Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz I - A

Linear Algebra I

1. **[8 Marks]** Consider the linear system

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y - z &= 8 \\ x - y - z &= -8 \end{aligned}$$

(a) Write the linear system in its vector form,

(b) Solve the linear system by reducing its augmented matrix to reduced row-echelon form.

2. **[12 Marks]**

(a) Write the definition of *Echelon form of a matrix.*

(b) Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER" :

- (—) The only fundamental question about a linear system is the existence.

- (—) Elementary ro operations can be applied only to the augmented matrix of a linear system.

- (—) If matrices A and B are row equivalent, they have the same reduced echelon form

- (—) If u and v are nonzero vectors in $\mathbb{R}^3$, with v not a multiple of u, then $\text{Span}\{u, v\}$ is the line in $\mathbb{R}^3$ that contains u, v, and 0.

- (—) A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the echelon form of A.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz I - B

Linear Algebra I

1. [**8 Marks**] Consider the linear system

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 10 \\ 3x_1 - x_2 + 7x_3 + 4x_4 &= 1 \\ -5x_1 + 3x_2 - 15x_3 - 6x_4 &= 9 \end{aligned}$$

(a) Write the linear system in its vector form,

(b) Solve the linear system by reducing its augmented matrix to reduced row-echelon form.

2. **[12 Marks]**

(a) What are the *Elementary Roww Operations.*?

(b) Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER" :

- (—) Every elementary row operation is not reversible.

- (—) Two lines in the plane can only intersect in a single point.

- (—) If a system of linear equations has no free variables, then it has a unique solution.

- (—) Asking whether a vector b is in Span$\{v_1, \ldots, v_p\}$ amounts to asking whether the linear system with augmented matrix $[v_1\ v_2 \ldots v_p\ \ b]$ has a solution.

- (—) A pivot column in a matrix A is a a column of A that contains a zero every where.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz I - C

Linear Algebra I

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1. [**8 Marks**] Consider the linear system

$$\begin{aligned} 3x - y + 7z &= 0 \\ 4x - 2y + 8z &= 1 \\ x - y + z &= 1 \\ 6x - 4y + 10z &= 3 \end{aligned}$$

(a) Write the linear system in its vector form,

(b) Solve the linear system by reducing its augmented matrix to reduced row-echelon form.

2. **[12 Marks]**

(a) Write the definition of the *Span Set*.

(b) Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER" :

- (—) Two linear systems are equivalent if they have the same number of unknowns.

- (—) Leading entries are not always in the same positions in any echelon form ontained from a given matrix.

- (—) Span$\{v_1, \ldots, v_p\}$ contains every scalar multiple of v_1.

- (—) If the augmented matrix [A b] can be transformed by elementary row operations into reduced echelon form, then the equation $Ax = b$ is consistent.

- (—) Two linear systems are called equivalent if they have the same number of unkowns.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz I - D

Linear Algebra I

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1. [**8 Marks**] Consider the linear system

$$\begin{aligned} 2x_2 + 3x_3 - 4x_4 &= 1 \\ 2x_3 + 3x_4 &= 4 \\ 2x_1 + 2x_2 - 5x_3 + 2x_4 &= 4 \\ 2x_1 - 6x_3 + 9x_4 &= 7 \end{aligned}$$

(a) Write the linear system in its vector form,

(b) Solve the linear system by reducing its augmented matrix to reduced row-echelon form.

2. **[12 Marks]**

(a) Write the definition of a *Linear Combination*

(b) Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER" :

- (—) The largest possible number of pivots a 4×6 matrix can have are 6.
- (—) Whenever a system has free variables, the solution set contains a unique solution.
- (—) Two vectors are equal if they are of the same size.
- (—) If a system of linear equations has two different solutions, it must have infinitely many solutions.
- (—) If the augmented matrices of the two linear system are row equivalent, then the two systems are the same.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz II - A

Linear Algebra I

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1. **[7 Marks]** For which rational numbers λ does the homogeneous system

$$\begin{aligned} x + (\lambda - 3)y &= 0 \\ (\lambda - 3)x + y &= 0 \end{aligned}$$

has a non trivial solution?

2. **[13 Marks]**

(a) **[7 Marks]** For what value(s) of α will the vectors

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ \alpha \\ -9 \end{bmatrix}$$

be linearly dependent? **Justify your answer**.

(b) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A is an $m \times n$ matrix and the equation $Ax = b$ is consistent for every b in $\mathbb{R}^m$, then A has m pivot columns.

- (—) If an $m \times n$ matrix A has a pivot position in every row, then the equation Ax has a unique solution for each b in $\mathbb{R}^m$.

- (—) If A and B are row equivalent $m \times n$ matrices and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz II - B

Linear Algebra I

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1. **[7 Marks]** Solve the homogeneous system

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

2. **[13 Marks]**

(a) **[7 Marks]** For what value(s) of α will the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ \alpha \end{bmatrix}$$

be linearly dependent? **Justify your answer**.

(b) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A and B are row equivalent $m \times n$ matrices and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

- (—) If u and v are in $\mathbb{R}^m$, then $-u$ is in $Span\{u, v\}$.

- (—) If $\{u, v, w\}$ is linearly independent, then u, v, and w are not in $\mathbb{R}^2$.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz II - C

Linear Algebra I

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1. **[7 Marks]** Find the condition(s) on the coefficients a_{ij} such that the homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

has an infinite number of solutions.

2. **[13 Marks]**

(a) **[7 Marks]** For what value(s) of α will the vectors

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} \alpha \\ 1 \\ 2 \end{bmatrix}$$

be linearly dependent? **Justify your answer**.

(b) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If $\{u, v, w\}$ is linearly independent, then u, v, and w are not in $\mathbb{R}^2$.

- (—) In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

- (—) If u and v are in $\mathbb{R}^m$, then $-u$ is in $Span\{u, v\}$

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz II - D

Linear Algebra I

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1. **[7 Marks]** Solve the homogeneous system

$$\begin{aligned} x_1 - x_2 + 7x_3 - x_4 &= 0 \\ 2x_1 + 3x_2 - 8x_3 + x_4 &= 0 \end{aligned}$$

2. **[13 Marks]**

(a) **[7 Marks]** For what value(s) of α will the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ \alpha \\ 4 \end{bmatrix}$$

be linearly dependent? **Justify your answer**.

