Sultan Qaboos University-College of Science
Department of Mathematics and Statistics
MATH 2202 - Linear Algebra I
Spring Semester 2010 - Final Exam

Date: 24th May, 2010 **Time Allowed: 150 mins.**
Total Marks: 100

ATTEMPT ALL SEVEN QUESTIONS AND SHOW THE DETAILS OF YOUR WORK.

Q1: (*a*) [6 marks] For what value(s) of h and k does the following system of equations

$$\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$$

(i) have no solution (ii) have infinitely many solutions (iii) have a unique solution.

(*b*) [4 marks] Use the determinant to find the value(s) of a such that the following vectors are linearly dependent in $\mathbb{R}^3$.

$$\begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} -a \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ a+2 \end{bmatrix}$$

Q2: (*a*) [5 marks] Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that $A^3 = 0$. Use matrix algebra laws to compute the product $(I - A)(I + A + A^2)$.

(*b*) [5 marks] Let A and B be two 2×2 matrices such that

$$B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}.$$

Find A.

Q3: (*a*) [8 marks] Let a, b and c be real numbers. Show that

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

(*b*) [8 marks] Find the value(s) of k such that the matrix

$$A = \begin{bmatrix} 1 & 2 & k \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & k \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

has an inverse.

Q4: (*a*) [8 marks] Let

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{c} a - 3b + c = 0 \\ a = 3d + c \end{array} \right\}.$$

Find two matrices A and B such that $H = Nul A$ and $H = Col B$.

(*b*) [8 marks] Determine whether the following sets are subspaces of $\mathbb{R}^3$. Justify your answer.

$$(i) H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a = b + c \right\}, \quad (ii) H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a^2 = b^2 + c^2 \right\}.$$

Q5: Let $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and let $\boldsymbol{\beta} = \{-1+t+t^2,\ 1-t+t^2,\ 1+t-t^2\}$.

(a) [6 marks] Compute the adjoint (adjugate) of P , and then obtain P^{-1}.

(b) [3 marks] Show that $\boldsymbol{\beta}$ is a basis for P_2, the set of real polynomials of degree at most 2.

(c) [3 marks] Use P^{-1} to find the coordinate vector of the polynomial $p(t) = t + 2t^2$ relative to the basis $\boldsymbol{\beta}$.

Q6: Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

(a) [3 marks] Is $u = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ an eigenvector of A? Justify your answer.

(b) [5 marks] Show that $\lambda = -1$ is an eigenvalue of A? Find the eigenvector of A that is corresponding to this eigenvalue.

(c) [4 marks] Given the characteristic equation of $A : \lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$, use this equation to find all eigenvalues of A.

(d) [10 marks] Diagonalize A.

Q7: [14 marks] Two of the following statements are TRUE while the other two are FALSE. Prove the true statements and explain why the others are false.

(a) If a matrix A is similar to a matrix B then $\det A = \det B$.

(b) If A is a 3×5 matrix and B is a 5×3 matrix such that dim $NulA$ =dim $NulB$, then $rankA = rankB + 2$.

(c) If A and B are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.

(d) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and A the standard matrix for T, then T is onto if and only if the columns of A are linearly independent.