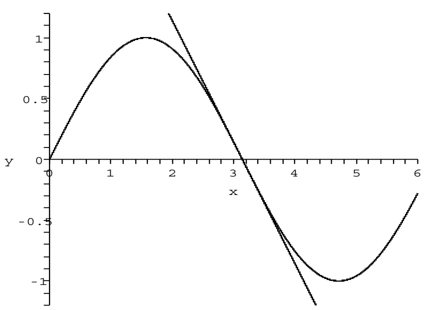
Chapter 2

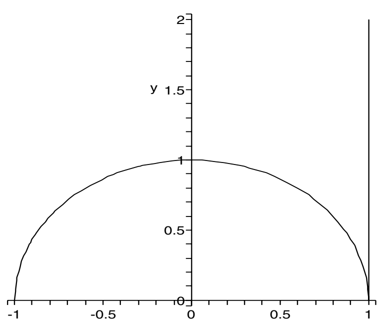
Differentiation

2.1 Tangent Lines and Velocity

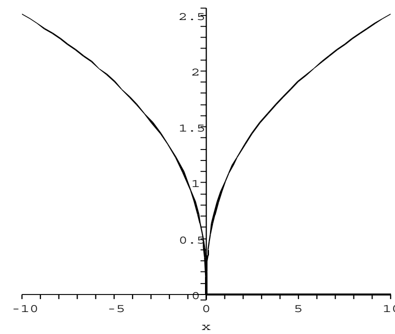
1.



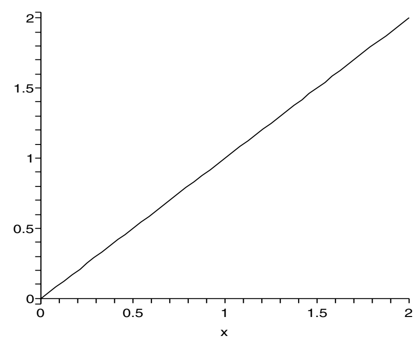
2. The tangent line is vertical and coincides with the y-axis:



3. The tangent line is vertical and coincides with the y-axis:



4. The tangent line overlays the line:



5. At $x = 1$ the slope of the tangent line appears to be about -1.

6. The slope at $x = 1$ is approximately -3.

7. C, B, A, D. At the point labeled C, the slope is very steep and negative. At point B, the slope is zero and at point A, the slope is just more than zero. The slope of the line tangent to point D is large and positive.

8. In order of increasing slope: D (large negative), C (small negative), B (small positive), and A (large positive).

9. (a) Points $(1, 0)$ and $(2, 6)$.
Slope is $\frac{6-0}{1} = 6$.

(b) Points $(2, 6)$ and $(3, 24)$.
Slope is $\frac{24-6}{1} = 18$.

(c) Points $(1.5, 1.875)$ and $(2, 6)$.
Slope is $\frac{6-1.875}{.5} = 8.25$.

(d) Points $(2, 6)$ and $(2.5, 13.125)$. Slope is $\frac{13.125-6}{.5} = 14.25$.

(e) Points $(1.9, 4.959)$ and $(2, 6)$. Slope is $\frac{6-4.959}{.1} = 10.41$.

(f) Points $(2, 6)$ and $(2.1, 7.161)$. Slope is $\frac{7.161-6}{.1} = 11.61$.

(g) Slope seems to be approximately 11.

10. (a) Points $(1, \sqrt{2})$ and $(2, \sqrt{5})$. Slope is $\frac{\sqrt{5}-\sqrt{2}}{2-1} \approx 0.5040$.

(b) Points $(2, \sqrt{5})$ and $(3, \sqrt{10})$. Slope is $\frac{\sqrt{10}-\sqrt{3}}{3-2} \approx 0.9262$.

(c) Points $(1.5, 1.8028)$ and $(2, 2.2361)$. Slope is $\frac{2.2361-1.8028}{2-1.5} \approx 0.8666$.

(d) Points $(2, 2.2361)$ and $(2.5, 2.2693)$. Slope is $\frac{2.2693-2.2361}{2.5-2} \approx 0.9130$.

(e) Points $(1.9, 2.1471)$ and $(2, 2.2361)$. Slope is $\frac{2.2361-2.1471}{2-1.9} \approx 0.8898$.

(f) Points $(2, 2.2361)$ and $(2.1, 2.3259)$. Slope is $\frac{2.3259-2.2361}{2.1-2} \approx 0.8987$.

(g) Slope seems to be approximately 0.89.

11. (a) Points $(1, .54)$ and $(2, -.65)$. Slope is $\frac{-.65-.54}{1} = -1.19$.

(b) Points $(2, -.65)$ and $(3, -.91)$. Slope is $\frac{-.91-(-.65)}{1} = -.26$.

(c) Points $(1.5, -.628)$ and $(2, -.654)$. Slope is $\frac{-.654-(-.628)}{.5} = -.05$.

(d) Points $(2, -.65)$ and $(2.5, 1.00)$. Slope is $\frac{1.00-(-.65)}{.5} = 3.3$.

(e) Points $(1.9, -.89)$ and $(2, -.65)$. Slope is $\frac{-.65-(-.89)}{.1} = 2.4$.

(f) Points $(2, -.654)$ and $(2.1, -.298)$. Slope is $\frac{-.298-(-.654)}{.1} = 3.56$.

(g) Slope seems to be approximately 3.

12. (a) Points $(1, -2.1850)$ and $(2, 1.1578)$. Slope is $\frac{1.1578-(-2.1850)}{2-1} \approx 3.3429$.

(b) Points $(2, 1.1578)$ and $(3, -0.2910)$. Slope is $\frac{-0.2910-1.1578}{3-2} \approx -1.4488$.

(c) Points $(1.5, -0.1425)$ and $(2, 1.1578)$. Slope is $\frac{1.1578-(-0.1425)}{2-1.5} \approx -2.6007$.

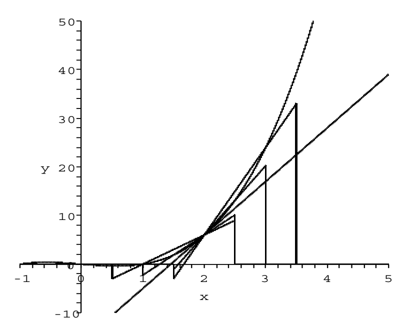
(d) Points $(2, 1.1578)$ and $(2.5, -3.3805)$. Slope is $\frac{-3.3805-1.1578}{2.5-2} \approx -9.0767$.

(e) Points $(1.9, 0.7736)$ and $(2, 1.1578)$. Slope is $\frac{1.1578-0.7736}{2-1.9} \approx 3.8427$.

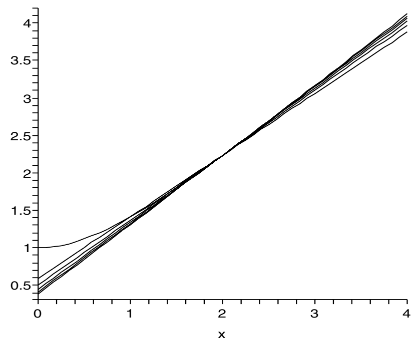
(f) Points $(2, 1.1578)$ and $(2.1, 1.7778)$. Slope is $\frac{1.7778-1.1578}{2.1-2} \approx 6.1996$.

(g) Slope seems to be approximately 4.68.

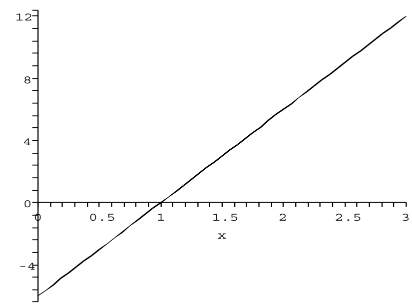
13.

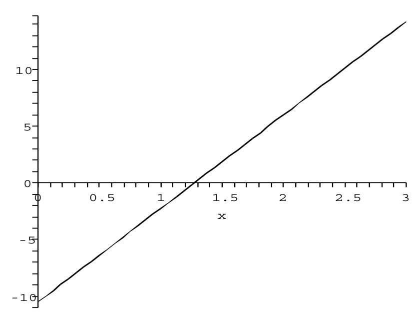


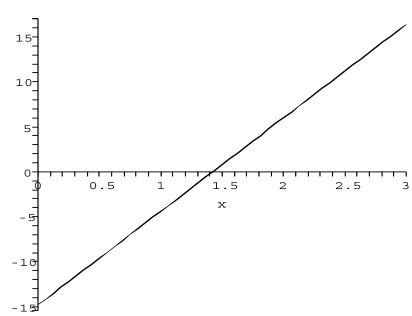
14. All the lines are very close to the tangent line:



15. The sequence of graphs should look like:

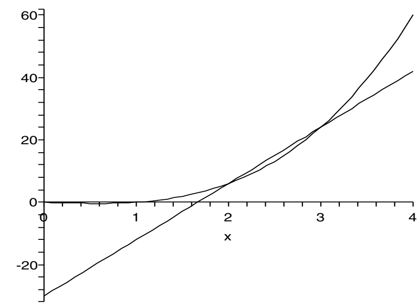


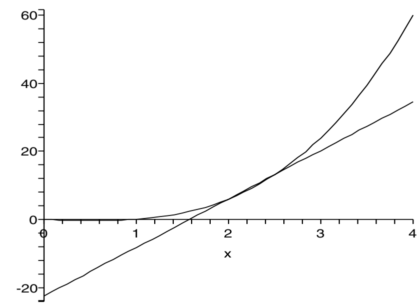


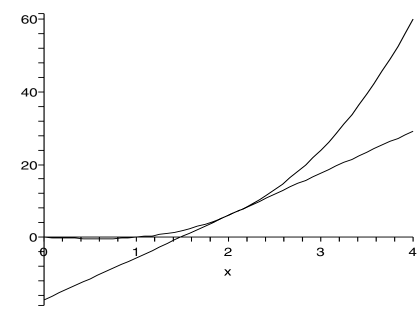


The third secant line is indistinguishable from the tangent line.

16. The sequence of graphs should look like:

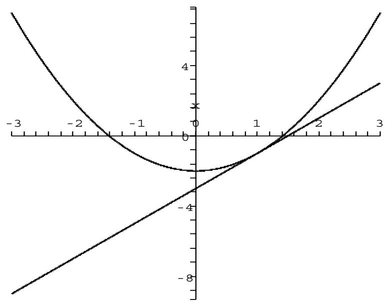






The third secant line is indistinguishable from the tangent line.

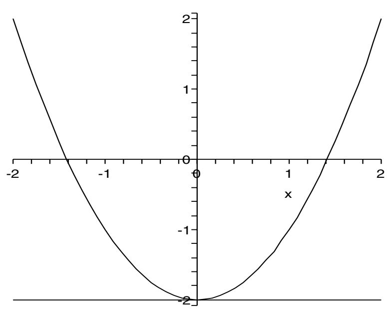
17. Slope is
$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \frac{(1+h)^2-2-(-1)}{h}$
$= \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0} h+2 = 2.$
Tangent line is $y-(-1) = 2(x-1)$ or $y = 2x-3$.



18. Slope is

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{h^2}{h} = 0.$$

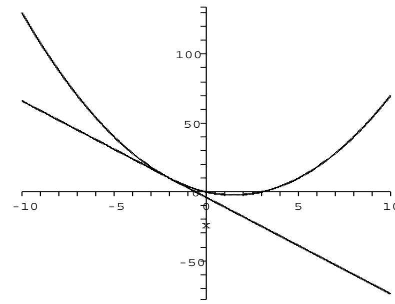
Tangent line is $y = -2$.



19. Slope is

$$\lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$$
$$= \lim_{h\to 0} \frac{(-2+h)^2 - 3(-2+h) - (10)}{h}$$
$$= \lim_{h\to 0} \frac{4 - 4h + h^2 + 6 - 3h - 10}{h}$$
$$= \lim_{h\to 0} \frac{-7h + h^2}{h} = \lim_{h\to 0} -7 + h = -7.$$

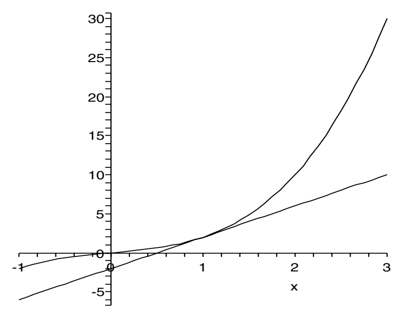
Tangent line is $y - 10 = -7(x+2)$ or $y = -7x - 4$.



20. Slope is

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3)+(1+h)-2}{h}$$
$$= \lim_{h\to 0} \frac{4h+3h^2+h^3}{h}$$
$$= \lim_{h\to 0} 4 + 3h + h^2 = 4.$$

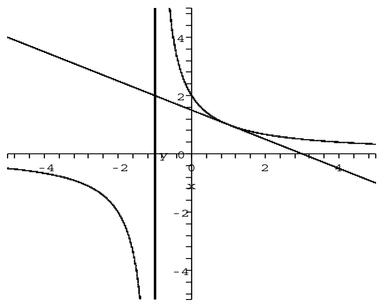
Tangent line is $y = 4(x-1)+2$.



21. Slope is

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{(1+h)+1} - \frac{2}{1+1}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{2+h} - 1}{h}$$
$$= \lim_{h\to 0} \frac{\left(\frac{2-(2+h)}{2+h}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\left(\frac{-h}{2+h}\right)}{h}$$
$$= \lim_{h\to 0} \frac{-1}{2+h} = \frac{-1}{2}$$

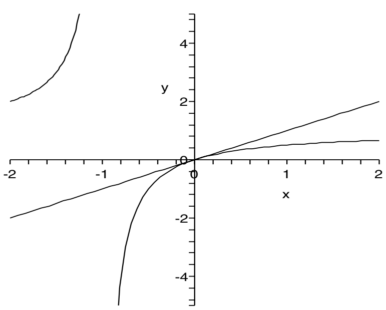
Tangent line is $y - 1 = -\frac{1}{2}(x-1)$ or $y = -\frac{x}{2} + \frac{3}{2}$.



22. Slope is

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{h}{h-1}-0}{h}$$
$$=\lim_{h\to 0}\frac{1}{h-1}=-1.$$

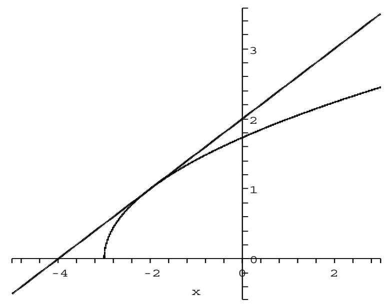
Tangent line is $y=-x$.



23. Slope is

$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(-2+h)+3}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\cdot\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$
$$=\lim_{h\to 0}\frac{(h+1)-1}{h(\sqrt{h+1}+1)}$$
$$=\frac{1}{\sqrt{h+1}+1}=\frac{1}{2}$$

Tangent line is $y-1=\frac{1}{2}(x+2)$ or $y=\frac{1}{2}x+2$.



