SULTAN QABOOS UNIVERSITY - COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 2202 - Linear Algebra I

Date: 12.1.2010 | **FINAL EXAM - FALL 2009** | Time: 150 minutes

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Answer the following questions (show all work).

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1. [15 marks] Let $A = \begin{bmatrix} 1 & 1 & -2 & 3 & -1 \\ -4 & -4 & 2 & 0 & -2 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Show that A is row equivalent to B.
   (b) Do the columns of A span $\mathbb{R}^4$? Justify your answer.
   (c) Find bases for $Col\ A$ and $Nul\ A$.
   (d) Find $\dim(Nul\ A)$ and $Rank\ A$.

2. [9 marks] Let $A,\ B,\ X$ be three $n \times n$ matrices such that B is invertible.

   (a) Solve for X, the equation: $B(A^TA + X)B = I$.
   (b) If $B^T = B$, show that : $X^T = X$.
   (c) If $\det A = 2$ and $\det B = -3$, find $\det(A^3B^{-1}A^TB^2)$.

3. [10 marks] Determine whether the following sets are subspaces of $\mathbb{R}^3$.

   $W_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 :\ x_2 = x_1 + x_3 - 1 \right\}$, $W_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 :\ x_3 \geqslant 0 \right\}$,

   and $W_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 :\ x_3 = 2x_2,\ x_1 + x_2 = 0 \right\}$.

4. [7 marks] Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4;\ \left\{ \begin{array}{r} 3a - 3b = -c \\ 4b + 2a - c = 2d \end{array} \right. \right\}$.

   (a) Find two matrices A and B such that $W = Nul\ A$ and $W = Col\ B$.
   (b) Use the answer of part (a) to show that W is a subspace of $\mathbb{R}^4$.

5. [9 marks] Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

   (a) Use cofactor expansion to show that: $\det(A - \lambda I) = (1 - \lambda)(2 + \lambda)^2$.
   (b) Find the eigenvalues and the corresponding eigenvectors of A.

PLEASE TURN OVER FOR THE REMAINING QUESTIONS $\ggg$

6. [marks] Let $P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$, and $\mathfrak{B} = \{1-t^2, t-t^2, 2-2t+t^2\}$.

   (a) Find P^{-1}.

   (b) Show that $\mathfrak{B}$ is a basis for $\mathbb{P}_2$ (the set of real polynomials of degree at most 2).

   (c) Use P^{-1} to find the coordinate vector of the polynomial $p(t) = 3 + t - 6t^2$ relative to $\mathfrak{B}$.

7. [8 marks] Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most two, and define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(0) \\ p(\frac{1}{2}) \end{bmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find a polynomial $p \in \mathbb{P}_2$ that spans the kernel of T.

8. [10 marks] The following statements are true. Explain why?

   (a) If $B = [\mathbf{b}_1\ \mathbf{b}_2\ ...\ \mathbf{b}_n]$, then $\{A\mathbf{b}_1, A\mathbf{b}_2, ..., A\mathbf{b}_n\}$ spans $Col\ AB$.

   (b) If A is an invertible $n \times n$ matrix, then $\dim(Nul\ A) = 0$.

   (c) If λ is an eigenvalue of a square matrix A, then λ^2 is an eigenvalue of the matrix A^2.

   (d) Let A be an $n \times n$ matrix. If $A^3 = 0$, then $(I + A)(I - A + A^2) = 0$, and $(I + A)^{-1} = I - A + A^2$.

Total: 80 marks