(b) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If u and v are in $\mathbb{R}^m$, then $-u$ is in $Span\{u, v\}$

- (—) If u, v and w are nonzero vectors in $\mathbb{R}^2$, then w is a linear combination of u and v.

- (—) If w is a linear combination of u and v in $\mathbb{R}^n$, then u is a linear combination of v and w.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz III - A

Linear Algebra I

1. [**7 Marks**] Find A^{-1}, if

$$A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

2. **[13 Marks]**

(a) **[2 Marks]** Write the definition of : Elementary matrix.

(b) **[5 Marks]** Can a square matrix with two identical columns be invertible? **Why or Why not**?

(c) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A and B are 3×3 and $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$, then $AB = [Ab_1 + Ab_2 + Ab_3]$.

- (—) If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

- (—) If the linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A has n pivot positions.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz III - B

Linear Algebra I

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1. [**7 Marks**] Find the third column of the of A^{-1}, if

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

2. **[13 Marks]**

(a) **[2 Marks]** Write the definition of : Elementary matrix.

(b) **[5 Marks]** Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? **Why or why not**?

(c) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A and B are 2×2 with columns a_1, a_2, and b_1, b_2, respectively, then $A = \begin{bmatrix} a_1 b_1 & a_2 b_2 \end{bmatrix}$.

- (—) If A is an invertible $n \times n$ matrix, then the equation $Ax = b$ is consistent for each b in $\mathbb{R}^n$.

- (—) If there is a b in $\mathbb{R}^n$ such that the equation $Ax = b$ is inconsistent, then the transformation $x \mapsto Ax$ is not one-to-one.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz III - C

Linear Algebra I

1. [**7 Marks**] Find the second column of the of A^{-1}, if

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

2. **[13 Marks]**

(a) [**2 Marks**] Write the definition of : Elementary matrix.

(b) [**5 Marks**] If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $Dx = b$? **Why**?

(c) [**6 Marks**] Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) The second row of AB is the second row of A multiplied on the right by B.
- (—) If A can be row reduced to the identity matrix, then A must be invertible.
- (—) If the equation $Ax = 0$ has a nontrivial solution, then A has fewer than n pivot positions.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz II - D

Linear Algebra I

1. [**7 Marks**] Find the first column of the of A^{-1}, if

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

2. **[13 Marks]**

(a) **[2 Marks]** Write the definition of : Elementary matrix.

(b) **[5 Marks]** If the equation $Gx = y$ has more than one solution for some y in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$? **Why or why not**?

(c) **[6 Marks]** Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

- (—) If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

- (—) If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 4 - A

Linear Algebra I

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1. **[10 Marks]** Use row operations to show that

$$\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix} = 0$$

2. **[10 Marks]**

Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

- (—) If two rows of a 3×3 matrix A are the same, then $\det A = 0$.

- (—) If B is produced by interchanging two rows of A, then $\det B = \det A$.

- (—) If A and B are two $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$.

- (—) If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.

Sultan Qaboos University – College of Science

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH2202, FALL SEMESTER 2007, QUIZ 4 - B

Linear Algebra I

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1. **[10 Marks]** Use row operations to show that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = 0$$

2. **[10 Marks]**

Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

- (—) If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.

- (—) If A is a 3×3 matrix, then $\det 5A = 5 \det A$.

- (—) If B is produced by multiplying row 3 of A by 5, then $\det B = 5. \det A$.

- (—) If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 4 - C

Linear Algebra I

1. **[10 Marks]** Use row operations to show that

$$\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = 0$$

2. **[10 Marks]**

Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If B is produced by interchanging two rows of A, then $\det B = \det A$.

- (—) If B is produced by multiplying row 3 of A by 5, then $\det B = 5. \det A$.

- (—) If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.

- (—) $\det A^T = -\det A$.

- (—) $\det(-A) = -\det A$.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 4 - D

Linear Algebra I

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1. **[10 Marks]** Use row operations to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix} = 0$$

2. **[10 Marks]**

Mark each of the following (T)rue or (F)alse - "JUSTIFY YOUR ANSWER:"

- (—) If A and B are two $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$.

- (—) $\det(-A) = -\det A$.

- (—) $\det A^T A \geq 0$.

- (—) If $\det A^3 = 0$, then A is not invertible matrix.

- (—) if A is invertible, then $(\det A)(\det A^{-1}) = 1$

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 5 - A

Linear Algebra I

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1. [**8 Marks**] Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $u = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is u in Nul A? Is u in col A? Justify your answer.

2. **[6 Marks]** Suppose an $n \times n$ matrix A is invertible. What can you say about Col A? About Nul A ?

3. **[6 Marks]** Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show that W is a subspace of $\mathbb{R}^3$.

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 5 - B

Linear Algebra I

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1. [**6 Marks**]

2. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$. Then every vector in H is a linear combination of v_1 and v_2 because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Is $\{v_1, v_2\}$ a basis for H ?

3. **[6 Marks]** Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show that W is a subspace of $\mathbb{R}^3$.

4. **[8 Marks]** Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{v_1, v_2\}$ is a basis for $\mathbb{R}^3$. Is $\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$?

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 5 - C

Linear Algebra I

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1. [**8 Marks**] Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $u = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is u in Nul A? Is u in col A? Justify your answer.

2. **[6 Marks]** Let $v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

3. **[6 Marks]** Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $w = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose that the equations $Ax = v$ and $Ax = w$ are both consistent. What can you say about the equation $Ax = v + w$?

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

MATH2202, Fall Semester 2007, Quiz 5 - D

Linear Algebra I

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1. [**6 Marks**] Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$. Then every vector in H is a linear combination of v_1 and v_2 because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Is $\{v_1, v_2\}$ a basis for H ?

2. **[8 Marks]** Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{v_1, v_2\}$ is a basis for $\mathbb{R}^3$. Is $\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$?

3. **[6 Marks]** Suppose an $n \times n$ matrix A is invertible. What can you say about Col A? About Nul A ?

Sultan Qaboos University – College of Science

DEPARTMENT OF MATHEMATICS AND STATISTICS

FALL SEMESTER 2007, 1^{st} MID-TERM EXAM

MATH2202 - Linear Algebra I

Date: 10-10-07 Total Marks = 40 Time allowed: 60 min.