24. Slope is

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(1+2h+h^2)+1}-\sqrt{2}}{h}$$

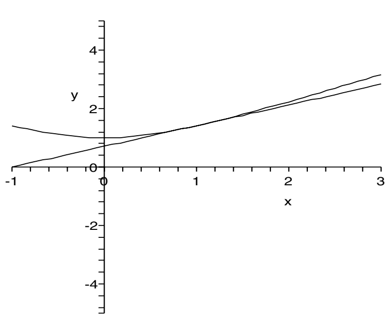
We then multiply by

$$\frac{(\sqrt{2+2h+h^2}+\sqrt{2})}{(\sqrt{2+2h+h^2}+\sqrt{2})}$$

to get

$$\lim_{h\to 0}\frac{(2+2h+h^2)-2}{h(\sqrt{2+2h+h^2}+\sqrt{2})}$$
$$=\lim_{h\to 0}\frac{h(2+h)}{h(\sqrt{2+2h+h^2}+\sqrt{2})}$$
$$=\lim_{h\to 0}\frac{2+h}{(\sqrt{2+2h+h^2}+\sqrt{2})}$$
$$=\frac{2}{2\sqrt{2}}=\frac{\sqrt{2}}{2}.$$

Tangent line is $y=\frac{\sqrt{2}}{2}(x-1)+\sqrt{2}$.



25. Numerical evidence suggests that

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=1$$

while

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -1.$$

Since these are not equal, there is no tangent line. A graph makes it apparent that this function has a "corner" at $x = 1$.

26. Tangent line does not exist at $x = 1$ because the function is not defined there.

27. Numerical evidence suggests that

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$
$$= 0$$

Since the slope of the tangent line from the left equals that from the right and the function appears to be continuous in the graph, we conjecture that the tangent line exists and has slope 0.

28. Tangent line does not exist at $x = 1$ because the function has a sharp corner there, causing the limit of slopes to fail to exist.

29. Looking at the graph, we see that there is a jump discontinuity at $a = 0$. Thus there cannot be a tangent line, as the tangent line from the left would be different from the tangent line from the right.

30. Tangent line does not exist at $x = 0$ because the function is not defined there. Tangent line would exist with slope -2 if the function were defined to be 0 at $x = 0$.

31. (a) Points $(0, 10)$ and $(2, 74)$. Average velocity is $\frac{64-0}{2} = 32$.

(b) Second point $(1, 26)$. Average velocity is $\frac{64-26}{1} = 48$.

(c) Second point $(1.9, 67.76)$. Average velocity is $\frac{74-67.76}{.1} = 62.4$.

(d) Second point $(1.99, 73.3616)$. Average velocity is $\frac{74-73.3616}{.01} = 63.84$.

(e) The instantaneous velocity seems to be approaching 64.

32. (a) Points $(0, 0)$ and $(2, 26)$. Average velocity is $\frac{26-0}{2-0} = 13$.

(b) Second point $(1, 4)$. Average velocity is $\frac{26-4}{2-1} = 22$.

(c) Second point $(1.9, 22.477)$. Average velocity is $\frac{26-22.477}{2-1.9} = 35.23$.

(d) Second point $(1.99, 25.6318)$. Average velocity is $\frac{26-25.6318}{2-1.99} = 36.8203$.

(e) The instantaneous velocity seems to be approaching 37.

33. (a) Points $(0, 0)$ and $(2, \sqrt{20})$. Average velocity is $\frac{\sqrt{20}-0}{2-0} = 2.236068$.

(b) Second point $(1, 3)$. Average velocity is $\frac{\sqrt{20}-3}{2-1} = 1.472136$.

(c) Second point $(1.9, \sqrt{18.81})$. Average velocity is $\frac{\sqrt{20}-\sqrt{18.81}}{2-1.9} = 1.3508627$.

(d) Second point $(1.99, \sqrt{19.8801})$. Average velocity is $\frac{\sqrt{20}-\sqrt{19.8801}}{2-1.99} = 1.3425375$.

(e) One might conjecture that these numbers are approaching 1.34. The exact limit is $\frac{6}{\sqrt{20}} \approx 1.341641$.

34. (a) Points $(0, 0)$ and $(2, 47.9426)$. Average velocity is $\frac{47.9426-0}{2-0} = 23.9713$.

(b) Second point $(1, 24.7404)$. Average velocity is $\frac{47.9426-24.7404}{2-1} = 23.2022$.

(c) Second point $(1.9, 45.7338)$. Average velocity is $\frac{47.9426-45.7338}{2-1.9} = 22.0871$.

(d) Second point $(1.99, 47.7230)$. Average velocity is $\frac{47.9426-47.7230}{2-1.99} = 21.9545$.

(e) The instantaneous velocity seems to be decreasing to slightly less than 22.

35. (a) Velocity at time $t = 1$ is
$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{-16(1+h)^2 + 5 - (-11)}{h}$$
$$= \lim_{h\to 0} \frac{-16 - 32h - 16h^2 + 5 + 11}{h}$$
$$= \lim_{h\to 0} \frac{-32h - 16h^2}{h}$$
$$= \lim_{h\to 0} -32 - 16h = -32.$$

(b) Velocity at time $t = 2$ is
$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h\to 0} \frac{-16(4 + 4h + h^2) + 5 + 59}{h}$$
$$= \lim_{h\to 0} \frac{-64 - 64h - 16h^2 + 64}{h}$$
$$= \lim_{h\to 0} -64 - 16h = -64.$$

36. (a) Velocity at time $t = 0$ is
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{h+16} - 4}{h} \cdot \frac{\sqrt{h+16} + 4}{\sqrt{h+16} + 4}$$
$$= \lim_{h\to 0} \frac{(h+16) - 16}{h(\sqrt{h+16} + 4)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+16} + 4} = 1/8.$$

(b) Velocity at time $t = 2$ is
$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$
Multiplying by
$$\frac{\sqrt{h+18} + \sqrt{18}}{\sqrt{h+18} + \sqrt{18}}$$
gives
$$= \lim_{h\to 0} \frac{(h+18) - 18}{h(\sqrt{h+18} + \sqrt{18})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+18} + \sqrt{18}} = \frac{1}{2\sqrt{18}}.$$

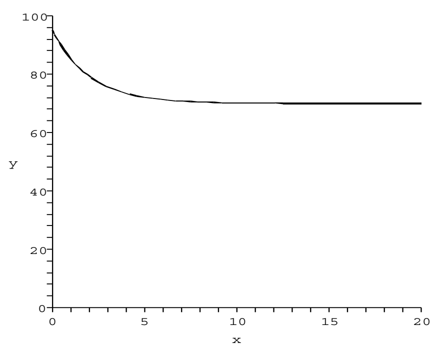
37. The slope of the tangent line at $p = 1$ is approximately
$$\frac{-20 - 0}{2 - 0} = -10$$
which means that at $p = 1$, the freezing temperature of water decreases by 10 degrees Celsius per 1 atm increase in pressure. The slope of the tangent line at $p = 3$ is approximately
$$\frac{-11 - (-20)}{4 - 2} = 4.5$$
which means that at $p = 3$, the freezing temperature of water increases by 4.5 degrees Celsius per 1 atm increase in pressure.

38. The slope of the tangent line at $v = 50$ is approximately
$$\frac{47 - 28}{60 - 40} = .95.$$
This means that at an initial speed of 50 mph, the range of the soccer kick increases by .95 yards per 1 mph increase in initial speed.

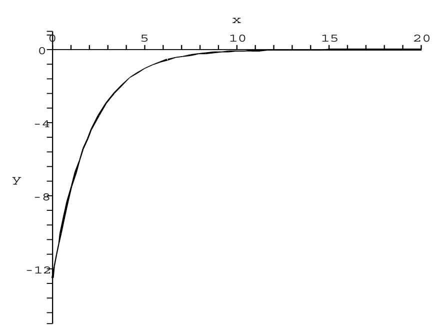
39. The hiker reached the top at the highest point on the graph (about 1.75 hours). The hiker was going the fastest on the way up at this point. The hiker was going the fastest on the way down at the point where the tangent line has the least (i.e. most negative) slope, at about 3 hours, at the end of the hike. Where the graph is level, the hiker was either resting, or walking on flat ground.

40. The tank is the fullest at the first spike (at around 8am). The tank is the emptiest just before this at the lowest dip (around 7am). The tank is filling up the fastest where the graph has the steepest positive slope (in between 7 and 8am). The tank is emptying the fastest just after 8am where the graph has the steepest negative slope. The level portions most likely represent night, when water usage is at a minimum.

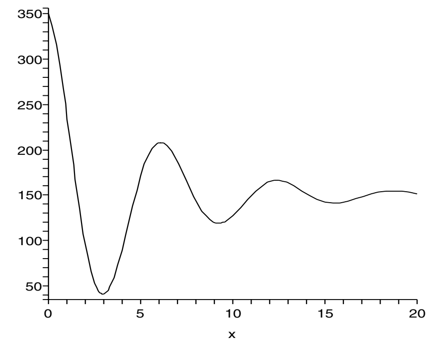
41. A possible graph of the temperature with respect to time:



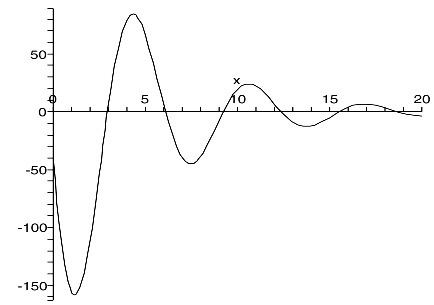
Graph of the rate of change of the temperature:



42. Possible graph of bungee-jumper's height:



A graph of the bungee-jumper's velocity:



43. (a) To say that

$$\frac{f(4)-f(2)}{2} = 21{,}034$$

per year is to say that the average rate of change in the bank balance between Jan. 1, 2002 and Jan. 1, 2004 was 21,034 ($ per year).

(b) To say that

$$2[f(4)-f(3.5)] = 25{,}036$$

(note that $2[f(4)-f(3.5)] = \frac{f(4)-f(3.5)}{1/2}$) per year is to say that the average rate of change between July 1, 2003 and Jan. 1, 2004 was 25,036 ($ per year).

(c) To say that

$$\lim_{h\to 0}\frac{f(4+h)-f(4)}{h} = \$30{,}000$$

is to say that the instantaneous rate of change in the balance on Jan. 1, 2004 was 30,000 ($ per year).

44. (a) $\frac{f(40)-f(38)}{2} = -2103$ represents the average rate of depreciation between 38 and 40 thousand miles.

(b) $\frac{f(40)-f(39)}{2} = -2040$ represents the average rate of depreciation between 39 and 40 thousand miles.

(c) $\lim\limits_{h\to 0} \frac{f(40+h)-f(40)}{h} = -2000$ represents the instantaneous rate of depreciation in the value of the car when it has 40 thousand miles.

45. We are given $\theta(t) = 0.4t^2$. We are advised that θ is measured in radians, and that t is time. Let us assume that t is measured in seconds.

Three rotations corresponds to $\theta = 6\pi$. Proceeding, if $\theta(t) = 6\pi$ then $0.4t^2 = 6\pi$ and solving for t yields $t = \sqrt{15\pi} \approx 6.865$ (seconds).

At that exact moment of time (call it a) , the exact angular velocity is

$$\lim_{h\to 0} \frac{\theta(a+h)-\theta(a)}{h}$$
$$= \lim_{h\to 0} \frac{.4(\sqrt{15\pi}+h)^2 - 6\pi}{h}$$
$$= \lim_{h\to 0} \frac{.4(15\pi + 2h\sqrt{15\pi} + h^2) - 6\pi)}{h}$$
$$= \lim_{h\to 0} \frac{.8h\sqrt{15\pi} + .4h^2}{h}$$
$$= \lim_{h\to 0} .8\sqrt{15\pi} + .4h = .8\sqrt{15\pi} \approx 5.492$$

and the units would be *radians per second.*

46. First find the time corresponding to two rotations: $4\pi = 0.4t^2 \Rightarrow t \approx 5.6050$.

Now the angular velocity is

$$\lim_{h\to 0} \frac{\theta(5.6+h)-\theta(5.6)}{h}$$
$$= \lim_{h\to 0} \frac{0.4(5.6+h)^2 - 0.4(5.6)^2}{h}$$
$$= \lim_{h\to 0} \frac{4.48h + 0.4h^2}{h} = 4.48.$$

The third rotation is helpful because the angular velocity increases.

47. $$v_{avg} = \frac{f(s)-f(r)}{s-r}$$
$$= \frac{as^2+bs+c-(ar^2+br+c)}{s-r}$$
$$= \frac{a(s^2-r^2)+b(s-r)}{s-r}$$
$$= \frac{a(s+r)(s-r)+b(s-r)}{s-r}$$
$$= a(s+r)+b$$

Let $v(r)$ be the velocity at $t = r$. We have

$$v(r) = \lim_{h\to 0} \frac{f(r+h)-f(r)}{h}$$
$$= \lim_{h\to 0} \frac{a(r^2+2rh+h^2)+bh-ar^2}{h}$$
$$= \lim_{h\to 0} \frac{h(2ar+ah+b)}{h}$$
$$= \lim_{h\to 0} 2ar+ah+b = \text{2ar + b.}$$

So $v(r) = 2ar + b$. The same argument shows that $v(s) = 2as + b$.

Finally,

$$\frac{v(r)+v(s)}{2} = \frac{(2ar+b)+(2as+b)}{2}$$
$$\frac{2a(s+r)+2b}{2} = a(s+r)+b = v_{avg}$$

48. $f(t) = t^3 - t$ works with $r = 0$, $s = 2$. The average velocity between r and s is $\frac{6-0}{2-0} = 3$. The instantaneous velocity at r is

$$\lim_{h\to 0} \frac{(0+h)^3-(0+h)-0}{h} = 0,$$

and the instantaneous velocity at s is

$$\lim_{h\to 0}\frac{(2+h)^3-(2+h)-6}{h}$$
$$=\lim_{h\to 0}\frac{8+12h+6h^2+h^3-2-h-6}{h}$$
$$=\lim_{h\to 0} 11+6h+h^2=11,$$

so the average between the instantaneous velocities is 5.5.

49. Let $x=h+a$. Then $h=x-a$, and clearly

$$\frac{f(a+h)-f(a)}{h}=\frac{f(x)-f(a)}{x-a}.$$

It is also clear that $x\to a$ if and only if $h\to 0$. Therefore if one of the two limits exists, then so does the other and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

50. For exercise 17,

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$
$$=\lim_{x\to 1}\frac{(x^2-2)-(-1)}{x-1}$$
$$=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=2.$$

For exercise 19,

$$\lim_{x\to -2}\frac{f(x)-f(-2)}{x+2}$$
$$\lim_{x\to -2}\frac{(x^2-3x)-10}{x+2}$$
$$=\lim_{x\to -2}\frac{(x-5)(x+2)}{x+2}=-7.$$

51. First, compute the slope of the tangent line. Using the result of #49, it is convenient to assume x is near but not exactly 1/2, and write

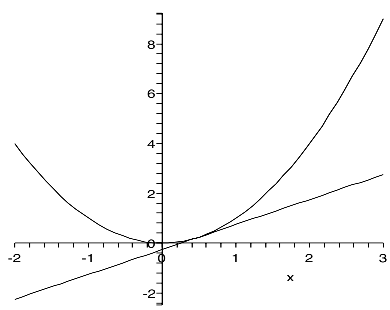
$$\lim_{x\to 1/2}\frac{f(x)-f(1/2)}{x-(1/2)}=\frac{x^2-(1/4)}{x-(1/2)}$$
$$=\lim_{x\to 1/2}\frac{(x-(1/2))(x+(1/2))}{x-(1/2)}$$
$$=\lim_{x\to 1/2} x+(1/2)=1$$

Next, we quickly write the equation of the tangent line in point-slope form:

$y-(1/4)=1(x-(1/2))$ or $y=x-(1/4)$.

