Answers to Test II
MATH 2202

(1) [5 marks]

$$\det\left(C^3 A^{-1} B^T (C^T)^{-1}\right) = \det C^3 \det A^{-1} \det B^T \det (C^T)^{-1}$$
$$= (\det C)^3 \frac{1}{\det A} \det B \frac{1}{\det C^T}$$
$$= (\det C)^3 \frac{\det B}{\det A} \frac{1}{\det C}$$
$$= \frac{(\det C)^2 \det B}{\det A}$$
$$= \frac{(-3)^2\, 2}{2} = 9.$$

(2) [7 marks]

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} a+3b-c=0 \\ a+b+c-d=0 \end{array} \right\}$$
$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

So $H = \text{Nul } A$, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Now, Solve $[A : 0]$, $\begin{bmatrix} 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix} \overset{-R_1+R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \end{bmatrix}$

$-2x_2 + 2x_3 - x_4 = 0 \Rightarrow x_2 = x_3 - \frac{1}{2}x_4$.

$x_1 + 3(x_3 - \frac{1}{2}x_4) - x_3 = 0 \Rightarrow x_1 + 2x_3 - \frac{3}{2}x_4 = 0 \Rightarrow x_1 = -2x_3 + \frac{3}{2}x_4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + \frac{3}{2}x_4 \\ x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$ Hence

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

(5) [6 marks]

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$ And

$$0 = \det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & x & y \\ 0 & x - x_1 & y - y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \end{bmatrix} \Rightarrow$$

$$(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1) = 0 \Rightarrow$$

$$(x - x_1)(y_2 - y_1) = (y - y_1)(x_2 - x_1) \Rightarrow$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

(6) [6 marks]

For the solution, see the book page 209, Example 5.

(3) [10 marks]

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \overset{\frac{1}{2}R_1}{\sim} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \begin{array}{c} 2R_1+R_2 \\ \sim \\ -3R_1+R_3 \end{array} \begin{bmatrix} -1 & 2 & -1 & 2 \\ 0 & -2 & -5 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 2 \\ 0 & -2 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ So $\operatorname{Col} A = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$.

Next, $2x_2 = -5x_3 - 3x_4 \Rightarrow x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$. And

$$-x_1 + 2\left(-\frac{5}{2}x_3 - \frac{3}{2}x_4\right) - x_3 + 2x_4 = 0 \Rightarrow$$

$$x_1 = -6x_3 - x_4.$$ Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 - x_4 \\ -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$ So

$$\operatorname{Nul} A = \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(4) [6 marks]

Compute

$$D = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= 2(0-2) - (-3-6) + (-1-0) = -4 + 9 - 1 = 4.$$

$$A_1(b) = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = 4\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= 4(0-2) - (6+4) + (2-0) = -8 - 10 + 2 = -16$$

$$A_2(b) = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 2(6+4) - 4(-3-6) + (2-6)$$

$$= 20 + 36 - 4 = 52.$$

So $x_1 = \frac{-16}{4} = -4$, $x_2 = \frac{52}{4} = 13$ and $2x_3 = 2 + x_1 = -2$

Hence $x_3 = -1$.