Answer all questions and show the details of your work

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1. [**12 pts**] Consider the system

$$\begin{aligned} x + 2y - 3z &= 4, \\ 3x - y + 5z &= 2, \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$

For which rational numbers a does the system have :

(a) no solutions,

(b) exactly one solution, compute the solution for those cases.

(c) infinitely many solutions.?

2. [**4 + 4 pts**]

(a) Determine whether the vectors $v_1 = (1, 1, 2), v_2 = (1, 0, 1)$ and $v_3 = (2, 1, 3)$ span the vectors in $\mathbb{R}^3$. **Explain**.

(b) Suppose that $\{v_1, v_2, v_3\}$ is a linearly independent subset of R^m. **Show that** the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also linearly independent.

3. [**3 + 7 pts**]

(a) Prove that, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

(b) Given the linear transformation T, defined by

$$T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$$

(1) Find the standard matrix of the transformation,

(2) What is the domain and the codomain of the transformation,

(3) Find x such that $T(x) = (1, 2, 3)$,

(4) Is T a one-to-one or onto.? **Why**?

4. [**4 + 6 pts**]

(a) Write the definitions of: a linear combination - a linearly independent set.

(b) Mark each of the following **T**rue or **F**alse. *Justify your Answer.*

(1) If $\{u, v, w\}$ is linearly independent, then u, v, and w are in $\mathbb{R}^2$.

(2) A transformation T is linear if and only if $T(cv) = cT(v)$, for any vector v and any scalar c.

(3) The columns of the standard matrix for a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ are the images of the columns of the $n \times n$ identity matrix.

GOOD LUCK

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

Fall Semester 2007, 1^{st} Mid-Term Exam

MATH2202 - Linear Algebra I

Date: 10-10-07 Total Marks = 40 Time allowed: 60 min.

Answer all questions and show the details of your work

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1. [**12 pts**] Consider the system

$$\begin{aligned} x + 2y - 3z &= 4, \\ 3x - y + 5z &= 2, \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$

For which rational numbers a does the system have

(a) no solutions,

(b) exactly one solution, compute the solution for those cases.

(c) infinitely many solutions.?

Solution: Simplify the augmented matrix of the given linear system, using row operations

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$$\xrightarrow[R_2 \to -\frac{1}{7}R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

From which we will have

Case I: $a^2 - 16 \neq 0$. i.e. $a \neq \pm 4$. Then

$$\xrightarrow[\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 2R_3}]{R_3 \to \frac{1}{a^2-16}R_2} \begin{bmatrix} 1 & 0 & 0 & \frac{8a+25}{7(a+4)} \\ 0 & 1 & 0 & \frac{10a+54}{7(a+4)} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{bmatrix}$$

and we get the unique solution

$$x = \frac{8a+25}{7(a+4)}, y = \frac{10a+54}{7(a+4)}, z = \frac{1}{a+4}.$$

Case II: If $a = -4$. Then we will have

$$\begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & -8 \end{bmatrix},$$

so our system is inconsistent.

Case III: If $a = 4$. Then we will have

$$\begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we find that the system is consistent, with complete solution

$$x = \frac{8}{7} - z, y = \frac{10}{7} + 2z$$

where z is a free variable.

2. [**4 + 4 pts**]

(a) Determine whether the vectors $v_1 = (1, 1, 2), v_2 = (1, 0, 1)$ and $v_3 = (2, 1, 3)$ span the vectors in $\mathbb{R}^3$. **Explain**?

Solution: We must determine whether an arbitrary vector $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ can be expressed as a linear combination

$$b = k_1 v_1 + k_2 v_2 + k_3 v_3$$

of the vectors v_1, v_2, and v_3. Expressing this vector equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

or

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

The problem thus reduces to determining whether this linear system is consistent for all values of b_1, b_2 and b_3. This system is consistent for all b_1, b_2 and b_3 if and only if the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

has a pivot in every row, Theorem 4, section 1.4. Reduceing the cofficient matrix to its REF, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

that means, their is no a pivot in every row. Hence the corresponding system is inconsistent, and then the vectors v_1, v_2, and v_3 do not span $\mathbb{R}^3$.

(b) Suppose that $\{v_1, v_2, v_3\}$ is a linearly independent subset of R^m. **Show that** the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also linearly independent.

Solution: In order the vectors of $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ be linearly independent, then

$$a_1 v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) = 0$$

that is

$$(a_1 + a_2 + a_3)v_1 + (a_2 + a_3)v_2 + a_3 v_3 = 0$$

and since $\{v_1, v_2, v_3\}$ is linearly independent, then

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_2 + a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

that is $a_1 = a_2 = a_3 = 0$, and hence the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also linearly independent.

3. [**3 + 7 pts**]

(a) **Prove that**, if $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then T is one to one if and only if the equation $T(x) = 0$ has only the trivial solution.

Solution: Since T is linear, then $T(0) = 0$.

- If T is one-to-one, then the equation $T(x) = 0$ has at most one solution and hence only trivial solution.

- If T is not one-to-one, then there is b that is the image of at least two different vectors in $\mathbb{R}^n$ - say u and v, such that $T(u) = b$ and $T(v) = b$. But since T is linear,

$$T(u - v) = T(u) - T(v) = b - b = 0$$

The vector $u - v$ is not a zero vector, since $u \neq v$. Hence the equation $T(x) = 0$ has more than one solution. So, either the two conditions are both true or thery are both false.

(b) Given the linear transformation T, defined by

$$T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$$

(1) **Find** the standard matrix of the transformation,

(2) **What** is the domain and the codomain of the transformation,

(3) **Find** x such that $T(x) = (1, 2, 3)$,

(4) **Is** T a one to one or onto.? **Why**?

Solution: **(1)** To find the standard matrix, rewrite the linear transformation in the form

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then the standard matrix is

$$A = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

(2) The transformation T will have the form, $T : \mathbb{R}^4 \to \mathbb{R}^3$, then the domain of T is $\mathbb{R}^4$ and the codomain $\mathbb{R}^3$.