The location of the tree is the point $(x,y)=(1,3/4)$ and this point is indeed on the tangent line. The tree will be hit if the car gets that far (that being something we have no way of knowing).

52. It is clear from the graph that no other tangent line will pass through the point $(1,\frac{3}{4})$. No other lines through this point will be tangent to the curve $y=x^2$.



2.2 The Derivative

1. Using (2.1):

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{3(1+h)+1-(4)}{h}$$
$$=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3$$

Using (2.2):

$$\lim_{b\to 1}\frac{f(b)-f(1)}{b-1}$$
$$=\lim_{b\to 1}\frac{3b+1-(3+1)}{b-1}$$

$$= \lim_{b\to 1} \frac{3b-3}{b-1}$$
$$= \lim_{b\to 1} \frac{3(b-1)}{b-1} = \lim_{b\to 1} 3 = 3$$

2. Using (2.1):

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{3(1+h)^2+1-4}{h}$$
$$= \lim_{h\to 0} \frac{6h+3h^2}{h}$$
$$= \lim_{h\to 0} 6+3h = 6.$$

Using (2.2):

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to 1} \frac{(3x^2+1)-4}{x-1}$$
$$= \lim_{x\to 1} \frac{3(x-1)(x+1)}{x-1}$$
$$= \lim_{x\to 1} 3(x+1) = 6.$$

3. Using (2.1): Since

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{3(1+h)+1}-2}{h}$$
$$= \frac{\sqrt{4+3h}-2}{h} \cdot \frac{\sqrt{4+3h}+2}{\sqrt{4+3h}+2}$$
$$= \frac{4+3h-4}{h(\sqrt{4+3h}+2)} = \frac{3h}{h(\sqrt{4+3h}+2)}$$
$$= \frac{3}{\sqrt{4+3h}+2}, \text{ we have:}$$
$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{3}{\sqrt{4+3h}+2}$$
$$= \frac{3}{\sqrt{4+3(0)}+2} = \frac{3}{4}.$$

Using (2.2): Since

$$\frac{f(b)-f(1)}{b-1}$$
$$= \frac{\sqrt{3b+1}-2}{b-1}$$
$$= \frac{(\sqrt{3b+1}-2)(\sqrt{3b+1}+2)}{(b-1)(\sqrt{3b+1}+2)}$$
$$= \frac{(3b+1)-4}{(b-1)\sqrt{3b+1}+2}$$
$$= \frac{3(b-1)}{(b-1)\sqrt{3b+1}+2}$$
$$= \frac{3}{\sqrt{3b+1}+2}, \text{ we have:}$$
$$f'(1) = \lim_{b\to 1} \frac{f(b)-f(1)}{b-1}$$
$$= \lim_{b\to 1} \frac{3}{\sqrt{3b+1}+2}$$
$$= \frac{3}{\sqrt{4}+2} = \frac{3}{4}.$$

4. Using (2.1):

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3}{(2+h)+1}-1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3}{3+h}-\frac{3+h}{3+h}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-h}{3+h}}{h}$$
$$= \lim_{h\to 0} \frac{-1}{3+h} = -\frac{1}{3}.$$

Using (2.2):

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{3}{x+1}-1}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{3}{x+1}-\frac{x+1}{x+1}}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{-(x-2)}{x+1}}{x-2}$$
$$= \lim_{x\to 2} \frac{-1}{x+1} = -\frac{1}{3}.$$

5. $\displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{3(x+h)^2+1-(3(x)^2+1)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2+6xh+3h^2+1-(3x^2+1)}{h}$$
$$= \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} 6x+3h = 6x$$

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) + 1 - f(x)}{h}$
$= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h}$
$= \lim_{h\to 0} \frac{h(2x+h-2)}{h} = 2x - 2.$

7. $\lim_{b\to x} \frac{f(b)-f(x)}{b-x}$
$= \lim_{b\to x} \frac{\frac{3}{b+1} - \frac{3}{x+1}}{b-x}$
$= \lim_{b\to x} \frac{\frac{3(x+1)-3(b+1)}{(b+1)(x+1)}}{b-x}$
$= \lim_{b\to x} \frac{-3(b-x)}{(b+1)(x+1)(b-x)}$
$= \lim_{b\to x} \frac{-3}{(b+1)(x+1)} = \frac{-3}{(x+1)^2}$

8. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{\frac{2}{2(x+h)-1} - \frac{2}{2x-1}}{h}$
$= \lim_{h\to 0} \frac{\frac{2(2x-1)-2(2x+2h-1)}{(2x+2h-1)(2x-1)}}{h}$
$= \lim_{h\to 0} \frac{\frac{-4h}{(2x+2h-1)(2x-1)}}{h}$
$= \lim_{h\to 0} \frac{-4}{(2x+2h-1)(2x-1)}$
$= \frac{-4}{(2x-1)^2}$

9. $\lim_{b\to x} \frac{f(b)-f(x)}{b-x}$
$= \lim_{b\to x} \frac{\sqrt{3b+1} - \sqrt{3x+1}}{b-x}$
Multiplying by
$\frac{\sqrt{3b+1} + \sqrt{3x+1}}{\sqrt{3b+1} + \sqrt{3x+1}}$
gives
$\lim_{b\to x} \frac{(3b+1)-(3x+1)}{(b-x)(\sqrt{3b+1} + \sqrt{3x+1})}$
$= \lim_{b\to x} \frac{3(b-x)}{(b-x)(\sqrt{3b+1} + \sqrt{3x+1})}$
$= \lim_{b\to x} \frac{3}{(\sqrt{3b+1} + \sqrt{3x+1})}$
$= \frac{3}{2\sqrt{3x+1}}$

10. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{2(x+h)+3-(2x+3)}{h}$
$= \lim_{h\to 0} \frac{2h}{h} = 2.$

11. $\lim_{b\to x} \frac{f(b)-f(x)}{b-x}$
$= \lim_{b\to x} \frac{b^3 + 2b - 1 - (x^3 + 2x - 1)}{b-x}$
$= \lim_{b\to x} \frac{b^3 - x^3 + 2b - 2x}{b-x}$
$= \lim_{b\to x} \frac{(b-x)(b^2 + bx + x^2 + 2)}{b-x}$
$= \lim_{b\to x} b^2 + bx + x^2 + 2$
$= 3x^2 + 2$

12. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{(x+h)^4 - 2(x+h)^2 + 1 - f(x)}{h}$
$= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 - 4x - 2h$
$= 4x^3 - 4x.$

13. The function has negative slope for $x < 0$, positive slope for $x > 0$, and zero slope at $x = 0$. Its slope function (derivative) can only be (c).

14. (e). The graph (e) is zero in two places and negative in between. The graph of exercise 18 is flat in two places, and decreasing between.

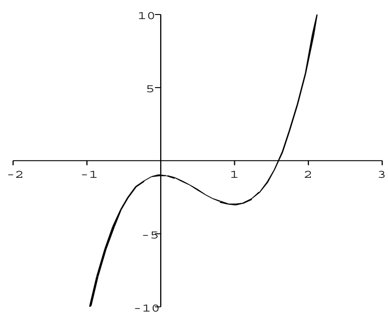
15. Here, moving from left to right, the slope goes from negative to positive to negative to positive. Its slope function (derivative) can only be (a) .

16. (d). Graph is decreasing everywhere so the derivative will be negative everywhere.

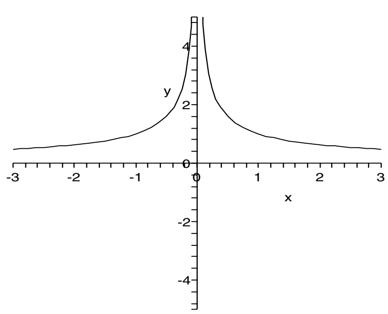
17. The graph is increasing to the left of the jump and decreasing to the right. The derivative of this function must be (b) which is postive to the left of the jump and negative to the right.

18. (f). The graph (f) is zero in two places and positive in between. The graph of exercise 22 is flat in two places, and increasing between.

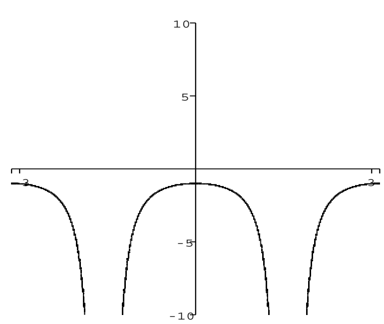
19. The derivative should look like:



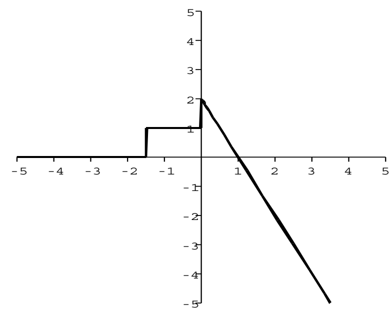
20. The derivative should look like:



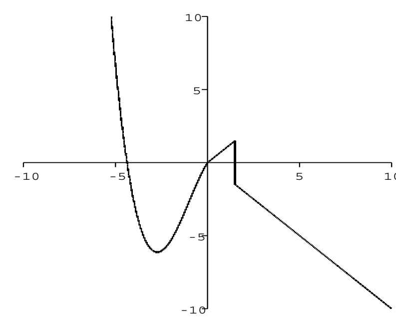
21. The derivative should look like:



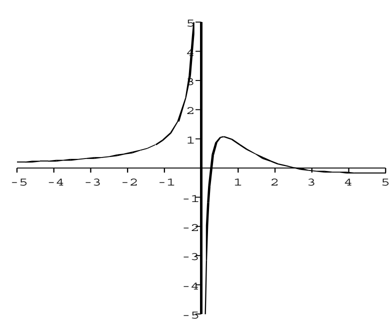
22. The derivative should look like:



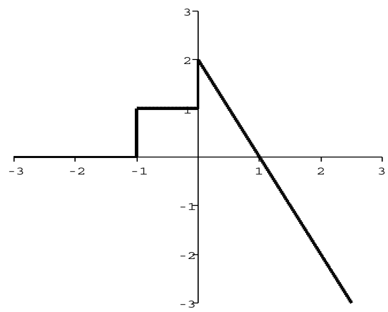
23. One possible graph of $f(x)$:



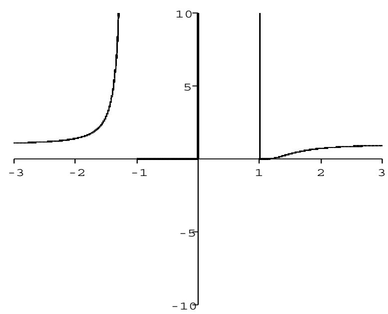
24. One possible graph of $f(x)$:

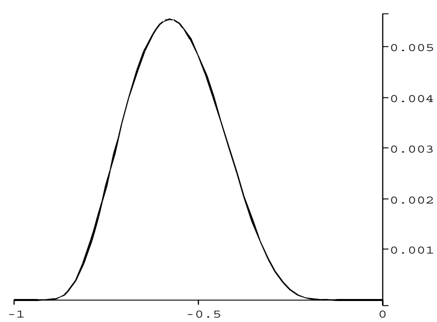


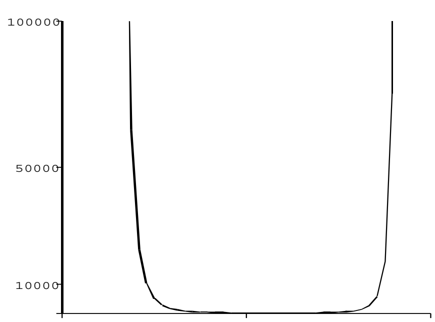
25. $f(x)$ is not differentiable at $x = 0$ or $x = 2$. The graph looks like:



26. $f(x)$ is not differentiable at $x = 0$ or $x = \pm 1$. We give three different graphs of different regions because of the differences in scale:







27. $f(x) = x^p \Longrightarrow f'(x) = px^{p-1}$.

If $p \geq 1$, then $p - 1 \geq 0$, so $f'(0) = 0$. Also, if $p = 0$, then $f(x) = 1$, so $f'(0) = 0$. However, if $p < 1$ but $p \neq 0$, then

$$f'(x) = \frac{p}{x^{1-p}}$$

where $1 - p \geq 0$, and so $f'(0)$ does not exist.

28. Let $u = ch$ so $h = \frac{u}{c}$. Then we have

$$\begin{aligned} &\lim_{h \to 0} \frac{f(a+ch) - f(a)}{h} \\ &= \lim_{\frac{u}{c} \to 0} \frac{f(a+u) - f(a)}{\frac{u}{c}} \\ &= \lim_{u \to 0} \frac{f(a+u) - f(a)}{\frac{u}{c}} \\ &= \lim_{u \to 0} c \left(\frac{f(a+u) - f(a)}{u} \right) \\ &= c \lim_{u \to 0} \frac{f(a+u) - f(a)}{u} = cf'(a) \end{aligned}$$

29. $$\begin{aligned} &\lim_{x \to a} \frac{[f(x)]^2 - [f(a)]^2}{x^2 - a^2} \\ &= \lim_{x \to a} \frac{[f(x) - f(a)][f(x) + f(a)]}{(x-a)(x+a)} \\ &= \left[\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)} \right] \left[\lim_{x \to a} \frac{f(x) + f(a)}{(x+a)} \right] \\ &= f'(a) \cdot \frac{2f(a)}{2a} \\ &= \frac{f(a)f'(a)}{a} \end{aligned}$$

30. We know that the limit $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$ exists. Since $f(x) < 0$ for all x we know that $\frac{f(x)}{x} > 0$ for all $x < 0$ and $\frac{f(x)}{x} < 0$ for all $x > 0$. The only way for this to be true and for $\lim_{x \to 0} \frac{f(x)}{x}$ to exist is if $f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0$.

31. We estimate the derivative at $x = 60$ as follows:

$$\frac{3.9 - 2.4}{80 - 40} = \frac{1.5}{40} = 0.0375$$

For every increase of 1 revolution per second of topspin, there is an increase of 0.0375° in margin of error.

32. We estimate the derivative at $x = 8.5$ as follows:

$$\frac{1.04 - .58}{9 - 8} = 0.46$$

For every increase of 1 foot in height of serving point, there is an increase of 0.46° in margin of error.

