MATH 2202 - Fall 2009

MIDTERM EXAM 2 SOLUTIONS

1. (a) $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \begin{array}{l} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{2}R_1 \end{array} \sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 5R_2 \end{array}$

Hence,

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -5 & 1 \end{bmatrix}$, and $U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$.

It follows that:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -1-2 & 2-1 \\ 3 & -3+10 & 6+5-6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} = A.$$

(b) $A\mathbf{x} = \mathbf{b} \Longleftrightarrow \begin{cases} L\mathbf{y} = \mathbf{b} \\ U\mathbf{x} = \mathbf{y} \end{cases}$, therefore

$$[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & -5 \\ \frac{3}{2} & -5 & 1 & 7 \end{bmatrix} \begin{array}{l} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{2}R_1 \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 5R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{bmatrix}$$

$\Rightarrow y_1 = 0,\ y_2 = -5,\ y_3 = -18$ i.e $\mathbf{y} = (0, -5, -18)^T$

$$[U\ \mathbf{y}] = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} \frac{1}{2}R_1 \\ \\ -\frac{1}{6}R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -6 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{c} R_1 - 2R_3 \\ R_2 + R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{c} R_1 - 2R_3 \\ -\frac{1}{2}R_2 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array}$$

$\Rightarrow x_1 = -5,\ x_2 = 1,\ x_3 = 3$ i.e $\mathbf{x} = (-5, 1, 3)^T$.

2. (a) $\det A = \begin{vmatrix} 2 & 4 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & 1 & 2 \end{vmatrix}$

$$= 2 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -2 \times 1 \times 5 \times 1 = -10$$

Hence,

$\det(-A) = (-1)^4 \det A = -10$, and $\det(A^T)^2 A^{-1} = (\det A)^2 \dfrac{1}{\det A} = \det A = -10$.

(b) $C_{44} = \begin{vmatrix} 2 & 4 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 0 \Rightarrow A^{-1}_{44} = dfracC_{44}\det A = 0.$

3. (a) Let A and $B = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_p]$ be matrices of dimension $m \times n$ and $n \times p$, respectively. Then,
$AB = [A\mathbf{b}_1, A\mathbf{b}_2, ..., A\mathbf{b}_p]$.
Using the linearty of the matrix A we get
$c_1 A\mathbf{b}_1 + c_2 A\mathbf{b}_2 + ... + c_p A\mathbf{b}_p = 0 \Rightarrow A[c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_p\mathbf{b}_p] = 0$
$\Rightarrow c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_p\mathbf{b}_p = 0 \Rightarrow c_1 = c_2 = ... = c_p = 0$ since $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_p$ are linearly independent.
Hence, $A\mathbf{b}_1, A\mathbf{b}_2, ..., A\mathbf{b}_p$ are also linearly independent.

(b) $\det A^T A = \det I \Rightarrow \det A^T \det A = 1 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$.

(c) $A^2 - 8A + 2I = 0 \Rightarrow A^2 - 8A = -2I \Rightarrow -\dfrac{1}{2}A^2 + 4A = I \Rightarrow \left(-\dfrac{1}{2}A + 4I\right) A = I$
$\Rightarrow A$ is invertible and $A^{-1} = -\dfrac{1}{2}A + 4I$.

4. (a) Since $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} S & XA_{22} \\ 0 & A_{22} \end{bmatrix}$,
then $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} S & XA_{22} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} = \begin{bmatrix} S + XA_{22}Y & XA_{22} \\ A_{22}Y & A_{22} \end{bmatrix}$.
Hence, $XA_{22} = A_{12}$, $A_{22}Y = A_{21}$, and $S + XA_{22}Y = A_{11}$.
Since A_{22} is invertible, it follows that $X = A_{12}A_{22}^{-1}$ and $Y = A_{22}^{-1}A_{21}$.
Substituting for X and Y in the partitioned matrix we get
$S + A_{12}A_{22}^{-1}A_{22}A_{22}^{-1}A_{21} = A_{11} \Rightarrow S = A_{11} - A_{12}A_{22}^{-1}A_{21}$.
which is the given value for S.

(b) U,V are $n \times n$ matrices such that $U^TU = I$, $V^TV = I \Rightarrow U, U^T, V, V^T$ are invertible, and $U^{-1} = U^T$, $(V^T)^{-1} = V$. In addition D is invertible, because it is a diagonal matrix with positive entries $\sigma_1, \sigma_2, ..., \sigma_n$ on the diagonal ($\det D = \sigma_1.\sigma_2 ... \sigma_n \neq 0$). Hence, $A = UDV^T$ is also invertible, because it is a product of invertible matrices. Further, $A^{-1} = (UDV^T)^{-1} = (V^T)^{-1}D^{-1}U^{-1} = VD^{-1}U^T$.