(3) To find x such that $T(x) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, solve the linear system corresponding to the augmented matrix

$$\begin{bmatrix} 7 & 2 & -1 & 1 & \mathbf{1} \\ 0 & 1 & 1 & 0 & \mathbf{2} \\ -1 & 0 & 0 & 0 & \mathbf{3} \end{bmatrix}.$$

Reducing the augmented matrix to REF, we will have

$$\begin{bmatrix} 7 & 2 & -1 & 1 & \mathbf{1} \\ 0 & 1 & 1 & 0 & \mathbf{2} \\ -1 & 0 & 0 & 0 & \mathbf{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{1}{3} & 8 \\ 0 & 0 & 1 & -\frac{1}{3} & -6 \end{bmatrix}$$

from which we will have

$$x_1 = -3, x_2 = 8 - \frac{1}{3}x_4, x_3 = -6 + \frac{1}{3}x_4 \text{ with } x_4 \text{ free.}$$

(4) Since the vector $(1, 2, 3)$ in $\mathbb{R}^3$ represent an image for more than one vectors in $\mathbb{R}^4$ under T, then the give linear transformation is ont $\mathbb{R}^3$.

4. [**4 + 6 pts**]

(a) Write the definitions of: a linear combination - a linearly independent set.

Solution:

- Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ and given any scalars $c_1, c_2, \ldots, c_p$, the vector y defined by

$$y = c_1v_1, c_2v_2, \ldots, c_pv_p$$

is called a **linear combination** of $v_1, v_2, \ldots, v_p$ with weights $c_1, c_2, \ldots, c_p$.

- An indexed set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

(b) Mark each of the following **T**rue or **F**alse. *Justify your Answer.*

(1) If $\{u, v, w\}$ is linearly independent, then u, v, and w are in $\mathbb{R}^2$.

(FALSE) Any set of three vectors in $\mathbb{R}^2$ would have to be linearly dependent, by Theorem 8 in section 1.7.

(2) A transformation T is linear if and only if $T(cv) = cT(v)$, for any vector v and any scalar c.

(FALSE) A transformation T is linear if and only if $T(cv + du) = cT(v) + dT(u)$, for any vectors v, u and any scalars c, d.

(3) The columns of the standard matrix for a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ are the images of the columns of the $n \times n$ identity matrix.

(FALSE) The columns of the standard matrix for a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ are the images of the columns of the $m \times m$ identity matrix.

GOOD LUCK

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

Fall Semester 2007, 2^{nd} Mid-Term Exam

MATH2202 - Linear Algebra I

Date: 14-11-07 | Total Marks = 40 | Time allowed: 60 min.

Answer all questions and show the details of your work

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1. [**4 + 4 pts**]

   (a)- Let A, B and C be three $n \times n$ invertible matrices. **Simplify**
   $$A^T B\,(CB)^{-1}\left(A^{-1}C^T\right)^T.$$

   (b)- **Find** the matrix X, if
   $$\left(X^T - 3\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

2. [**3 + 2 + 5 pts**] Let
   $$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

   (a)- Use **I**nvertible **M**atrix **T**heorem to **decide** whether A is invertible or not.

   (b)- **Is** b spanned by the columns of A? Justify your Answer.

   (c)- Use Cramer's rule to **solve** the system $Ax = b$.

3. [**4 + 6 pts**]

   (a) **Prove that**, if $v_1, v_2, \ldots, v_s$ are in a vector space V, then $Span\{v_1, v_2, \ldots, v_s\}$ is a subspace of V.

   (b) Use determinants to **find** the value(s) of α for which the matrix
   $$\begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$
   has no inverse.

4. [**4 + 8 pts**]

   (a) **Write** the definition of a Vector Space. **What** are the conditions for a subset S of a vector space V to be a subspace of V.

   (b) **Mark** each of the following **T**rue or **F**alse. *Justify your Answer.*

   (1) $V = \{(x, y) : y \leq 0; x, y \in \mathbb{R}\}$ with the usual addition and scalar multiplication of vectors is a vector space.

   (2) $\mathbb{R}^2$ with addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and the ordinary scalar multiplication is a vector space.

   (3) If $V = \mathbb{R}^2$, then $H = \left\{(x, y) : x^2 + y^2 \leq 1\right\}$ is a subspace of V.

   (4) If $V = \mathbb{M}_{22}$, then $H = \left\{A \in \mathbb{M}_{22} : A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\right\}$ is a subspace of V.

GOOD LUCK

Sultan Qaboos University – College of Science

DEPARTMENT OF MATHEMATICS AND STATISTICS

FALL SEMESTER 2007, 2^{nd} MID-TERM EXAM

MATH2202 - Linear Algebra I

Date: 14-11-07 Total Marks = 40 Time allowed: 60 min.

Answer all questions and show the details of your work

---

1. [**4 + 4 pts**]

(a)- Let A, B and C be three $n \times n$ invertible matrices. **Simplify**

$$A^T B\,(CB)^{-1}\left(A^{-1}C^T\right)^T.$$

Solution:

$$\begin{aligned} A^T B\,(CB)^{-1}\left(A^{-1}C^T\right)^T &= A^T B\left(B^{-1}C^{-1}\right)\left(\left(C^T\right)^T\left(A^{-1}\right)^T\right) \\ &= A^T\left(BB^{-1}\right)\left(C^{-1}C\right)\left(A^T\right)^{-1} \\ &= A^T\left(A^T\right) = I_n. \end{aligned}$$

The second equality is a consequence of the fact that matrix multiplication is associative, the fact that $\left(C^T\right)^T = C$ and $\left(A^{-1}\right)^T = \left(A^T\right)$.

(b)- Find the matrix X, if

$$\left(X^T - 3\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Solution: If X satisfies the given equation, then taking the inverse of both sides, we will have

$$\begin{aligned} X^T - 3\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}. \end{aligned}$$

Hence

$$X^T = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

transposing both sides, we obtain

$$X = \left(\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 9 & 5 \\ -2 & 6 \end{bmatrix}^T = \begin{bmatrix} 9 & -2 \\ 5 & 6 \end{bmatrix}.$$

2. [**3 + 2 + 5 pts**] Let

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

(a)- Use **I**nvertible **M**atrix **T**heorem to **decide** whether A is invertible or not.

Solution: By the IMT, the matrix A is invertible if it is equivalent to I_3 and hence has three pivots.

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix A has three pivot positions and hence is invertible.

