Model Answers to Test 1
Linear Algebra MATH 2202

1) [5 marks]

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \underset{3R_1+R_3}{\overset{5R_1+R_2}{\sim}} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$$

There is NO h makes $v_3 \in \text{span}\{v_1, v_2\}$.

Reason: If $\begin{bmatrix} 2 \\ 1 \\ h+6 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, then $1=0$, a contradiction.

All values of h make the vectors are linearly dependent, because for all values of h, we have free variable, namely, x_2.

2) [7 marks]

$$\begin{bmatrix} -1 & 2 & 11 \\ 3 & -1 & -13 \\ 2 & 3 & 6 \end{bmatrix} \underset{2R_1+R_3}{\overset{3R_1+R_2}{\sim}} \begin{bmatrix} -1 & 2 & 11 \\ 0 & 5 & 20 \\ 0 & 7 & 28 \end{bmatrix} \underset{\frac{1}{7}R_3}{\overset{\frac{1}{5}R_2}{\sim}} \begin{bmatrix} -1 & 2 & 11 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \overset{-R_2+R_3}{\sim} \begin{bmatrix} -1 & 2 & 11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $x_2 = 4$ and $x_1 = -3$, that is

$$\begin{bmatrix} 11 \\ -13 \\ 6 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

This gives that

$$T\left(\begin{bmatrix} 11 \\ -13 \\ 6 \end{bmatrix}\right) = T\left(-3 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = -3\, T\left(\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}\right) + 4\, T\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)$$

$$= -3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

3) [4 marks]

$$\det \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix} = a + 2 - a^2 \neq 0.$$

That is $a^2 - a - 2 \neq 0$.

Hence $(a-2)(-a+1) \neq 0$ gives $a \neq 2$ and $a \neq -1$.

4) [5 marks]

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $AB = 0 = AC$ but $B \neq C$.

5) [7 marks]

Since $(A - AX)^{-1} = X^{-1}B$ and $A - AX$ is invertible,

$$I = (A - AX)(A - AX)^{-1} = \left((A - AX)X^{-1}\right)B.$$

So B is invertible. Next

$$A - AX = \left((A - AX)^{-1}\right)^{-1} = (X^{-1}B)^{-1} = B^{-1}X.$$

Hence $A = AX + B^{-1}X = (A + B^{-1})X$. Since A is invertible,

$$I = AA^{-1} = (A + B^{-1})(XA^{-1}).$$

Hence $A + B^{-1}$ is invertible.

So $(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = IX = X$.

6) [12 marks]

(a) Let $B = [b_1 \ b_2 \ \dots \ b_n]$. Then $AB = [Ab_1 \ Ab_2 \ \dots \ Ab_n]$.

Assume $\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = 0$. Then

$$A(\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n) = \alpha_1(Ab_1) + \alpha_2(Ab_2) + \dots + \alpha_n(Ab_n) = 0.$$

Since $\{Ab_1, Ab_2, \dots, Ab_n\}$ is linearly independent, $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$. Hence $\{b_1, b_2, \dots, b_n\}$ is linearly independent.

(b) Let $AB = BA = I$ and $AC = CA = I$. Then

$C = CI = C(AB) = (CA)B = IB = B$. Hence the inverse of A is unique.

(c) $T(x_1, x_2, x_3, x_4) = [2 \ 0 \ 3 \ -4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Now $A = [2 \ 0 \ 3 \ -4]$.

Since $[2 \ 0 \ 3 \ -4 : 0]$ has infinitely many solutions, T is not one-to-one. Every column of A spans $\mathbb{R}$. Hence T is onto.