33. Compute average velocities:

Time Interval	Average Velocity
(1.7, 2.0)	9.0
(1.8, 2.0)	9.5
(1.9, 2.0)	10.0
(2.0, 2.1)	10.0
(2.0, 2.2)	9.5
(2.0, 2.3)	9.0

Our best estimate of the velocity at $t = 2$ is 10.

34. Compute average velocities:

Time Interval	Average Velocity
$(1.7, 2)$	$\frac{7.0-4.6}{2-1.7} = 8$
$(1.8, 2)$	8.5
$(1.9, 2)$	9
$(2, 2.1)$	8
$(2, 2.2)$	8
$(2, 2.3)$	7.67

A velocity of between 8 and 9 seems to be a good guess.

35. We compile the rate of change in Ton-MPG over each of the four two-year intervals for which data is given:

intervals	rate of change
(1992,1994)	$\frac{45.7-44.9}{2} = .4$
(1994,1996)	.4
(1996,1998)	.4
(1998,2000)	.2

These rates of change are measured in Ton-MPG per year. Either the first or second (they happen to agree) could be used as an estimate for the one-year interval "1994" while only the last is a promising estimate for the one-year interval "2000". The mere fact that all these numbers are positive suggests that efficiency is improving, but the last number being smaller seems to suggest that the rate of improvement is slipping.

36. The average rate of change from 1992 to 1994 is 0.05, and from 1994 to 1996 is 0.1, so a good estimate of the rate of change in 1994 is 0.75. The average rate of change from 1998 to 2000 is -0.2, and this is a good estimate for the rate of change in 2000. Comparing to exercise 35, we see that the MPG per ton increased, but the actual MPG for vehicles decreased. The weight of vehicles must have increased, and if the weight remained constant then the actual MPG would have increased.

37. We prepare a table of values for the function $f(x) = x^x$ (when x is near 1). Difference quotients based at $x = 1$ are then compiled in the last column.

x	$y = x^x$	$\frac{y-1}{x-1}$
1.1000000	1.1105342	1.1053424
1.0100000	1.0101005	1.0100503
1.0010000	1.0010010	1.0010005
1.0001000	1.0001000	1.0001000
1.0000100	1.0000100	1.0000100
1.0000010	1.0000010	1.0000010
1.0000001	1.0000001	1.0000001

The evidence of this table strongly suggests that the difference quotients (essentially indistinguishable from the values themselves) are heading toward 1. If true, this would mean that $f'(1) = 1$.

38. Numerically estimate

$$\lim_{x\to\pi} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x\to\pi} \frac{x^{\sin x} - 1}{x - \pi}.$$

Computing this expression for values of x close to π, we see the limit is approximately 1.

39. The left-hand derivative is

$$\begin{aligned} D_- f(0) &= \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{2h + 1 - 1}{h} = 2 \end{aligned}$$

The right-hand derivative is

$$\begin{aligned} D_+ f(0) &= \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{3h + 1 - 1}{h} = 3 \end{aligned}$$

40. The left-hand derivative is

$$\begin{aligned} D_- f(0) &= \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{h^2 - 0}{h} = 0 \end{aligned}$$

The right-hand derivative is

$$\begin{aligned} D_+ f(0) &= \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{h^3 - 0}{h} = 0 \end{aligned}$$

41.
$$\begin{aligned} D_+ f(0) &= \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{k(h) - k(0)}{h} = k'(0). \\ D_- f(0) &= \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) \end{aligned}$$

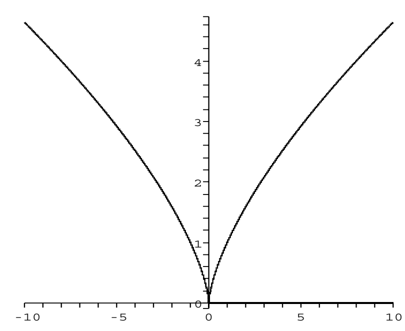
If $f(x)$ has a jump discontinuity at $x = 0$, it would be because its *left limit* at $x = 0$, namely $g(0)$, is not the same as the *value* which is $k(0)$. In that case there could be no left derivative (by Theorem 2.1) and one would have to reject the statement $D_- f(0) = g'(0)$.

42. The derivative $f'(0)$ exists if and only if the limit $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h}$ exists, and this limit exists if and only if the one-sided limits exist and are equal. But the one-sided limits are the left- and right-hand derivatives.

43. If $f'(x) > 0$ for all x, then the tangent lines all have positive slope, so the function is always sloping up.

44. If $f'(x) < 0$ for all x, then the tangent lines all have negative slope, so the function is always sloping down to the right.

45.



From the graph, we see that $f(x)$ appears continuous at $x = 0$, where it has both *limit* and *value* zero. However, when we try to compute its derivative at $x = 0$, we come to the difference quotient

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \frac{h^{2/3}}{h} = \frac{1}{h^{1/3}}$$

Clearly this expression has no finite limit as h approaches zero. The numbers get large without bound. We do sometimes say that the vertical line $x = 0$ is the tangent line, but as a line it has no *slope* (just as the function has no derivative).

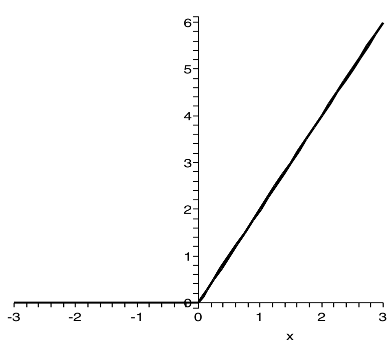
46. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$, and
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x = 0$,
so $\lim_{x\to 0} f(x) = 0$.

This equals $f(0)$, so the function is continuous.

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0,$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2.$$

These one sided derivatives are not equal, so the function is not differentiable at $x = 0$.

Graphically, we can see that the function is continuous, but has a sharp corner at $x = 0$ so is not differentiable there.

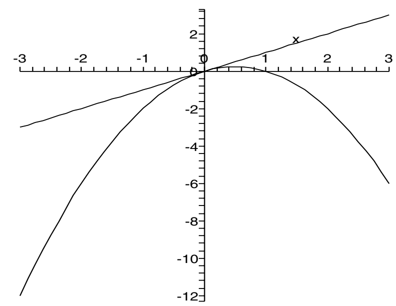


47. Let $f(x) = -1 - x^2$; then for all x, we have $f(x) \le x$. But at $x = -1$, we find $f(-1) = -2$ and

$$\begin{aligned} f'(-1) &= \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to 0} \frac{-1-(-1+h)^2-(-2)}{h} \\ &= \lim_{h\to 0} \frac{1-(1-2h+h^2)}{h} \\ &= \lim_{h\to 0} \frac{2h-h^2}{h} = \lim_{h\to 0} 2-h = 2 \end{aligned}$$

So, $f'(x)$ is not always less than 1.

48. This is not always true. For example, the function $f(x) = -x^2 + x$ satisfies the hypotheses, but $f'(x) > 1$ for all $x < 0$, as the following graph shows.



49. (a) meters per second

(b) items per dollar

50. (a) $c'(t)$ will represent the rate of change in amount of chemical, and will be measured in grams per minute.

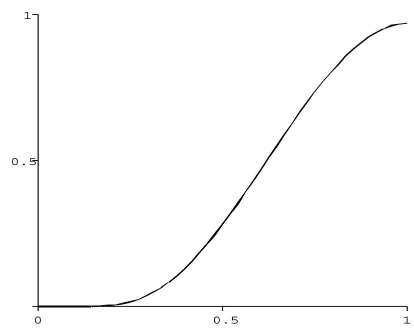
(b) $p'(x)$ will represent the rate of change of mass, and will be measured in kg per meter.

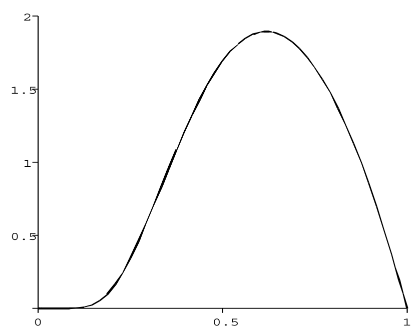
51. If $f'(t) < 0$, the function $f(t)$ is negatively sloped and decreasing, meaning the stock is losing value with the passing of time. This may be the basis for selling the stock if the current trend is expected to be a long term one.

52. You should buy the stock with value $g(t)$. It is cheaper because $f(t) > g(t)$, and growing faster because $f'(t) < g'(t)$ (or possibly declining more slowly).

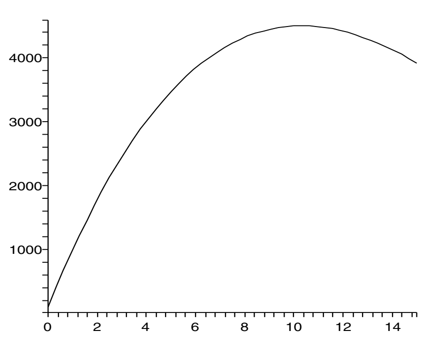
53. The following sketches are consistent with the hypotheses of infection rate rising, peaking, and returning to zero.

We started with the derivative $I'(t)$ (infection rate) and had to think backwards to construct the function $I(t)$. One can see in $I(t)$ the slope increasing up to the time of peak infection rate, thereafter the *slope* decreasing but not the *values.* They merely level off.

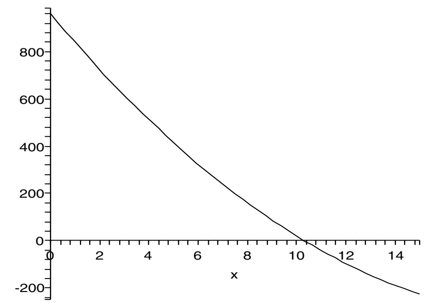




54. One possible graph of the population $P(t)$:



Graph of $P'(t)$:



55. Because the curve appears to be bending upward, the slopes of the secant lines (based at $x = 1$ and with upper endpoint beyond 1) will increase with the upper endpoint. This has also the effect that any one of these slopes is greater than the actual derivative. Therefore

$$f'(1) < \frac{f(1.5) - f(1)}{.5} < \frac{f(2) - f(1)}{1}$$

As to where $f(1)$ fits in this list, it seems necessary to read the graph and come up with estimates of $f(1)$ about 4, and $f(2)$ about 7. That would put the third number in the above list at about 3, comfortably less than $f(1)$.

56. Note that $f(0) - f(-1)$ is the slope of the secant line from $x = -1$ to $x = 0$ (about -1), and that $\frac{f(0)-f(-0.5)}{0.5}$ is the slope of the secant line from $x = -0.5$ to $x = 0$ (about -0.5). $f(0) = 3$ and $f'(0) = 0$.

In increasing order, we have $f(0) - f(-1)$, $\frac{f(0)-f(-0.5)}{0.5}$, $f'(0)$, and $f(0)$.

57. This is a tricky one. It happens that for the function $f(x) = x^2 - x$, the value at $x = 1$ is *zero* ($f(1) = 0$)! Because of this fact,

$$\frac{(1+h)^2 - (1+h)}{h} = \frac{f(1+h) - f(1)}{h}$$

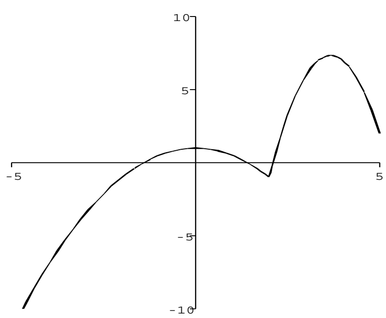
and the answer should be:
$f(x) = x^2 - x$ and $a = 1$.

58. $\lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$ is the derivative of the function
$f(x) = \sqrt{x}$ at $x = 4$.

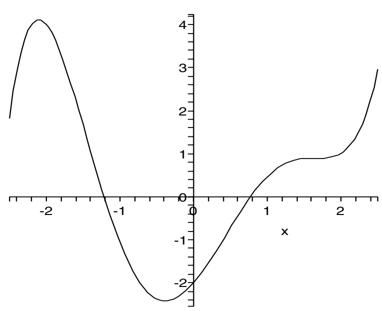
59. $\lim\limits_{h\to 0} \dfrac{\left(\frac{1}{2+h}\right) - \left(\frac{1}{2}\right)}{h}$ would be $f'(a)$ for
$f(x) = \dfrac{1}{x}$ and $a = 2$.

60. $\lim\limits_{h\to 0} \dfrac{(h-1)^2 - 1}{h}$ is the derivative of the function
$f(x) = x^2$ at $x = -1$.

61. One possible such graph:



62. One possible such graph:



63. We have:

$$f(t) = \begin{cases} 100 & 0 < t \le 20 \\ 100 + 10(t-20) & 20 < t \le 80 \\ 700 + 8(t-80) & 80 < t < \infty \end{cases}$$

This is another example of a *piecewise linear* function (this one is continuous), and although not differentiable at the transition times $t = 20$ or $t = 80$, elsewhere we have

$$f'(t) = \begin{cases} 0 & 0 < t < 20 \\ 10 & 20 < t < 80 \\ 8 & 80 < t < \infty \end{cases}$$

64. We estimate $f'(1)$ as follows:

$$f'(1) \approx \frac{9-13}{2-0} = -2$$

For every increase of one month (which corresponds to being one month younger than your comrades), the number of players in the English Premier League decreases by 2. This suggests that it if being an English Premier League soccer player is your goal, that you have a better chance at it if you are older.