(b)- **Is** b spanned by the columns of A? Justify your Answer.

Solution: Since the matrix A is invertible, then by the IMT, the equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^3$, and then columns of A span $\mathbb{R}^3$. Hence b spanned by the columns of A.

(c)- Use Cramer's rule to **solve** the system $Ax = b$.

Solution: We will have

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 5 & -1 \\ 3 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -1 & -1 \\ 4 & 3 & 3 \\ -2 & 0 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix}$$

for which we will have

$$\det(A) = -52, \det(A_1) = 0, \det(A_2) = 0 \text{ and } \det(A_3) = -52.$$

Applying Cramer's rule, we will have

$$x_1 = \frac{0}{-52} = 0; x_2 = \frac{0}{-52} = 0; x_3 = \frac{-52}{-52} = 1$$

3. [**4 + 6 pts**]

(a) Prove that, if $v_1, v_2, \ldots, v_s$ are in a vector space V, then $Span\{v_1, v_2, \ldots, v_s\}$ is a subspace of V.

Solution: In order to prove this, we need to check if the zero of V is in $Span\{v_1, v_2, \ldots, v_s\}$, and $Span\{v_1, v_2, \ldots, v_s\}$ is closed under addition and scalar multiplication of V.

1- 0 is in $Span\{v_1, v_2, \ldots, v_s\}$, since

$$0 = 0v_1 + 0v_2 + \cdots + 0v_p.$$

2- To show that $Span\{v_1, v_2, \ldots, v_s\}$ closed under vector addition, we need to choose two arbitrary vectors in $Span\{v_1, v_2, \ldots, v_s\}$

$$u = a_1v_1 + a_2v_2 + \cdots + a_pv_p$$

and

$$v = b_1v_1 + b_2v_2 + \cdots + b_pv_p$$

then

$$u + v = (a_1v_1 + a_2v_2 + \cdots + a_pv_p) + (v_1v_1 + b_2v_2 + \cdots + b_pv_p)$$

that is

$$u + v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_p + b_p)v_p$$

that means $u + v$ is in $Span\{v_1, v_2, \ldots, v_s\}$.

3- To show that $Span\{v_1, v_2, \ldots, v_s\}$ closed under scalar multiplication, we need to choose an arbitrary number c and an arbitrary vector in $Span\{v_1, v_2, \ldots, v_s\}$

$$v = b_1v_1 + b_2v_2 + \cdots + b_pv_p$$

then

$$cv = c(b_1v_1 + b_2v_2 + \cdots + b_pv_p)$$

that is

$$cv = (cb_1)v_1 + (cb_2)v_2 + \cdots + (cb_p)v_p)$$

so cv is in $Span\{v_1, v_2, \ldots, v_s\}$. Hence $Span\{v_1, v_2, \ldots, v_s\}$ is a subspace of V.

(b) Use determinants to **find** the value(s) of α for which the matrix

$$\begin{bmatrix} -\alpha & \alpha - 1 & \alpha + 1 \\ 1 & 2 & 3 \\ 2 - \alpha & \alpha + 3 & \alpha + 7 \end{bmatrix}$$

has no inverse.

Solution:

$$\begin{vmatrix} -\alpha & \alpha - 1 & \alpha + 1 \\ 1 & 2 & 3 \\ 2 - \alpha & \alpha + 3 & \alpha + 7 \end{vmatrix} \xrightarrow[R_3 = R_3 - (2-\alpha)R_2]{R_1 = R_1 + \alpha R_2} \begin{vmatrix} 0 & 3\alpha - 1 & 4\alpha + 1 \\ 1 & 2 & 3 \\ 0 & 3\alpha - 1 & 4\alpha + 1 \end{vmatrix} = 0$$

Hence, for all values of α, the matrix is not invertible.

4. [**4 + 8 pts**]

(a) Write the definition of a Vector Space. **What** are the conditions for a subset S of a vector space V to be a subspace of V.

Solution: - A non empty set V of objects, called vectors, on which are defined two binary operations, addition and scalar multiplication, is said to be a vector space if the following axioms are satisfied :

1. Closed : $\forall\ u, v \in V,\ u + v \in V$.
2. Commutative : $\forall\ u, v \in V,\ u + v = v + u$.
3. Associative : $\forall\ u, v, w \in V,\ u + (v + w) = (u + v) + w$.
4. There is a zero vector $0 \in V$ such that $\forall\ u \in V, u + 0 = u$.
5. $\forall\ u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$.
6. $\forall\ u \in V$, and r is any scalar, $ru \in V$.
7. $\forall\ u\ \&\ v \in V$, and r is any scalar, $r(u + v) = ru + rv$.
8. $\forall\ u \in V$, and r, s are any scalars, $r(su) = (rs)u$.
9. $\forall\ u \in V$, and r, s are any scalars, $(r + s)u = ru + su$.
10. $\forall\ u \in V,\ 1u = u$.

- A subset S of a vector space V is a subspace of V, if

1. The zero of V is in H,
2. H is closed under addition,
3. H is closed under scalar multiplication.

(b) **Mark** each of the following **T**rue or **F**alse. *Justify your Answer.*

(1) $V = \{(x, y) : y \le 0; x, y \in \mathbb{R}\}$ with the usual addition and scalar multiplication of vectors is a vector space.

FALSE No Additive Inverse exist, since if $(x, y) \in V, y < 0$ then $(-x, -y) \notin V$ since $-y > 0$. Also, it is not closed under scalar multiplication, since if $\alpha < 0$ and $y < 0$, then $\alpha y > 0$.

(2) $\mathbb{R}^2$ with addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and the ordinary scalar multiplication is a vector space.

FALSE This is not a vector space, since under scalar multiplication, for any scalar r and $\forall\ u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{R}^2, r(u + v) \neq ru + rv$ for example, if $r \neq 1$, then

$$r(u+v) = r((x_1, y_1)+(x_2, y_2)) = (rx_1+rx_2+r, ry_1+ry_2+r) \neq ru+rv = (rx_1+rx_2+1, ry_1+ry_2+1)$$

(3) If $V = \mathbb{R}^2$, then $H = \{(x, y) : x^2 + y^2 \le 1\}$ is a subspace of V.

FALSE This is not a subspace, since $(1, 0) \in H$, but $2(1, 0) = (2, 0) \notin H$.

