Solutions to Test 1 of MATH 2202 Linear Algebra I

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(1) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix} \underset{-3R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{bmatrix}$

$\overset{-2R_2+R_3}{\sim} \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & 8-2a & b-6 \end{bmatrix}.$

(i) The system has no solution when $8-2a=0$ (hence $a=4$) and $b-6\neq 0$ (that is $b\neq 6$).

(ii) The system has a unique solution when $8-2a\neq 0$ (hence $a\neq 4$) and for any value of b.

(iii) The system has infinitely many solutions if $8-2a=0$ (hence $a=4$) and $b-6=0$ (hence $b=6$).

(2) $\begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 2 & 6 & 2 & 9 & 12 \\ 0 & 0 & 1 & 5 & -2 \end{bmatrix} \overset{-2R_1+R_2}{\sim} \begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 0 & 0 & 2 & 15 & -4 \\ 0 & 0 & 1 & 5 & -2 \end{bmatrix}$

$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 0 & 0 & 1 & 5 & -2 \\ 0 & 0 & 2 & 15 & 4 \end{bmatrix} \overset{-2R_2+R_3}{\sim} \begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 0 & 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}$

(a) T maps $\mathbb{R}^5$ onto $\mathbb{R}^3$ since its standard matrix has a pivot in every row. So columns of A span $\mathbb{R}^3$.

(b) $A \sim \begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 0 & 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$

$x_1 = -3x_2 - 8x_5 \qquad x_4 = 0$

$x_3 = 2x_5 \qquad x_2, x_5$ are free variables

So

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 - 8x_5 \\ x_2 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -8 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(c) $$Ap = \begin{bmatrix} 1 & 3 & 0 & -3 & 8 \\ 2 & 6 & 2 & 9 & 12 \\ 0 & 0 & 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

Parametric equation for solutions of $Ax=b$ is

$$x = \begin{bmatrix} 6 \\ 1 \\ -1 \\ 0 \\ 7 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -8 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad r, s \in \mathbb{R}.$$

(d) T is not 1-1 because the columns of A are linearly dependent (five vectors in $\mathbb{R}^3$).

(3) $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $u = \begin{bmatrix} h \\ 6 \end{bmatrix}$

(a) $b \in \text{span}\{v, u\}$ if $Ax=b$ with $A = [v \ u]$ has a solution, $b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

$$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}.$$

$Ax=b$ has a solution if $6-3h \neq 0$ (that is $h \neq 2$).

(b) $\{v, u\}$ span $\mathbb{R}^2$ when we have a pivot in every row. This happens when $h \neq 2$.

(c) $\{v, u\}$ are linearly dependent when they are multiple of one another. This happens when $h = 2$.

(4) (a) $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1+3x_2+2x_3\\x_3\\3x_1+8x_2+3x_3\end{bmatrix} = \begin{bmatrix}1&3&2\\0&0&1\\3&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

So the standard matrix of T is $A = \begin{bmatrix}1&3&2\\0&0&1\\3&8&3\end{bmatrix}$.

(b) First find A^{-1}.

$$[A : I_3] = \left[\begin{array}{ccc|ccc}1&3&2&1&0&0\\0&0&1&0&1&0\\3&8&3&0&0&1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&3&2&1&0&0\\0&-1&-3&-3&0&1\\0&0&1&0&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&3&2&1&0&0\\0&1&3&3&0&-1\\0&0&1&0&1&0\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&3&0&1&-2&0\\0&1&0&3&-3&-1\\0&0&1&0&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&0&0&-8&7&3\\0&1&0&3&-3&-1\\0&0&1&0&1&0\end{array}\right]$$

So $A^{-1} = \begin{bmatrix}-8&7&3\\+3&-3&-1\\0&1&0\end{bmatrix}$, and $T^{-1}(x) = A^{-1}x$. Hence

$$T^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-8x_1+7x_2+3x_3\\3x_1-3x_2-x_3\\x_2\end{bmatrix}.$$

(5) (a) Since A is invertible $AA^{-1} = A^{-1}A = I$. Hence $I = I^T = (AA^{-1})^T = (A^{-1}A)^T$. So $I = (A^{-1})^T A^T = A^T(A^{-1})^T$, and hence $(A^T)^{-1} = (A^{-1})^T$.

(b) Let $B = [b_1\ b_2 \dots b_n]$. Then $AB = [Ab_1\ Ab_2 \dots Ab_n]$.

Let $\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = 0$, where $\alpha_i \in \mathbb{R}$. Then

$$\alpha_1(Ab_1) + \alpha_2(Ab_2) + \dots + \alpha_n(Ab_n) = A(\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n) = A.0 = 0$$

Since $[Ab_1\ Ab_2 \dots Ab_n]$ are linearly independent, $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, and hence columns of B are linearly independent.