2.3 Computation of Derivatives: Power Rule

1.
$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3) - \frac{d}{dx}(2x) + \frac{d}{dx}(1) \\ &= 3x^2 - 2\frac{d}{dx}(x) + 0 \\ &= 3x^2 - 2(1) \\ &= 3x^2 - 2 \end{aligned}$$

2. $f'(x) = 9x^8 - 15x^4 + 8x - 4$

3.
$$\begin{aligned} f'(t) &= \frac{d}{dt}(3t^3) - \frac{d}{dt}\left(2\sqrt{t}\right) \\ &= 3\frac{d}{dt}(t^3) - 2\frac{d}{dt}\left(t^{1/2}\right) \\ &= 3(3t^2) - 2\left(\frac{1}{2}t^{-1/2}\right) \\ &= 9t^2 - \frac{1}{\sqrt{t}} \end{aligned}$$

4. $f(s) = 5s^{1/2} - 4s^2 + 3$, so
$$f'(s) = \frac{5}{2}s^{-1/2} - 8s = \frac{5}{2\sqrt{s}} - 8s$$

5. $$f'(x) = \frac{d}{dx}\left(\frac{3}{x}\right) - \frac{d}{dx}(8x) + \frac{d}{dx}(1) = 3\frac{d}{dx}(x^{-1}) - 8\frac{d}{dx}(x) + 0 = 3(-x^{-2}) - 8(1) = -\frac{3}{x^2} - 8$$

6. $f(x) = 2x^{-4} - x^3 + 2$, so
$$f'(x) = -8x^{-5} - 3x^2 = -\frac{8}{x^5} - 3x^2$$

7. $$h'(x) = \frac{d}{dx}\left(\frac{10}{\sqrt{x}}\right) - \frac{d}{dx}(2x) = 10\frac{d}{dx}\left(x^{-1/2}\right) - 2\frac{d}{dx}(x) = 10\left(-\frac{1}{2}x^{-3/2}\right) - 2(1) = -5x^{-3/2} - 2 = \frac{-5}{x\sqrt{x}} - 2$$

8. $h(x) = 12x - x^2 - 3x^{-1/2}$, so
$$h'(x) = 12 - 2x + \frac{3}{2}x^{-3/2} = 12 - 2x + \frac{3}{2\sqrt{x^3}}$$

9. $$f'(s) = \frac{d}{ds}\left(2s^{3/2}\right) - \frac{d}{ds}\left(3s^{-1/3}\right) = 2\frac{d}{ds}\left(s^{3/2}\right) - 3\frac{d}{ds}\left(s^{-1/3}\right) = 2\left(\frac{3}{2}s^{1/2}\right) - 3\left(-\frac{1}{3}s^{-4/3}\right) = 3s^{1/2} + s^{-4/3} = 3\sqrt{s} + \frac{1}{\sqrt[3]{s^4}}$$

10. $f'(t) = 3\pi t^{\pi-1} - 2.6t^{0.3}$

11. $$f'(x) = \frac{d}{dx}\left(2\sqrt[3]{x}\right) + \frac{d}{dx}(3) = 2\frac{d}{dx}\left(x^{1/3}\right) + 0 = 2\left(\frac{1}{3}x^{-2/3}\right) = \frac{2}{3}x^{-2/3} = \frac{2}{3\sqrt[3]{x^2}}$$

12. $f(x) = 4x - 3x^{2/3}$, so
$$f'(x) = 4 - 2x^{-1/3} = 4 - \frac{2}{\sqrt[3]{x}}$$

13. $f(x) = x(3x^2 - \sqrt{x}) = 3x^3 - x^{3/2}$ so
$$f'(x) = 3\frac{d}{dx}(x^3) - \frac{d}{dx}\left(x^{3/2}\right) = 3(3x^2) - \left(\frac{3}{2}x^{1/2}\right) = 9x^2 - \frac{3}{2}\sqrt{x}$$

14. $f(x) = 3x^3 + 3x^2 - 4x - 4$, so
$$f'(x) = 9x^2 + 6x - 4$$

15. $$f(x) = \frac{3x^2 - 3x + 1}{2x} = \frac{3x^2}{2x} - \frac{3x}{2x} + \frac{1}{2x} = \frac{3}{2}x - \frac{3}{2} + \frac{1}{2}x^{-1}$$ so
$$f'(x) = \frac{d}{dx}\left(\frac{3}{2}x\right) - \frac{d}{dx}\left(\frac{3}{2}\right) + \frac{d}{dx}\left(\frac{1}{2}x^{-1}\right) = \frac{3}{2}\frac{d}{dx}(x) - 0 + \frac{1}{2}\frac{d}{dx}(x^{-1}) = \frac{3}{2}(1) + \frac{1}{2}(-1x^{-2}) = \frac{3}{2} - \frac{1}{2x^2}$$

16. $f(x) = 4x^{3/2} - x^{1/2} + 3x^{-1/2}$, so
$$f'(x) = 6x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$$

17. $$f'(x) = \frac{d}{dx}(x^4 + 3x^2 - 2) = 4x^3 + 6x$$
$$f''(x) = \frac{d}{dx}(4x^3 + 6x) = 12x^2 + 6$$

18. $f(x) = x^6 - \sqrt{x} = x^6 - x^{1/2}$ so
$$\frac{df}{dx} = \frac{d}{dx}\left(x^6 - x^{1/2}\right) = 6x^5 - \frac{1}{2}x^{-1/2}$$

$$\begin{aligned}\frac{d^2f}{dx^2} &= \frac{d}{dx}\left(6x^5 - \frac{1}{2}x^{-1/2}\right)\\ &= 30x^4 - \frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\right)\\ &= 30x^4 + \frac{1}{4}x^{-3/2}\\ &= 30x^4 + \frac{1}{4x\sqrt{x}}\end{aligned}$$

19. $f(x) = 2x^4 - 3x^{-1/2}$ so
$\frac{df}{dx} = 8x^3 + \frac{3}{2}x^{-3/2}$
$\frac{d^2f}{dx^2} = 24x^2 - \frac{9}{4}x^{-5/2}$

20. $f(t) = 4t^2 - 12 + \frac{4}{t^2} = 4t^2 - 12 + 4t^{-2}$

so $f'(t) = \frac{d}{dt}(4t^2 - 12 + 4t^{-2})$
$= 8t^2 - 0 + 4(-2t^{-3})$
$= 8t^2 - 8t^{-3}$
$f''(t) = \frac{d}{dt}(8t - 8t^{-3})$
$= 8 - 8(-3t^{-4})$
$= 8 + 24t^{-4}$
$f'''(t) = \frac{d}{dt}(8+24t^{-4}) = 0+24(-4t^{-5})$
$= -96t^{-5} = \frac{-96}{t^5}$

21. $f'(x) = 4x^3 + 6x$
$f''(x) = 12x^2 + 6$
$f'''(x) = 24x$
$f^{(4)}(x) = 24$

22. $f'(x) = 10x^9 - 12x^3 + 2$
$f''(x) = 90x^8 - 36x^2$
$f'''(x) = 720x^7 - 72x$
$f^{(4)}(x) = 5040x^6 - 72$
$f^{(5)}(x) = 30240x^5$

23. $f(x) = \frac{x^2 - x + 1}{\sqrt{x}}$
$= x^{3/2} - x^{1/2} + x^{-1/2}$ so
$f'(x) = \frac{d}{dx}\left(x^{3/2} - x^{1/2} + x^{-1/2}\right)$
$= \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$
$f''(x) =$
$\frac{d}{dx}\left(\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right)$
$= \frac{3}{4}x^{-1/2} + \frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$
$f'''(x) =$
$\frac{d}{dx}\left(\frac{3}{4}x^{-1/2} + \frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}\right)$
$= -\frac{3}{8}x^{-3/2} - \frac{3}{8}x^{-5/2} - \frac{15}{8}x^{-7/2}$
$= -\frac{3(x^2 + x + 5)}{8x^3\sqrt{x}}$

24. $f(t) = t^3 + t^{5/2} - t - t^{1/2}$
$f'(t) = 3t^2 + \frac{5}{2}t^{3/2} - 1 - \frac{1}{2}t^{-1/2}$
$f''(t) = 6t + \frac{15}{4}t^{1/2} + \frac{1}{4}t^{-3/2}$
$f'''(t) = 6 + \frac{15}{8}t^{-1/2} - \frac{3}{8}t^{-5/2}$
$f^{(4)}(t) = -\frac{15}{16}t^{-3/2} + \frac{15}{16}t^{-7/2}$

25. $s(t) = -16t^2 + 40t + 10$
$v(t) = s'(t) = -32t + 40$
$a(t) = v'(t) = s''(t) = -32$

26. $s(t) = 12t^3 - 6t - 1$
$v(t) = s'(t) = 36t^2 - 6$
$a(t) = s''(t) = 72t$

27. $s(t) = \sqrt{t} + 2t^2 = t^{1/2} + 2t^2$
$v(t) = s'(t) = \frac{1}{2}t^{-1/2} + 4t$
$a(t) = v'(t) = s''(t) = -\frac{1}{4}t^{-3/2} + 4$

28. $s(t) = 10 - 10t^{-1}$
$v(t) = s'(t) = 10t^{-2}$
$a(t) = s''(t) = -20t^{-3}$

29. $v(t) = -32t + 40$, $v(1) = 8$, going up. $a(t) = -32$, $a(1) = -32$, speed decreasing.

30. $v(t) = -32t + 40$, $v(2) = -24$, going down.
$a(t) = -32$, $a(2) = -32$, speed increasing.

31. $v(t) = 20t - 24$, $v(2) = 16$, going up.
$a(t) = 20$, $a(1) = 20$, speed increasing.

32. $v(t) = 20t - 24$, $v(1) = -4$, going down.
$a(t) = 20$, $a(1) = 20$, speed decreasing.

33. $f(x) = 4\sqrt{x} - 2x$, $a = 4$
$f(4) = 4\sqrt{4} - 2(4) = 0$

$$f'(x) = \frac{d}{dx}\left(4x^{1/2} - 2x\right)$$
$$= 2x^{-1/2} - 2 = \frac{2}{\sqrt{x}} - 2$$

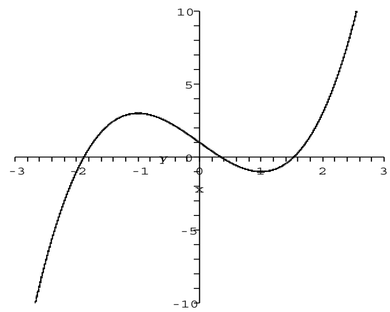
$f'(4) = 1 - 2 = -1$
The equation of the tangent line is
$y = -1(x - 4) + 0$ or $y = -x + 4$.

34. $f(2) = 1$.
$f'(x) = 2x - 2$,
$f'(2) = 2$.
Line through $(2, 1)$ with slope 2 is
$y = 2(x - 2) + 1$.

35. $f(x) = x^2 - 2$, $a = 2$, $f(2) = 2$
$f'(x) = 2x$
$f'(2) = 4$
The equation of the tangent line is
$y = 4(x - 2) + 2$ or $y = 4x - 6$.

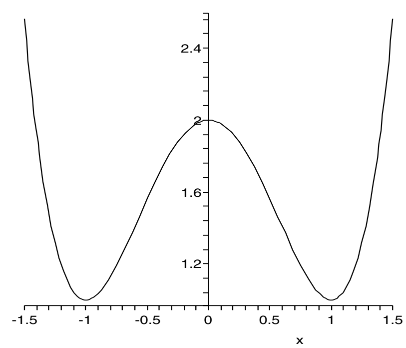
36. Tangent line to a line is always the same line, $y = 3x + 4$.

37. $f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3$
The tangent line to $y = f(x)$ is horizontal when $f'(x) = 0$:
$3x^2 - 3 = 0$
$\iff 3(x^2 - 1) = 0$
$\iff 3(x + 1)(x - 1) = 0$
$\iff x = -1$ or $x = 1$.



The graph shows that the first is a relative maximum, the second is a relative minimum.

38. Tangent line is horizontal where $f'(x) = 0$.
$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1) = 0$ when $x = \pm 1$ or 0.

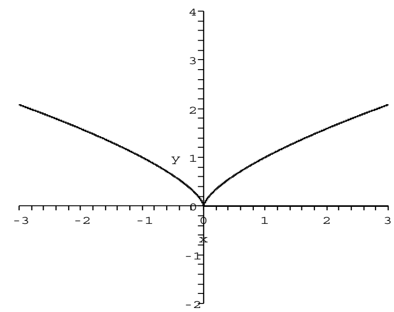


The graph shows that the first and last are relative minimums, while the middle ($x = 0$) is a relative maximum.

39. $f(x) = x^{2/3}$

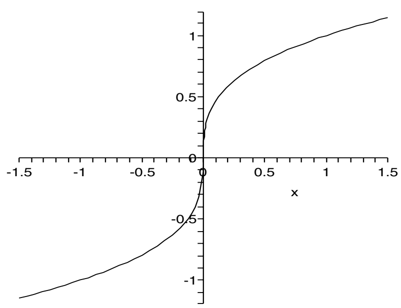
$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

The slope of the tangent line to $y = f(x)$ does not exist where the derivative is undefined, which is only when $x = 0$.



In this case, because the function is continuous, we might say that the tangent line is the vertical line $x = 0$. The feature at $x = 0$ is sometimes known as a *cusp*.

40. $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$ is undefined at $x = 0$.



The graphical significance of this point is that there is a vertical tangent here.

41. As regards the (a) function, its derivative would be negative for all negative x and positive for all positive x. Since no such function appears among the pictures, this (a) function has to be the one whose derivative is absent from the list. There being no f''' in the list, (a) has to be f''.

This same (a) function is negative for a certain interval of the form $(-a, a)$, and the (c) function is decreasing on a similar type of interval. Thus the (a) function (f'') is apparently the derivative of the (c) function. It follows that (c) must be f'.

This leaves (b) for f itself, and our identifications are consistent in every respect.

42. Curve (b) is the function $f(x)$, curve (a) is the derivative $f'(x)$, and curve (c) is the second derivative $f''(x)$.

43. $f(x) = \sqrt{x} = x^{1/2}$
$f'(x) = \dfrac{1}{2}x^{-1/2}$
$f''(x) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x^{-3/2}$
$f'''(x) = \left(\dfrac{1}{2}\right)\left(\dfrac{-1}{2}\right)\left(\dfrac{-3}{2}\right)x^{-5/2}$
$f^{(n)}(x) = (-1)^{n-1}\dfrac{\Pi_n}{2^n}x^{-(2n-1)/2}$
in which Π_n is the product of the first $n-1$ *odd* integers (starting from 1 and ending at $2n-3$). Recall that the product of *all* the whole numbers from 1 to n is denoted by $n!$. If one were to multiply Π_n by product of the $n-1$ *even* numbers (from 2 to $2n-2$), one would get $(2n-2)!$ (in the numerator). Of course, one would have to do the same to the denominator, but this product of the new numbers could be written in the form $2^{n-1}(n-1)!$ A final form for an answer could be
$f^{(n)}(x) = (-1)^{n-1}\frac{(2n-2)!}{2^{2n-1}(n-1)!}x^{(2n-1)/2}$.

44. $f'(x) = -2x^{-3}$
$f''(x) = 6x^{-4}$
$f'''(x) = -24x^{-5}$. The pattern is
$f^{(n)}(x) = (-1)^n(n+1)!x^{-n-2}$.

45. $f(x) = ax^2 + bx + c \Rightarrow f(0) = c$
$f'(x) = 2ax + b \Rightarrow f'(0) = b$
$f''(x) = 2a \Rightarrow f''(0) = 2a$
Given $f''(0) = 3$, we learn $2a = 3$, or $a = 3/2$. Given $f'(0) = 2$ we learn

$2 = b$, and given $f(0) = -2$, we learn $c = -2$. In the end
$f(x) = ax^2 + bx + c = \frac{3}{2}x^2 + 2x - 2$.

46. $f(x) = ax^2 + bx + c$. $f(0) = 0 \Rightarrow c = 0$.
$f'(x) = 2ax + b$. $f'(0) = 5 \Rightarrow b = 5$.
$f''(x) = 2a$. $f''(0) = 1 \Rightarrow a = \frac{1}{2}$.
So $f(x) = \frac{1}{2}x^2 + 5x$.

47. For $y = \frac{1}{x}$, we have $\frac{d}{dx} = -\frac{1}{x^2}$. Thus, the slope of the tangent line at $x = a$ is $-\frac{1}{a^2}$.

When $a = 1$, the slope of the tangent line at $(1, 1)$ is -1, and the equation of the tangent line is $y = -x + 2$. The tangent line intersects the axes at $(0, 2)$ and $(2, 0)$. Thus, the area of the triangle is $\frac{1}{2}(2)(2) = 2$.