(4) If $V = \mathbb{M}_{22}$, then $H = \left\{ A \in \mathbb{M}_{22} : A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right\}$ is a subspace of V.

True This is a subspace of M_{22}, since

1- The zero of M_{22}, is in H, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2- For any $u = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix}$, $u + v = \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{bmatrix} \in H$.

3- For any $u = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$ and a scalar r, $ru = \begin{bmatrix} 0 & ra_1 \\ rb_1 & 0 \end{bmatrix} \in H$.

GOOD LUCK

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

Fall Semester 2007, Final Exam

MATH2202 - LINEAR ALGEBRA I

Date: 29-12-07 Time allowed: $2\frac{1}{2}$ Hrs.

ANSWER ALL QUESTIONS AND SHOW THE DETAILS OF YOUR WORK

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1. [**10 pts**] **Complete** the following *definitions* :

   (a) A set $\{v_1, \ldots, v_p\}$ in a vector space V is **linearly dependent** if ... ...

   (b) Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{b_1, \ldots, b_p\}$ is a **basis** of H if ... ...

   (c) A scalar λ is an **eigenvalue** of a matrix A if ... ...

   (d) The **null space** of an $m \times n$ matrix A, is ... ...

   (e) Suppose $\beta = \{b_1, \ldots, b_n\}$ is a basis for a vector space V and x is in V. The β**-coordinates of** x are ... ...

---

2. [**15 pts**]

   (a) [**5 pts**] **List** five statements that are each equivalent to the statement that an $n \times n$ matrix A is invertible. The following concepts *should be included*, one in each statment: **(i)** the trivial solution, **(ii)** the column space, **(iii)** basis, **(iv)** rank, and **(v)** eigenvalue.

   (b) [**7 pts**] **Find** the general solution of the following system of equations. **Write** your answer in parametic vector form.

$$\begin{aligned} x_1 - 5x_2 - 9x_3 + 8x_4 &= -7 \\ x_2 + 3x_3 - 4x_4 &= 2 \\ 2x_2 + 6x_3 - 8x_4 &= 4 \end{aligned}$$

   (c) [**3 pts**] **Construct** a 3×3 matrix A, with nonzero entries, and a vector b in $\mathbb{R}^3$ such that b is *not* in the set spanned by the columns of A. **Explain Your Answer**.

---

3. [**10 pts**]

   (a) [**3 pts**] **Show** that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

   (b) [**7 pts**] Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

      (i) **Show** that $A^3 = 0$.

      (ii) Use matrix algebra to **compute** the product $(I - A)(I + A + A^2)$.

---

4. [**15 pts**]

   (a) [**7 pts**] Let $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$.

      (i) Find a matrix B that is row equivalent to A.

      (ii) Find bases for Nul A and Col A.

   (b) [**3 pts**] **Explain** why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible.

   (c) [**5 pts**] Let $\beta = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ be a basis for $\mathbb{P}_2$. **Find** the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to β.

PLEASE TURN OVER

5. [**10 pts**]

(a) [**3 pts**] **Show that** if AB is invertible, then B is also invertible.

(b) [**4 pts**] **Use the properties of the determinant** to **show that**

$$\det\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (z-x)(z-y)(y-x)$$

(c) [**3 pts**] **Prove that** if a vector space V has a basis $\beta = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

---

6. [**10 pts**]

(a) [**7 pts**] Let

$$H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

(i) Explain why H is a subspace of $\mathbb{R}^4$,

(ii) Find a basis for H,

(iii) State the dimension of H.

(b) [**3 pts**] If a 3×8 matrix A has rank 3, **find** dim Nul A, dim Row A, and rank A^T.

---

7. [**6 pts**]

(a) [**3 pts**] Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

(b) [**3 pts**] **Prove** that if $v_1, \ldots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is lineary independent.

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8. [**9 pts**] Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigenvectors of the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

(i) **Compute** the eigenvalues of A.

(ii) **Find** matrices P and D that diagonalize the matrix A,and write the equation that relates A to P and D.

(iii) **Verify** your answer.

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9. [**15 pts**] Mark each of the following **True** or **False**. **Justify Your Answer**.

(a) If A is an $m \times n$ matrix and the equation $Ax = b$ is consistent for some b, then the columns of A span $\mathbb{R}^m$.

(b) Suppose that v_1, v_2, and v_3 are in $\mathbb{R}^5$, v_2 is not a multiple of v_1, and v_3 is not a linear combination of v_1 and v_2. Then $\{v_1, v_2, v_3\}$ is linearly independent.

(c) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

(d) If A and B are $m \times n$, then both AB^T and A^TB are defined.

(e) If $AB = I$, then A is invertible.

(f) If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.

(g) If B is produced by interchanging two rows of A, then $detB = detA$.

(h) If $\dim V = p$, then there exists a spanning set of $p+1$ vectors in V.

(i) A row replacement operation on a matrix A does not change the eigenvalues of A.

(j) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

GOOD LUCK

Sultan Qaboos University – College of Science

Department of Mathematics and Statistics

Fall Semester 2007, Final Exam

MATH2202 - Linear Algebra I

Date: 29-12-07 Time allowed: $2\frac{1}{2}$ Hrs.

ANSWER ALL QUESTIONS AND SHOW THE DETAILS OF YOUR WORK

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1. [**10 pts**] **Complete** the following *definitions* :

(a) A set $\{v_1, \ldots, v_p\}$ in a vector space V is **linearly dependent** if ... ...

(Solution:) A set $\{v_1, \ldots, v_p\}$ in a vector space V is **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1v_1 + c_2v_2 + \ \ldots \ + c_pv_p = 0.$$

(b) Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{b_1, \ldots, b_p\}$ is a **basis** of H if ... ...

(Solution:) Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{b_1, \ldots, b_p\}$ is a **basis** of H if

(i) β is a linearly independent set, and

(ii) the subspace spanned by β coincides with H; that is

$$H = Span\{b_1, \ldots, b_p\}.$$

(c) A scalar λ is an **eigenvalue** of a matrix A if ... ...

(Solution:) A scalar λ is an **eigenvalue** of a matrix A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an *eigenvector corresponding to* λ.

(d) The **null space** of an $m \times n$ matrix A, is ... ...

