin{bmatrix} 4 & 1 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \overset{-R_2+R_3}{\sim} \begin{bmatrix} 4 & 1 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{-R_2+R_1}{\sim} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$

(b) No, because A does not have a pivot position in row 3.

(c) Rank A = No. of pivot columns = 2

dim Nul A = No. of non pivot columns = 1.

(d) Bases for Col $A = \left\{ \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \right\}$

Bases for Row $A = \{ (1, 0, 1), (0, 1, 2) \}$.

(e) $\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$

(f) By (e) $\begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This means that the nontrivial solution of $Ax=0$ is $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Since dim Nul $A = 1$, then any solution of $Ax=0$ is a multiple of u, i.e.

$x = tu$, $t \in \mathbb{R}$. Basis for Nul $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(g) $Ap = b \Rightarrow$ Any solution to $Ax = b$ is given by

$$x = tu + p = t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$ where $t \in \mathbb{R}$.

(3)

(a) The set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \leq 0 \right\}$ is not a subspace of $\mathbb{R}^2$ because it is not closed under vector addition. For example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin V$.

(b) It is easy to see that W is the set of all linear combination of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

So $W = \text{span}\{v_1, v_2, v_3\}$. Hence W is a subspace of $\mathbb{R}^3$.

Since $v_3 = 2v_1 - v_2$, $W = \text{span}\{v_1, v_2\}$. v_1 and v_2 are linearly independent since neither v_1 is a multiple of v_2 neither v_2 is a multiple of v_1.

So $\{v_1, v_2\}$ is basis for W.

(c) $H = \text{Nul } A$, where $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 4 \end{bmatrix}$. So H is a subspace of $\mathbb{R}^4$. To determine $\dim H = \dim \text{Nul } A$, Solve $Ax = 0$. So $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & 0 & -4 & 0 \end{bmatrix} \overset{-3R_1+R_2}{\sim} \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 9 & -10 & 0 \end{bmatrix}$.

$\dim H = \dim \text{Nul } A =$ The number of nonpivot columns of $A = 1$.

(4)

(a) $T(1+t+t^2+t^3) = \begin{bmatrix} 1+0+0^2+0^3 \\ 1+1+1^2+1^3 \\ 1+2+2^2+2^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \end{bmatrix}$.

(b) For any polynomials $p, q \in \mathbb{P}_3$ and for any scalar α, we have

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \\ (p+q)(2) \end{bmatrix} = \begin{bmatrix} p(0)+q(0) \\ p(1)+q(1) \\ p(2)+q(2) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \\ q(2) \end{bmatrix}$$

$$= T(p) + T(q),$$

$$T(\alpha p) = \begin{bmatrix} (\alpha p)(0) \\ (\alpha p)(1) \\ (\alpha p)(2) \end{bmatrix} = \begin{bmatrix} \alpha p(0) \\ \alpha p(1) \\ \alpha p(2) \end{bmatrix} = \alpha \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} = \alpha T(p).$$

(c) $\operatorname{Ker} T = \left\{ p \in \mathbb{P}_3 : T(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$= \left\{ p \in \mathbb{P}_3 : \begin{bmatrix} p(0) = 0 \\ p(1) = 0 \\ p(2) = 0 \end{bmatrix} \right\}$

Choose $p(t) = t(t-1)(t-2) = 2t - 3t^2 + t^3$

(5) (a) We have $Av_1 = 2v_1$ and $Av_2 = v_2$. Then v_1 and v_2 are eigenvectors of A corresponding respectively to the eigenvalues $\lambda = 2$ and $\lambda = 1$.

(b) We have $A = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$

(c) $A^{2011} = PD^{2011}P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{2011} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3(2^{2011}) & 1 \\ 2(2^{2011}) & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3(2^{2011}) - 2 & -3(2^{2011}) + 3 \\ 2(2^{2011}) - 4 & -2(2^{2011}) + 6 \end{bmatrix}$.

(6) (a)

$$0 = |A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ 5 & 5 & 4-\lambda & -5 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\begin{vmatrix} -1-\lambda & 0 & 0 \\ 5 & 4-\lambda & -5 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 \begin{vmatrix} 4-\lambda & -5 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)^3(4-\lambda).$$

This gives that $(\lambda+1)^3(\lambda-4) = 0$.

(b) The eigenvalues of A is $\lambda = -1$ (with multiplicity 3) and $\lambda = 4$. For $\lambda = -1$, solve $(A+I)x = 0$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Then}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The basis for the eigenspace of $\lambda = -1$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 4$, Solve $(A - 4I)x = 0$.

$$\begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 5 & 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \overset{-R_1+R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\overset{-R_2+R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ So}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$ The basis of the eigenspace of

$\lambda = 4$ is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(c) Let $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & -1 & -1 & +1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

To check $A = PDP^{-1}$, it is enough to check

$AP = PD = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

(7) (a) A is similar to B means there exists an invertible matrix P such that $B = P^{-1}AP$. Now if λ is an eigenvalue of A, then

$0 = |B - \lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda I P|$
$= |P^{-1}(A - \lambda I)P| = |P^{-1}|\,|A - \lambda I|\,|P| = \frac{1}{|P|}|A - \lambda I|\,|P|$
$= |A - \lambda I|$.

So λ is an eigenvalue of A.

(b) $\text{Nul } A = \{ x \in \mathbb{R}^n : Ax = 0 \}$.

- obviosly $A \cdot 0 = 0$. Hence $0 \in \text{Nul } A$.
- Let $u, v \in \text{Nul } A$ and α any scalar. Then $Au = 0$, $Av = 0$. Hence $A(u+v) = Au + Av = 0$, so $u + v \in \text{Nul } A$ and
- $A(\alpha u) = \alpha Au = \alpha \cdot 0 = 0$. So $\alpha u \in \text{Nul } A$. Hence $\text{Nul } A$ is a subspace of $\mathbb{R}^n$.

(c) $\text{Rank } A$ = No. of pivot columns of A

$\dim \text{Nul } A$ = No. of non pivot columns of A

So n = Columns of A = No. of pivot Columns of A + No. of non pivot columns of A

= rank A + dim Nul A.

(d) $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix}$$

$$= (b-a)(a-c) - (a-b)(c-a) = 0.$$

So the matrix is not invertible.

(8) (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$ (distinct), so A is diagonalizable but $\det A = 0$. So A is not invertible.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $\det A = 1$. So A is invertible. Next, $0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$ so $\lambda_1 = \lambda_2 = 1$. Solve $(A-I)x = 0$. Then

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ gives x_1 free, $x_2 = 0$. So

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_2$. The basis of the eigenspace of $\lambda = 1$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. A cannot be diagonalized.

(c) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = 1$, $\det B = -1$. Now $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Hence $\det(A+B) = 0 = 1 + (-1) = \det A + \det B$.