When $a = 2$, the slope of the tangent line at $(2, \frac{1}{2})$ is $-\frac{1}{4}$, and the equation of the tangent line is $y = -\frac{1}{4}x + 1$. The tangent line intersects the axes at $(0, 1)$ and $(4, 0)$. Thus, the area of the triangle is $\frac{1}{2}(4)(1) = 2$.

In general, the equation of the tangent line is $y = -\left(\frac{1}{a^2}\right)x + \frac{2}{a}$. The tangent line intersects the axes at $(0, \frac{2}{a})$ and $(2a, 0)$. Thus, the area of the triangle is

$$\frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$$

48. For $y = \frac{1}{x^2} = x^{-2}$, we have
$f'(x) = -2x^{-3} = -2/x^3$.
Thus, the slope of the tangent line at $x = a$ is $-2/x^3$.

When $a = 1$, the slope of the tangent line at $(1, 1)$ is -2, and the equation of the tangent line is $y = -2x + 3$. The tangent line intersects the axes at $(0, 3)$ and $(\frac{3}{2}, 0)$. Thus the area of the triangle is $\frac{1}{2}(3)(\frac{3}{2}) = \frac{9}{4}$.

When $a = 2$, the slope of the tangent line at $(2, \frac{1}{4})$ is $-\frac{1}{4}$, and the equation of the tangent line is $y = -\frac{1}{4}x + \frac{3}{4}$. The tangent line intersects the axes at $(0, \frac{3}{4})$ and $(3, 0)$. Thus the area of the triangle is $\frac{1}{2}(\frac{3}{4})(3) = \frac{9}{8}$.

Since $\frac{9}{4} \neq \frac{9}{8}$, we see that the result for exercise 47 does not hold here.

49. (a)
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\max_{a\le t\le x+h} f(t) - \max_{a\le t\le x} f(t)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[f(x+h) - f(x)\right] \\ &= f'(x) \end{aligned}$$

(b)
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\max_{a\le t\le x+h} f(t) - \max_{a\le t\le x} f(t)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[f(a) - f(a)\right] \\ &= 0 \end{aligned}$$

50. (a)
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\min_{a\le t\le x+h} f(t) - \min_{a\le t\le x} f(t)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[f(a) - f(a)\right] \\ &= 0 \end{aligned}$$

(b)
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\min_{a\le t\le x+h} f(t) - \min_{a\le t\le x} f(t)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[f(x+h) - f(x)\right] \\ &= f'(x) \end{aligned}$$

51. If $d(t)$ represents the national debt, then $d'(t)$ represents the rate of change of the national debt. The debt itself, by implication, is increasing and therefore $d'(t) > 0$.

Since the rate of increase has been reduced, this implies $d''(t)$ is being reduced. We cannot conclude anything about the size of $d(t)$.

52. $m'(t) = 6t$ kg per meter. $m'(t)$ represents the rate the mass is increasing as t increases. This is the linear density of the rod.

53. $w(b) = cb^{3/2}$
$w'(b) = \frac{3c}{2}b^{1/2} = \frac{3c\sqrt{b}}{2}$
$w'(b) > 1$ when
$\frac{3c\sqrt{b}}{2} > 1, \ \sqrt{b} > \frac{2}{3c}$
$b > \frac{4}{9c^2}$.

Since c is constant, when b is large enough, b will be greater than $\frac{4}{9c^2}$. After this point, when b increases by 1 unit, the leg width w is increasing by more than 1 unit, so that leg width is increasing faster than body length.

This puts a limitation on the size of land animals since, eventually, the body will not be long enough to accomodate the width of the legs.

54. World Record Times – Men's Track

Dist.	Time	Ave	$f(d)$
400	43.18	9.26	9.25
800	101.11	7.91	8.17
1000	131.96	7.58	7.86
1500	206.00	7.28	7.32
2000	284.79	7.02	6.95

Here, distance is in meters, time is in seconds and hence average in meters per second.

The function $f(d)$ is quite close to predicting the average speed of world record pace.

$v'(d)$ represents the rate of change in average speed over d meters per meter. $v'(d)$ tells us how much $v(d)$ would change if d changed to $d + 1$.

55. We can approximate $f'(2000) \approx \frac{9039.5-8690.7}{2001-1999} = 174.4$. This is the rate of change of the GDP in billions of dollars per year.

To approximate $f''(2000)$, we first estimate $f'(1999) \approx \frac{9016.8-8347.3}{2000-1998} = 334.75$ and $f'(1998) \approx \frac{8690.7-8004.5}{1999-1997} = 343.1$.

Since these values are decreasing, $f''(2000)$ is negative. We estimate $f''(2000) \approx \frac{174.4-334.75}{2000-1999} = -160.35$. This represents the rate of change of the rate of change of the GDP over time. In 2000, the GDP is increasing by a rate of 343.1 billion dollars per year, but this increase is decreasing by a rate of 160.35 billion dollars-per-year per year.

56. $f'(2000)$ can be approximated by the average rate of change from 1995 to 2000. $f'(2000) \approx \frac{4619-4353}{2000-1995} = 53.2$. This is the rate of change of weight of SUVs over time. In 2000 the weight of SUVs is increasing by 53.2 pounds per year.

Similarly approximate $f'(1995) \approx 32.8$ and $f'(1990) \approx 26.8$.

The second derivative is definitely positive. We can approximate $f''(2000) \approx \frac{53.2-32.8}{2000-1995} = 4.08$. This is the rate of change in the rate of change of the weight of SUVs. Not only are SUVs getting heavier at a rate of 53.2 pounds per year, this rate is itself increasing at a rate of about 4 pounds-per-year per year.

57. Newton's Law states that force equals mass times acceleration. That is, if $F(t)$ is the driving force at time t, then $m \cdot f''(t) = m \cdot a(t) = F(t)$ in which m is the mass, appropriately unitized. The third derivative of the distance function is then $f'''(t) = a'(t) = \frac{1}{m}F'(t)$. It is both the derivative of the acceleration and directly proportional to the rate of change in force. Thus an abrupt change in acceleration or "jerk" is the direct consequence of an abrupt change in force.

58. $Q'(x) = 500L^{1/3}x^{-1/2}$, and $Q'(40) = \frac{500L^{1/3}}{\sqrt{40}}$. This is the rate of change in the daily output as capital investment changes. As capital investment increases, the daily output increases, and $Q'(40)$ tells us how fast the daily output is increasing when the capital investment is \$40,000.

59–62 Commentary: At this stage, finding a function whose derivative is given, is a matter of thinking backward, or of anticipation. When the derivative is a power, one anticipates that it could have arisen from differentiating a function which was also a power, but whose exponent was one higher. That is, to get to x^p, try cx^{p+1} where c is some constant. After that, it is a matter of testing and adjusting the constant c. The answer is never unique (why?), but anything offered can always be checked by differentiation.

59. Try $f(x) = cx^4$ for some constant c. Then $f'(x) = 4cx^3$ so c must be 1. One possible answer is x^4.

60. Try $f(x) = cx^5$ for some constant c. Then $f'(x) = 5cx^4$ so c must be 1. One possible answer is x^5.

61. $f'(x) = \sqrt{x} = x^{1/2}$
$f(x) = \dfrac{2}{3}x^{3/2}$ is one possible function.

62. If $f'(x) = x^{-2}$, then $f(x) = -x^{-1}$ is one possible function.

63.
$$\begin{aligned}
&\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}\\
&= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h^2} - \frac{[f(a)-f(a-h)]}{h^2}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(a+h)-f(a)}{h} - \frac{f(a)-f(a-h)}{h}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - \lim_{h\to 0}\frac{f(a)-f(a-h)}{h}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[f'(a)-f'(a-h)\right]
\end{aligned}$$

Now let $k=-h$ in the previous equation, to get

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}\\
&= \lim_{k\to 0}\frac{1}{-k}\left[f'(a)-f'(a+k)\right]\\
&= \lim_{k\to 0}\frac{1}{k}\left[f'(a+k)-f'(a)\right]\\
&= f''(a)
\end{aligned}$$

64. We have that

$$f(x) = \begin{cases} -x^2 & x<0 \\ x^2 & x\ge 0 \end{cases}.$$

Thus

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(h)-2f(0)+f(-h)}{h^2}\\
&= \lim_{h\to 0}\frac{f(h)+f(-h)}{h^2}\\
&= \lim_{h\to 0}\frac{h^2+(-h^2)}{h^2} = 0
\end{aligned}$$

and therefore exists.

On the other hand, we have

$$f'(x) = \begin{cases} -2x & x<0 \\ 2x & x\ge 0 \end{cases}$$

and

$$f''(x) = \begin{cases} -2 & x < 0 \\ 2 & x > 0 \end{cases}$$

but $f''(0)$ does not exist, since the limit from the left is -2 but the limit from the right is 2.

2.4 The Product and Quotient Rules

1. $f(x) = (x^2+3)(x^3-3x+1)$
$f'(x) = \frac{d}{dx}(x^2+3)\cdot(x^3-3x+1) + (x^2+3)\cdot\frac{d}{dx}(x^3-3x+1)$
$= (2x)(x^3-3x+1) + (x^2+3)(3x^2-3)$

2. $f(x) = (x^3-2x^2+5)(x^4-3x^2+2)$
$f'(x) = \frac{d}{dx}(x^3-2x^2+5)(x^4-3x^2+2) + (x^3-2x^2+5)\frac{d}{dx}(x^4-3x^2+2)$
$= (3x^2-4x)(x^4-3x^2+2) + (x^3-2x^2+5)(4x^3-6x)$

3. $f(x) = (\sqrt{x}+3x)\left(5x^2-\frac{3}{x}\right)$
$= (x^{1/2}+3x)(5x^2-3x^{-1})$
$f'(x) = \left(\frac{1}{2}x^{-1/2}+3\right)(5x^2-3x^{-1}) + (x^{1/2}+3x)(10x+3x^{-2})$

4. $f(x) = (x^{3/2}-4x)(x^4-3x^{-2}+2)$
$f'(x) = \frac{d}{dx}(x^{3/2}-4x)(x^4-3x^{-2}+2) + (x^{3/2}-4x)\frac{d}{dx}(x^4-3x^{-2}+2)$
$= (\frac{3}{2}x^{1/2}-4)(x^4-3x^{-2}+2) + (x^{3/2}-4x)(4x^3+6x^{-3})$

5. $f(x) = \frac{3x-2}{5x+1}$
$f'(x) = \frac{\left((5x+1)\frac{d}{dx}(3x-2)-(3x-2)\frac{d}{dx}(5x+1)\right)}{(5x+1)^2}$
$= \frac{3(5x+1)-(3x-2)5}{(5x+1)^2}$
$= \frac{15x+3-15x+10}{(5x+1)^2} = \frac{13}{(5x+1)^2}$

6. $f'(x) =$
$\frac{(x^2-5x+1)\frac{d}{dx}(x^2+2x+5)-(x^2+2x+5)\frac{d}{dx}(x^2-5x+1)}{(x^2-5x+1)^2}$
$= \frac{(x^2-5x+1)(2x+2)-(x^2+2x+5)(2x-5)}{(x^2-5x+1)^2}$

7. $f(x) = \frac{3x-6\sqrt{x}}{5x^2-2} = \frac{3(x-2x^{1/2})}{5x^2-2}$
$f'(x) =$
$3\frac{\left((5x^2-2)\frac{d}{dx}(x-2x^{1/2})-(x-2x^{1/2})\frac{d}{dx}(5x^2-2)\right)}{(5x^2-2)^2}$
$= 3\frac{\left((5x^2-2)(1-x^{-1/2})-(x-2x^{1/2})(10x)\right)}{(5x^2-2)^2}$
$= 3\frac{\left(\left(5x^2-2-5x^{3/2}+2x^{-1/2}\right)-(10x^2-20x^{3/2})\right)}{(5x^2-2)^2}$
$= \frac{3(-5x^2+15x^{3/2}+2x^{-1/2}-2)}{(5x^2-2)^2}$

8. $f(x) = \frac{6x-2x^{-1}}{x^2+x^{1/2}}$
$f'(x) =$
$\frac{(x^2+x^{1/2})\frac{d}{dx}(6x-2x^{-1})-(6x-2x^{-1})\frac{d}{dx}(x^2+x^{1/2})}{(x^2+x^{1/2})^2}$
$= \frac{(x^2+x^{1/2})(6+2x^{-2})-(6x-2x^{-1})(2x+\frac{1}{2}x^{-1/2})}{(x^2+x^{1/2})^2}$

9. $f(x) = \frac{(x+1)(x-2)}{x^2-5x+1} = \frac{x^2-x-2}{x^2-5x+1}$
$f'(x) =$
$\frac{\left((x^2-5x+1)\frac{d}{dx}(x^2-x-2)-(x^2-x-2)\frac{d}{dx}(x^2-5x+1)\right)}{(x^2-5x+1)^2}$
$= \frac{\left((x^2-5x+1)(2x-1)-(x^2-x-2)(2x-5)\right)}{(x^2-5x+1)^2}$
$= \frac{-4x^2+6x-11}{(x^2-5x+1)^2}$

10. $f(x) = \frac{x^2-2x}{x^2+5x}$
$f'(x) = \frac{(x^2+5x)\frac{d}{dx}(x^2-2x)-(x^2-2x)\frac{d}{dx}(x^2+5x)}{(x^2+5x)^2}$
$= \frac{(x^2+5x)(2x-2)-(x^2-2x)(2x+5)}{(x^2+5x)^2}$

11. We do not recommend treating this one as a quotient, but advise preliminary simplification.
$f(x) = \frac{x^2+3x-2}{\sqrt{x}}$
$= \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}}$
$= x^{3/2} + 3x^{1/2} - 2x^{-1/2}$
$f'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}$

12. $f(x) = \frac{2x}{x^2+1}$
$f'(x) = \frac{(x^2+1)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$
$= \frac{(x^2+1)(2)-(2x)(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$

13. We simplify instead of using the product rule.
$f(x) = x\left(\sqrt[3]{x}+3\right) = x^{4/3}+3x$
$f'(x) = \frac{4}{3}x^{1/3}+3$

14. We simplify instead of using the product rule.
$f(x) = \frac{1}{3}x^2 + 5x^{-2}$
$f'(x) = \frac{2}{3}x - 10x^{-3}$

15. $f(x) = (x^2-1)\frac{x^3+3x^2}{x^2+2}$
$f'(x) =$
$\frac{d}{dx}(x^2-1)\cdot(\frac{x^3+3x^2}{x^2+2})$
$+ (x^2-1)\cdot\frac{d}{dx}(\frac{x^3+3x^2}{x^2+2})$
We have
$\frac{d}{dx}(\frac{x^3+3x^2}{x^2+2}) =$
$\frac{(x^2+2)\frac{d}{dx}(x^3+3x^2)-(x^3+3x^2)\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$
$= \frac{(x^2+2)\cdot(3x^2+6x)-(x^3+3x^2)\cdot(2x)}{(x^2+2)^2}$
$= \frac{3x^4+6x^2+6x^3+12x-(2x^4+6x^3)}{(x^2+2)^2}$
$= \frac{x^4+6x^2+12x}{(x^2+2)^2}$
so $f'(x) =$
$(2x)\cdot(\frac{x^3+3x^2}{x^2+2}) + (x^2-1)\cdot\frac{x^4+6x^2+12x}{(x^2+2)^2}$

16. $f(x) = \frac{(x+2)(x-1)(x+1)}{x(x+1)} = \frac{x^2+x-2}{x}$
$= x+1-2x^{-1}$
So $f'(x) = 1+2x^{-2}$.