(Solution:) The **null space** of an $m \times n$ matrix A, written Null A, is the set of all solutions to the homogeneous equation $Ax = 0$. In set notation,

$$NulA = \{x : \ x \in \mathbb{R}^n \text{ and } Ax = 0\}.$$

(e) Suppose $\beta = \{b_1, \ldots, b_n\}$ is a basis for a vector space V and x is in V. The β**-coordinates of** x are ... ...

(Solution:) Suppose $\beta = \{b_1, \ldots, b_p\}$ is a basis for a vector space V and x is in V. The β**-coordinates of** x are the weights $c_1, \ldots, c_n$ such that

$$x = c_1b_1 + \ \ldots \ + c_nb_n.$$

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2. [**15 pts**]

(a) [**5 pts**] **List** five statements that are each equivalent to the statement that an $n \times n$ matrix A is invertible. The following concepts *should be included*, one in each statment: **(i)** the trivial solution, **(ii)** the column space, **(iii)** basis, **(iv)** rank, and **(v)** eigenvalue.

(Solution:) Let A be a square $n \times n$ matrix. The the following statments are equivalent.

(i) The equation $Ax = 0$ has only the trivial solution.

(ii) Col A $= \mathbb{R}^n$.

(iii) The columns of A form a basis for $\mathbb{R}^n$.

(iv) Rank of A is equal to n.

(v) The number 0 is not an eigenvalue of A.

(b) [**7 pts**] **Find** the general solution of the following system of equations. **Write** your answer in parametic vector form.

$$\begin{aligned} x_1 - 5x_2 - 9x_3 + 8x_4 &= -7 \\ x_2 + 3x_3 - 4x_4 &= 2 \\ 2x_2 + 6x_3 - 8x_4 &= 4 \end{aligned}$$

(Solution:) The augmented matrix of the given system is

$$\begin{bmatrix} 1 & -5 & -9 & 8 & -7 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 2 & 6 & -8 & 4 \end{bmatrix}$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 6 & -12 & 3 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we get the equivalent linear system

$$\begin{aligned} x_1 + 6x_3 - 12x_4 &= 3 \\ x_2 + 3x_3 - 4x_4 &= 2 \\ 0 &= 0. \end{aligned}$$

Solving for basic variables, we will have

$$\begin{aligned} x_1 &= 3 - 6x_3 + 12x_4 \\ x_2 &= 2 - 3x_3 + 4x_4 \end{aligned}$$

with both $x_3 \& x_4$ free, that is

$$x = \begin{bmatrix} 3 - 6x_3 + 12x_4 \\ 2 - 3x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 12 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

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(c) [**3 pts**] **Construct** a 3×3 matrix A, with nonzero entries, and a vector b in $\mathbb{R}^3$ such that b is *not* in the set spanned by the columns of A. **Explain Your Answer**.

(Solution:) The solution here is to construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

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3. [**10 pts**]

(a) [**3 pts**] **Show** that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

(Solution:) One possibility is to show that T does not map the zero vector into the zero vector, something that every linear transformation does do. We can easily check that

$$T(0,0) = (0,4,0) \neq (0,0,0).$$

(b) [**7 pts**] Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(i) **Show** that $A^3 = 0$.

(Solution:)

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$A^3 = AA^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(ii) Use matrix algebra to **compute** the product $(I - A)(I + A + A^2)$.

(Solution:)

$$\begin{aligned} (I - A)(I + A + A^2) &= I + A + A^2 - A(I + A + A^2) \\ &= I + A + A^2 - A - A^2 - A^3 \\ &= I - A^3 \end{aligned}$$

Since $A^3 = 0$, $(I - A)(I + A + A^2) = I - A^3 = I$.

4. [**15 pts**]

(a) [**7 pts**] Let $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$.

(i) Find a matrix B that is row equivalent to A.

(Solution:) Apply elementary row operations on the given matrix, we can get its equivalent form $B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(ii) Find bases for Nul A and Col A.

(Solution:) Since the matrix B in part (i) is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.$$

To find a basis for Nul A, we find the general solution of $Ax = 0$ in terms of the free variables:

$$x_1 = -6x_3 - 5x_4, x_2 = (-\frac{5}{2})x_4, \text{with} x_3 \text{and} x_4 \text{free.}$$

So

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) [**3 pts**] **Explain** why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible.

(Solution:) Suppose that A is invertible matrix. Then by Theorem 5 of section 2.2 "If A is an invertible $n \times n$ matrix, then for each $b \in \mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$" the equation $Ax = b$ has a solution, (in fact, a unique solution), for each b. By Theorem 4 of section 1.4, "Let A be an $n \times n$. If for each $b \in \mathbb{R}^n$, the equation $Ax = b$ has a solution, then each $b \in \mathbb{R}^n$ is a linear coimbination of the columns of A, and then columns of A span $\mathbb{R}^n$ ", the columns of A span $\mathbb{R}^n$

(c) [**5 pts**] Let $\beta = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ be a basis for $\mathbb{P}_2$. **Find** the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to β.

(Solution:) We must find c_1, c_2, and c_3 such that

$$c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) = p(t) = 1 + 4t + 7t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{array}{cccl} c_1+ & & c_3 & = 1 \\ & c_2+ & 2c_3 & = 4 \\ c_1+ & c_2+ & c_3 & = 7. \end{array}$$

We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{so} [p]_\beta = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

5. [**10 pts**]

(a) [**3 pts**] **Show that** if AB is invertible, then B is also invertible.

(Solution:) Let W be the inverse of AB. Then $WAB = I$ and $(WA)B = I$. By the invertible matrix theorem, "If an $n \times n$ matrix A is invertible, then there is an $n \times n$ matrix C, such that $CA = I$" ,the matrix B is invertible.