17. $\frac{d}{dx}[f(x)g(x)h(x)]$
$= \frac{d}{dx}[(f(x)g(x))\,h(x)]$
$= (f(x)g(x))\,h'(x)+h(x)\frac{d}{dx}(f(x)g(x))$
$= (f(x)g(x))\,h'(x)$
$+ h(x)\,(f(x)g'(x)+g(x)f'(x))$
$= f'(x)g(x)h(x)$
$+ f(x)g'(x)h(x)+f(x)g(x)h'(x)$
In the general case of a product of n functions, the derivative will have n terms to be added, each term a product of all but one of the functions multiplied by the derivative of the missing function.

18. The derivative of $g(x)^{-1} = \frac{1}{g(x)}$ is
$\frac{g(x)\frac{d}{dx}(1)-(1)\frac{d}{dx}g(x)}{g(x)^2} = -\frac{g'(x)}{g(x)^2}$
$= -g'(x)(g(x))^{-2}$
as claimed.

The derivative of $f(x)(g(x))^{-1}$ is then $f'(x)(g(x))^{-1}+f(x)(-g'(x)(g(x))^{-2})$.

19. $f'(x) = \left[\frac{d}{dx}(x^{2/3})\right](x^2-2)(x^3-x+1)$
$+x^{2/3}\left[\frac{d}{dx}(x^2-2)\right](x^3-x+1)$
$+ x^{2/3}(x^2-2)\frac{d}{dx}(x^3-x+1)$
$= \frac{2}{3}x^{-1/3}(x^2-2)(x^3-x+1)$
$+ x^{2/3}(2x)(x^3-x+1)$
$+ x^{2/3}(x^2-2)(3x^2-1)$

20. $f'(x) = 1(x^3-2x+1)(3-2/x)$
$+ (x+4)(3x^2-2)(3-2/x)$
$+ (x+4)(x^3-2x+1)(2/x^2)$.

21. $h(x) = f(x)g(x)$
$h'(x) = f'(x)g(x)+f(x)g'(x)$

(a) $h(1) = f(1)g(1)$
$= (-2)(1) = -2$
$h'(1) = f'(1)g(1)+f(1)g'(1)$
$= (3)(1)+(-2)(-2) = 7$
So the equation of the tangent line is
$y = 7(x-1)-2$.

(b) $h(0) = f(0)g(0)$
$= (-1)(3) = -3$
$h'(0) = f'(0)g(0)+f(0)g'(0)$
$= (-1)(3)+(-1)(-1)$
$= -2$
So the equation of the tangent line is
$y = -2x-3$.

22. $h(x) = \dfrac{f(x)}{g(x)}$
$h'(x) = \dfrac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

(a) $h(1) = \dfrac{f(1)}{g(1)} = \dfrac{-2}{1} = -2$
$h'(1) = \dfrac{f'(1)g(1)-f(1)g'(1)}{(g(1))^2}$
$= \dfrac{(3)(1)-(-2)(-2)}{(1)^2}$
$= -1$
So the equation of the tangent line is
$y = -1(x-1)-2$.

(b) $h(0) = \dfrac{f(0)}{g(0)} = \dfrac{-1}{3}$

$$h'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{(g(0))^2}$$
$$= \frac{(-1)(3) - (-1)(-1)}{(-1)^2}$$
$$= -4$$

So the equation of the tangent line is
$y = -4x - \frac{1}{3}$.

23. $h(x) = x^2 f(x)$
$h'(x) = 2xf(x) + x^2 f'(x)$

(a) $h(1) = 1^2 f(1) = -2$
$h'(1) = 2(1)f(1) + 1^2 f'(1)$
$= (2)(-2) + 3 = -1$
So the equation of the tangent line is
$y = -(x-1) - 2$.

(b) $h(0) = 0^2 f(0) = 0$
$h'(0) = 2(0)f(0) + 0^2 f'(0) = 0$
So the equation of the tangent line is
$y = 0$.

24. $h(x) = \dfrac{x^2}{g(x)}$

$$h'(x) = \frac{2xg(x) - x^2 g'(x)}{(g(x))^2}$$

(a) $h(1) = \dfrac{1^2}{g(1)} = \dfrac{1}{1} = 1$

$$h'(1) = \frac{2(1)g(1) - 1^2 g'(1)}{(g(1))^2}$$
$$= \frac{(2)(1) - (-2)}{(1)^2} = 4$$

So the equation of the tangent line is
$y = 4(x-1) + 1$.

(b) $h(0) = \dfrac{0^2}{g(0)} = \dfrac{0}{3} = 0$

$$h'(0) = \frac{2(0)g(0) - 0^2 g'(0)}{(g(0))^2}$$
$$= \frac{0}{(3)^2} = 0$$

So the equation of the tangent line is
$y = 0$.

25. The rate at which the quantity Q changes is Q'. Since the amount is said to be "decreasing at a rate of 4%" we have to ask "4% of *what*?" The answer in this type of context is usually 4% of *itself*. In other words, $Q' = -.04Q$. As for P, the 3% rate of increase would translate as $P' = .03P$. By the product rule, with $R = PQ$, we have:
$R' = (PQ)' = P'Q + PQ'$
$= (.03P)Q + P(-.04Q)$
$= -(.01)PQ = (-.01)R$.
In other words, revenue is decreasing at a rate of 1%.

26. Revenue will be constant when the derivative is 0. Substituting $Q' = -0.04Q$ and $P' = aP$ into the expression for R' gives
$R' = -0.04QP + aQP$
$= (-0.04 + a)QP$
This is zero when $a = 0.04$, so price must increase by 4%.

27. $R' = Q'P + QP'$
At a certain moment of time (call it t_0) we are given $P(t_0) = 20$ (\$/item)
$Q(t_0) = 20{,}000$ (items)
$P'(t_0) = 1.25$ (\$/item/year)
$Q'(t_0) = 2{,}000$ (items/year)
$\Rightarrow R'(t_0) = 2{,}000(20) + (20{,}000)1.25$
$= 65{,}000$ \$/year
So revenue is increasing by \$65,000/year at the time t_0.

28. We are given $P = \$14$, $Q = 12,000$ and $Q' = 1,200$. We want $R' = \$20,000$. Substituting these values into the expression for R' (see exercise 25) yields:
$20,000 = 1200 \cdot 14 + 12,000 \cdot P'$

Solve to get $P' = 0.27$ dollars per year.

29. If $u(m) = \dfrac{82.5m - 6.75}{m + .15}$ then using the quotient rule,

$$\frac{du}{dm} = \frac{(m + .15)(82.5) - (82.5m - 6.75)1}{(m + .15)^2} = \frac{19.125}{(m + .15)^2}$$

which is clearly positive. It seems to be saying that initial ball speed is an increasing function of the mass of the bat. Meanwhile,

$$u'(1) = \frac{19.125}{1.15^2} \approx 14.46$$
$$u'(1.2) = \frac{19.125}{1.35^2} \approx 10.49,$$

which suggests that the rate at which this speed is increasing is decreasing.

30. $$u'(M) = \frac{(M + 1.05)\frac{d}{dM}(86.625 - 45M)}{(M + 1.05)^2} - \frac{\frac{d}{dM}(M + 1.05)(86.625 - 45M)}{(M + 1.05)^2}$$
$$= \frac{(-45M - 47.25) - (86.625 - 45M)}{(M + 1.05)^2} = \frac{-133.875}{(M + 1.05)^2}$$

This quantity is negative. In baseball terms, as the mass of the baseball increases, the initial velocity decreases.

31. If $u(m) = \dfrac{14.11}{m + .05} = \dfrac{282.2}{20m + 1}$, then

$$\frac{du}{dm} = \frac{(20m + 1) \cdot 0 - 282.2(20)}{(20m + 1)^2} = \frac{-5644}{(20m + 1)^2}$$

This is clearly negative, which means that impact speed of the ball is a decreasing function of the weight of the club. It appears that the explanation may have to do with the stated fact that the speed of the club is inversely proportional to its mass. Although the lesson of Example 4.6 was that a heavier club makes for greater ball velocity, that was assuming a fixed club speed, quite a different assumption from this problem.

32. $u'(v) = \dfrac{0.2822}{0.217} \approx 1.3$. The initial speed of the ball increases 1.3 times more than the increase in club speed.

33. $$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$
$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{hg(h) - 0}{h} = \lim_{h\to 0} \frac{hg(h)}{h} = \lim_{h\to 0} g(h) = g(0)$$

since g is continuous at $x = 0$.
When $g(x) = |x|$, $g(x)$ is continuous but not differentiable at $x = 0$. We have
$f(x) = x|x| = \begin{cases} -x^2 & x < 0 \\ x^2 & x \ge 0. \end{cases}$ This is differentiable at $x = 0$.

34. This does not work. For example, suppose $a = 2$ and let $g(x) = |x - 2|$. Then

$$f(x) = x|x - 2| = \begin{cases} -x^2 + 2x & x < 2 \\ x^2 - 2x & x \ge 2 \end{cases}$$

so

$$f'(x) = \begin{cases} -2x + 2 & x < 2 \\ 2x - 2 & x > 2. \end{cases}$$

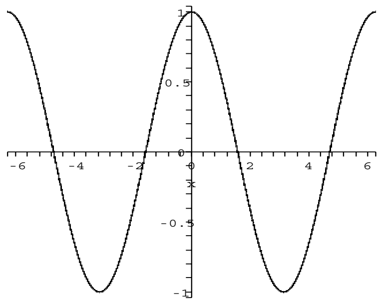
The left hand limit as x approaches 2 is -2 while the right hand limit is 2. Since these are not equal, $f(x)$ is not differentiable at $x = 2$.

35. Answers depend on CAS.

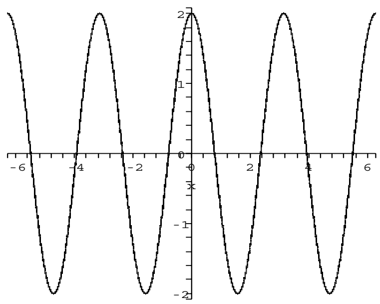
36. Answers depend on CAS.

37. For any constant k, the derivative of $\sin kx$ is $k \cos kx$.

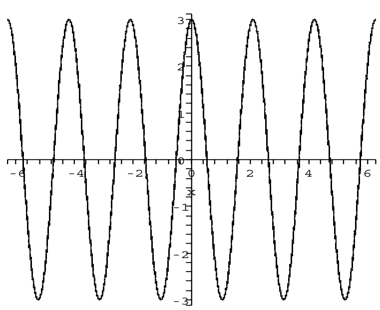
Graph of $\frac{d}{dx}\sin x$:



Graph of $\frac{d}{dx}\sin 2x$:

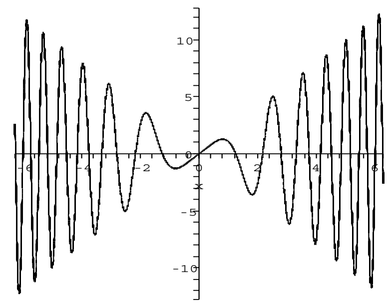


Graph of $\frac{d}{dx}\sin 3x$:

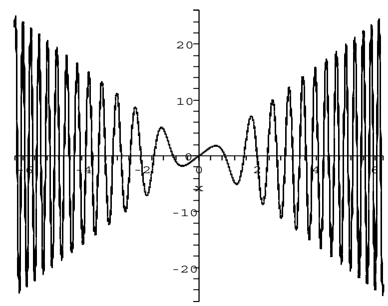


38. The derivative of $\sin kx^2$ is $2kx\cos kx^2$.

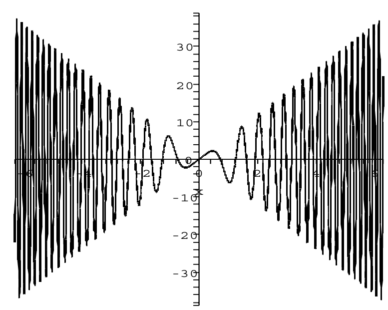
Graph of $\frac{d}{dx}\sin x^2$:



Graph of $\frac{d}{dx}\sin 2x^2$:



Graph of $\frac{d}{dx}\sin 3x^2$:



39. Using the quotient rule, we got a derivative in the form $\dfrac{3x}{2\sqrt{3x^3+x^2}}$ which could be written $\dfrac{3x}{2\sqrt{x^2(3x+1)}}$. One *could* then factor $\sqrt{x^2}$ out of the denominator as $|x|$ and use $\dfrac{x}{|x|} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ to rewrite the function as in the problem. CAS answers may vary.

40. The function $f(x)$ simplifies to $f(x) = 2x$, so $f'(x) = 2$. CAS answers vary, but should simplify to 2.

41. If $F(x) = f(x)g(x)$ then
$$\begin{aligned}F'(x) &= f'(x)g(x) + f(x)g'(x) \text{ and}\\ F''(x) &= f''(x)g(x) + f'(x)g'(x)\\ &\quad + f'(x)g'(x) + f(x)g''(x)\\ &= f''(x)g(x) + 2f'(x)g'(x)\\ &\quad + f(x)g''(x)\\ F'''(x) &= f'''(x)g(x) + f''(x)g'(x)\\ &\quad + 2f''(x)g'(x) + 2f'(x)g''(x)\\ &\quad + f'(x)g''(x) + f(x)g'''(x)\\ &= f'''(x)g(x) + 3f''(x)g'(x)\\ &\quad + 3f'(x)g''(x) + f(x)g'''(x)\end{aligned}$$
One can see obvious parallels to the binomial coefficients as they come from Pascal's Triangle:
$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
On this basis, one could correctly predict the pattern of the fourth or any higher derivative.

42. $F^{(4)}(x) = f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$.

43. If $g(x) = [f(x)]^2 = f(x)f(x)$, then
$$\begin{aligned}g'(x) &= f'(x)f(x) + f(x)f'(x)\\ &= 2f(x)f'(x).\end{aligned}$$

44. $g(x) = f(x)[f(x)]^2$, so
$$\begin{aligned}g'(x) &= f'(x)[f(x)]^2 + f(x)(2f(x)f'(x))\\ &= 3[f(x)]^2 f'(x).\end{aligned}$$

The derivative of $[f(x)]^n$ is $n[f(x)]^{n-1}f'(x)$.