(b) [**4 pts**] **Use the properties of the determinant** to **show that**

$$\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (z-x)(z-y)(y-x)$$

(Solution:)

$$\begin{aligned} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} &= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix} \\ &= (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} \\ &= (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix} \\ &= (y-x)(z-x)(z-y). \end{aligned}$$

(c) [**3 pts**] **Prove that** if a vector space V has a basis $\beta = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

(Solution:) Let $\{u_1, \ldots, u_p\}$ be a set in V with more than n vectors. The coordinate vectors $[u_1]_\beta, \ldots, [u_p]_\beta$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector. So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[u_1]_\beta + \ldots + c_p[u_p]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n.$$

Since the coordinate mapping is a linear transformation,

$$[c_1u_1 + \ldots + c_pu_p]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The zero vector on the right contains the n weights needed to build the vector $c_1u_1 + \ldots + c_pu_p$ from the basis vectors in β. That is $c_1u_1 + \ldots + c_pu_p = 0.b_1 + \ldots + 0.b_n = 0$. SInce the

6. [**10 pts**]

(a) [**7 pts**] Let

$$H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

(i) Explain why H is a subspace of $\mathbb{R}^4$,

(Solution:) H is a subspace of $\mathbb{R}^4$, since $H = Span\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \text{and} v_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}.$$

(ii) Find a basis for H,

(Solution:) Since $v_3 = -(1/3)v_1$, $\{v_1, v_2, v_3\}$ is linearly dependent. By the Spanning Set Theorem, v_3 may be removed from the set with no change in the span set, so $H = Span\{v_1, v_2\}$. Since v_1 and v_2 are not multiple of each other, $\{v_1, v_2\}$ is linearly independent and is thus a basis for H.

(iii) State the dimension of H.

(Solution:) The dimension of H is 2.

(b) [**3 pts**] If a 3×8 matrix A has rank 3, **find** dim Nul A, dim Row A, and rank A^T.

(Solution:) By the Rank Theorem, $dimNulA = 8 - rankA = 8 - 3 = 5$. Since $dimRowA = rankA, dimRowA = 3$. Since $rankA^T = dimColA^T = dimRowA, rankA^T = 3$.

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7. [**6 pts**]

(a) [**3 pts**] Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

(Solution:) Suppose that A^2 is the zero matrix. If $Ax = \lambda x$ for some $x \neq 0$, then $A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$. Since x is nonzero, λ must be zero. thus each eigenvalue of A is zero.

(b) [**3 pts**] **Prove** that if $v_1, \ldots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is lineary independent.

(Solution:) Suppose that $\{v_1, \ldots, v_r\}$ is linearly dependent. Since v_1 is nonzero, then one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index that v_{p+1} is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1 v_1 + \ldots + c_p v_p = v_{p+1} \tag{1}$$

Multiplying both sides of (1) by A and using the fact that $Av_k = \lambda_k v_k$ for each k, we obtain

$$c_1 A v_1 + \ldots + c_p A v_p = A v_{p+1}$$

that is

$$c_1 \lambda_1 v_1 + \ldots + c_p \lambda_p v_p = \lambda_{p+1} v_{p+1}. \tag{2}$$

Multiplying both sides of (1) by λ_{p+1} and subtracting the result from (2), we have

$$c_1(\lambda_1 - \lambda_{p+1}) v_1 + \ldots + c_p(\lambda_p - \lambda_{p+1}) v_p = 0. \tag{3}$$

Since $\{v_1, \ldots, v_p\}$ is linearly independent, the weights in (3) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \ldots, p$. But then (1) says that $v_{p+1} = 0$, which is impossible. Hence $\{v_1, \ldots, v_r\}$ cannot be linearly dependent and therefore must be linearly independent.

8. [**9 pts**] Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigenvectors of the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

(a) Compute the eigenvalues of A. To find the corresponding eignvalues, compute $Av_i, i = 1, 2, 3$

$$\begin{aligned}
Av_1 &= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\
Av_2 &= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \\
Av_3 &= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Then the corresponding eigenvalues are $\lambda_1 = 3, \lambda_2 = 2$ and $\lambda_3 = 1$.

(b) Find matrices P and D that diagonalize the matrix A,and write the equation that relates A to P and D.

(Solution:)

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$A = PDP^{-1}$$

(c) Verify your answer.

(Solution:)

$$\begin{aligned}
PDP^{-1} &= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 3 & -2 \\ 3 & -5 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} = A
\end{aligned}$$

Or simply one can check if $AP = PD$ or not, that is

$$\begin{aligned}
AP &= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 9 & 6 & 1 \\ 12 & 6 & 1 \end{bmatrix} \\
PD &= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 9 & 6 & 1 \\ 12 & 6 & 1 \end{bmatrix}.
\end{aligned}$$

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9. [**15 pts**] Mark each of the following **True** or **False**. **Justify Your Answer**.

(a) If A is an $m \times n$ matrix and the equation $Ax = b$ is consistent for some b, then the columns of A span $\mathbb{R}^m$.
FALSE. For the columns of A to span $\mathbb{R}^m$, the equation $Ax = b$ must be consistent for all $b \in \mathbb{R}^m$, not for just one vector $b \in \mathbb{R}^m$

(b) Suppose that v_1, v_2, and v_3 are in $\mathbb{R}^5$, v_2 is not a multiple of v_1, and v_3 is not a linear combination of v_1 and v_2. Then $\{v_1, v_2, v_3\}$ is linearly independent.
FALSE. The statment would be true if the condition v_1 is not zero were present. However, if $v_1 = 0$, then $\{v_1, v_2, v_3\}$ is linearly dependent, no matter what else might be true about v_2 and v_3.

(c) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.
FALSE. The plane must pass through the origin to be a subspace.

(d) If A and B are $m \times n$, then both AB^T and $A^T B$ are defined.
TRUE. If A and B are $m \times n$, then B^T has many rows as A has columns, so AB^T is defined. Also, $A^T B$ is defined because A^T has m columns and B has m rows.

(e) If $AB = I$, then A is invertible.
FALSE. A must be square in order to conclude from the equation $AB = I$ that A is invertible.

(f) If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.
TRUE. Given $AB = BA$, left-multiply by A^{-1} to get $B = A^{-1}BA$, and then right-multiply by A^{-1} to obtain $BA^{-1} = A^{-1}B$.

(g) If B is produced by interchanging two rows of A, then $detB = detA$.
FALSE. If two rows of A are interchanged to produce B, then $detB = -detA$.

(h) If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.
TRUE. Take any basis (which will contain p vectors) for V and adjoin the the zero vector to it.

(i) A row replacement operation on a matrix A does not change the eigenvalues of A.
FALSE. Row operations on a matrix usually change its eigenvalues.

(j) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.
FALSE. An $n \times n$ matrix A may be invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

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