45. $$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$
$$P + \frac{n^2a}{V^2} = \frac{nRT}{V - nb}$$
$$P = \frac{nRT}{V - nb} - \frac{n^2a}{V^2}$$
From this, we find with some difficulty
$$P'(V) = \frac{-nRT}{(V - nb)^2} + \frac{2n^2a}{V^3}$$
$$P''(V) = \frac{2nRT}{(V - nb)^3} + \frac{6n^2a}{V^4}.$$
Obviously, if $P'(V) = 0$, then
$$\frac{2na}{V^3} = \frac{RT}{(V - nb)^2}(= X)$$
in which X is a temporary name. If $P''(V)$ is *also* zero, then
$$0 = P''(V) = \frac{2nX}{(V - nb)} - \frac{3nX}{V}$$
$$= nX\left[\frac{2}{V - nb} - \frac{3}{V}\right] = \frac{nX(3nb - V)}{V(V - nb)},$$
$\Rightarrow V = 3nb,$ so $V - nb = 2nb,$ and
$$X = \frac{2na}{V^3} = \frac{2a}{27n^2b^3}.$$
$$RT = (V - nb)^2X = 4n^2b^2X = \frac{8a}{27b},$$
so $T = \dfrac{8a}{27bR}$, and since
$P = \dfrac{nRT}{V - nb} - \dfrac{n^2a}{V^2}$, we have
$$P = \frac{8an}{27b(2nb)} - \frac{n^2a}{9n^2b^2} = \frac{a}{27b^2}.$$
In summary,
$$(T_c, P_c, V_c) = \left(\frac{8a}{27bR}, \frac{a}{27b^2}, 3nb\right)$$
Substitute in the given numbers; in particular $T_c = 647°$ (Kelvin).

46. $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Without any activator there is no enzyme. With unlimited amount of activator, the amount of enzyme approaches 1.

47. $$f(x) = \frac{x^{2.7}}{1 + x^{2.7}}$$
$$\begin{aligned}f'(x) &= \frac{(1 + x^{2.7}) \cdot 2.7x^{1.7} - 2.7x^{1.7} \cdot (x^{2.7})}{(1 + x^{2.7})^2}\\ &= \frac{2.7x^{1.7}}{(1 + x^{2.7})^2}\end{aligned}$$
The fact that $0 < f(x) < 1$ when $x > 0$ suggest to us that f may

be some kind of concentration ratio or percentage-of-presence of the allosteric enzyme in some system. If so, the derivative would be interpreted as the rate of change in the concentration per unit of activator.

48. $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Without any inhibitor the amount of enzyme approaches 1. With unlimited amount of inhibitor, the amount of enzyme approaches 0.

$$f'(x) = -\frac{2.7x^{1.7}}{(1+x^{2.7})^2}$$

For positive x, f' is negative. Increase in the amount of inhibitor leads to a decrease in the amount of enzyme.

49. $\frac{d}{dx}\left[x^3 f(x)\right] = 3x^2 \cdot f(x) + x^3 f'(x)$

50. Quotient rule gives $\dfrac{x^2 f'(x) - 2xf(x)}{x^4}$.

51. Utilizing $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ (which is a special case of the power rule), we find

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{f(x)}\right) = \frac{f(x)\frac{1}{2\sqrt{x}} - \sqrt{x}f'(x)}{[f(x)]^2} = \frac{f(x) - 2xf'(x)}{2\sqrt{x}[f(x)]^2}.$$

52. Product rule gives

$$\frac{1}{2\sqrt{x}}f(x) + \sqrt{x}f'(x).$$

2.5 The Chain Rule

1. $f(x) = (x^3-1)^2$
Using the chain rule:
$f'(x) = 2(x^3-1)(3x^2) = 6x^2(x^3-1)$
Using the product rule:
$f(x) = (x^3-1)(x^3-1)$
$f'(x) = (3x^2)(x^3-1) + (x^3-1)(3x^2)$
$= 2(3x^2)(x^3-1)$
$= 6x^2(x^3-1)$
Using preliminary multiplication:
$f(x) = x^6 + 2x^3 + 1$
$f'(x) = 6x^5 + 6x^2$
$= 6x^2(x^3-1)$

2. $f(x) = (x^2+2x+1)(x^2+2x+1)$
Using the product rule
$f'(x) =$
$(2x+2)(x^2+2x+1) + (x^2+2x+1)(2x+2)$
Using the chain rule:
$f'(x) = 2(x^2+2x+1)(2x+2)$

3. $f(x) = (x^2+1)^3$
Chain rule:
$f'(x) = 3(x^2+1)^2 \cdot 2x$
Using preliminary multiplication:
$f(x) = x^6 + 3x^4 + 3x^2 + 1$
$f'(x) = 6x^5 + 12x^3 + 6x$

4. $f(x) = 16x^4 + 32x^3 + 24x^2 + 8x + 1$, so
$f'(x) = 64x^3 + 96x^2 + 48x + 8$
Using the chain rule:
$f'(x) = 4(2x+1)^3(2)$

5. $f(x) = \sqrt{x^2+4}$

$$f'(x) = \frac{1}{2\sqrt{x^2+4}} \cdot 2x = \frac{x}{\sqrt{x^2+4}}$$

6. $f(x) = (x^3+x-1)^3$
$f'(x) = 3(x^3+x-1)^2(3x^2+1)$

7. $f(x) = x^5\sqrt{x^3+2}$

$$f'(x) = x^5\frac{1}{2\sqrt{x^3+2}}3x^2 + 5x^4\sqrt{x^3+2} = \frac{3x^7 + 10x^4(x^3+2)}{2\sqrt{x^3+2}} = \frac{13x^7+20x^4}{2\sqrt{x^3+2}}$$

8. $f(x) = (x^3+2)x^{5/2}$
$f'(x) = 3x^2 \cdot x^{5/2} + (x^3+2)\frac{5}{2}x^{3/2}$

9. $f(x) = \dfrac{x^3}{(x^2+4)^2}$

$$f'(x) = \frac{3x^2(x^2+4)^2 - 2(x^2+4)(2x)x^3}{(x^2+4)^4} = \frac{3x^4+12x^2-4x^4}{(x^2+4)^3} = \frac{x^2(12-x^2)}{(x^2+4)^3}$$

10. $f(x) = \dfrac{x^2+4}{x^6}$

$$f'(x) = \frac{x^6 \cdot 2x - (x^2+4)6x^5}{x^{12}}$$

11. $f(x) = \dfrac{6}{\sqrt{x^2+4}} = 6(x^2+4)^{-1/2}$

$$f'(x) = -3(x^2+4)^{-3/2} \cdot 2x = \frac{-6x}{(x^2+4)^{3/2}}$$

12. $f(x) = (1/8)(x^3+4)^5$

$f'(x) = (5/8)(x^3+4)^4(3x^2)$

13. $f(x) = (\sqrt{x}+3)^{4/3}$

$$f'(x) = \frac{4(\sqrt{x}+3)^{1/3}}{3} \cdot \frac{1}{2\sqrt{x}} = \frac{2(\sqrt{x}+3)^{1/3}}{3\sqrt{x}}$$

14. $f'(x) = \dfrac{1}{2\sqrt{x}}(x^{4/3}+3)+\sqrt{x}\left(\dfrac{4}{3}\right)x^{1/3}$

15. $f(x) = \left(\sqrt{x^3+2}+2x\right)^{-2}$

$f'(x) =$

$$-2\left(\sqrt{x^3+2}+2x\right)^{-3}\left[\frac{3x^2}{2\sqrt{x^3+2}}+2\right] = -\frac{3x^2+4\sqrt{x^3+2}}{(\sqrt{x^3+2}+2x)^3 \cdot \sqrt{x^3+2}}$$

16. $f(x) = (64-12x^2+x^4)^{1/2}$

$f'(x) =$

$$\frac{1}{2}(64-12x^2+x^4)^{-1/2}(-24x+4x^3)$$

17. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{\sqrt{x^2+1} - x\left(\frac{1}{2\sqrt{x^2+1}}\right)2x}{x^2+1} = \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

18. $f'(x) = \frac{(x^2+1)2(x^2-1)2x-(x^2-1)^2 2x}{(x^2+1)^2}$

19. $f(x) = \sqrt{\dfrac{x}{x^2+1}}$

$$f'(x) = \frac{1}{2\sqrt{\frac{x}{x^2+1}}} \cdot \frac{(x^2+1)-2x^2}{(x^2+1)^2} = \frac{1-x^2}{2\sqrt{x}(x^2+1)^{3/2}}$$

20. $f'(x) = \left(\frac{1}{2\sqrt{(x^2+1)(\sqrt{x}+1)^3}}\right) \cdot$

$$\left(2x(\sqrt{x}+1)^3 + (x^2+1)3(\sqrt{x}+1)^2\frac{1}{2\sqrt{x}}\right)$$

21. $f(x) = \sqrt[3]{x\sqrt{x^4+2x\sqrt[4]{\dfrac{8}{x+2}}}}$

$$f(x) = \left(x\left[x^4+2x\left(\tfrac{8}{x+2}\right)^{1/4}\right]^{1/2}\right)^{1/3}$$

$$f'(x) = \tfrac{1}{3}\left(x\left[x^4+2x\left(\tfrac{8}{x+2}\right)^{1/4}\right]^{1/2}\right)^{-2/3} \cdot$$

$$\left(\left[x^4+2x\left(\tfrac{8}{x+2}\right)^{1/4}\right]^{1/2} + \right.$$

$$+ x\left(\tfrac{1}{2}\right)\left[x^4+2x\left(\tfrac{8}{x+2}\right)^{1/4}\right]^{-1/2} \cdot$$

$$\left[4x^3+2\left(\tfrac{8}{x+2}\right)^{1/4}\right.$$

$$\left.\left. + 2x\left(\tfrac{1}{4}\right)\left(\tfrac{8}{x+2}\right)^{-3/4}\left(\tfrac{-8}{(x+2)^2}\right)\right]\right)$$

22. $f(x) = \dfrac{3x^2+2\sqrt{x^3+4/x^4}}{(x^3-4)\sqrt{x^2+2}}$

$$f'(x) = \frac{\frac{d}{dx}(3x^2+2\sqrt{x^3+4/x^4})[(x^3-4)\sqrt{x^2+2}]}{(x^3-4)^2(x^2+2)}$$

$$- \frac{(3x^2+2\sqrt{x^3+4/x^4})\frac{d}{dx}((x^3-4)\sqrt{x^2+2})}{(x^3-4)^2(x^2+2)}$$

$$= \frac{\left(6x+\left(\frac{1}{\sqrt{x^3+4/x^4}}\right)(3x^2-16x^{-5})\right)(x^3-4)\sqrt{x^2+2}}{(x^3-4)^2(x^2+2)}$$

$$- \frac{(3x^2+2\sqrt{x^3+4/x^4})\left[3x^2\sqrt{x^2+2}+(x^3-4)\frac{1}{2\sqrt{x^2+2}(2x)}\right]}{(x^3-4)^2(x^2+2)}$$

23. $f(x)=\sqrt{x^2+16},\ a=3,\ f(3)=5$
$f'(x)=\dfrac{1}{2\sqrt{x^2+16}}(2x)=\dfrac{x}{\sqrt{x^2+16}}$
$f'(3)=\dfrac{3}{\sqrt{3^2+16}}=\dfrac{3}{5}$
So the tangent line is $y=\dfrac{3}{5}(x-3)+5$ or $y=\dfrac{3}{5}x+\dfrac{16}{5}$.

24. $f(-2)=\dfrac{3}{4}$
$f'(x)=\dfrac{-12x}{(x^2+4)^2}$
$f'(-2)=\dfrac{24}{64}=\dfrac{3}{8}$
The equation of the tangent line is $y=\dfrac{3}{8}(x+2)+\dfrac{3}{4}$.

25. $s(t)=\sqrt{t^2+8}$
$v(t)=s'(t)=\dfrac{2t}{2\sqrt{t^2+8}}=\dfrac{t}{\sqrt{t^2+8}}$ m/s
$v(2)=\dfrac{2}{\sqrt{12}}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$ m/s

26. $s(t)=\dfrac{60t}{\sqrt{t^2+1}}$
$v(t)=\dfrac{\sqrt{t^2+1}(60)-60t\frac{1}{2\sqrt{t^2+1}}2t}{t^2+1}$ m/s
$v(2)=\dfrac{60\sqrt{5}-\frac{240}{\sqrt{5}}}{5}=\dfrac{12\sqrt{5}}{5}$ m/s

27. For higher derivatives, fractional exponents will be required.
$f(x)=\sqrt{2x+1}=(2x+1)^{1/2}$
$f'(x)=\dfrac{1}{2}(2x+1)^{-1/2}\cdot 2=(2x+1)^{-1/2}$
$f''(x)=-\dfrac{1}{2}(2x+1)^{-3/2}(2)$
$=-(2x+1)^{-3/2}$
$f'''(x)=-\left(-\dfrac{3}{2}\right)(2x+1)^{-5/2}\cdot 2$
$=3(2x+1)^{-5/2}$
$f^{(4)}(x)=3\left(-\dfrac{5}{2}\right)(2x+1)^{-7/2}\cdot 2$
$=-15(2x+1)^{-7/2}$
$f^{(n)}(x)=$
$(-1)^{n+1}1\cdot 3\ldots(2n-3)(2x+1)^{-(2n-1)/2}$

28. $f(x)=\dfrac{2}{x+1}$
$f'(x)=\dfrac{-2}{(x+1)^2}$
$f''(x)=\dfrac{4}{(x+1)^3}$
$f'''(x)=\dfrac{-12}{(x+1)^4}$
$f^{(4)}(x)=\dfrac{48}{(x+1)^5}$
$f^{(n)}(x)=\dfrac{(-1)^n 2(n!)}{(x+1)^{n+1}}$

29. $h'(1)=f'(g(1))g'(1)$
$g(1)=4$, so $h'(1)=f'(4)g'(1)$.

From the table, we have:

$f'(4)\approx\dfrac{2-(-2)}{5-3}=2$, and
$g'(1)\approx\dfrac{6-2}{2-0)}=2$ so
$h'(1)\approx 4$.

30. $k'(1)=g'(f(1))f'(1)$
$f(1)=-2$, so $k'(1)=g'(-2)f'(1)$.

From the table, we have:

$f'(1)\approx\dfrac{-3-(-1)}{2-0}=-1$, and
$g'(-2)\approx\dfrac{2-6}{-1-(-3)}=-2$ so
$k'(1)\approx 2$.

31. $k'(3)=g'(f(3))f'(3)$
$f(3)=-2$, so $k'(3)=g'(-2)f'(3)$.

From the table, we have:

$f'(3)\approx\dfrac{0-(-3)}{4-2}=\dfrac{3}{2}$, and
$g'(-2)\approx\dfrac{2-6}{-1-(-3)}=-2$ so
$k'(1)\approx -3$.

32. $h'(3) = f'(g(3))g'(3)$
$g(3) = 4$, so $h'(3) = f'(4)g'(3)$.

From the table, we have:

$f'(4) \approx \dfrac{2-(-2)}{5-3} = 2$, and
$g'(3) \approx \dfrac{6-2}{2-4)} = -2$ so
$h'(1) \approx -4$.

33. $h'(x) = f'(g(x))g'(x)$
$h'(1) = f'(g(1))g'(1)$
$= f'(2)\cdot(-2) = -6$

34. $h'(x) = f'(g(x))g'(x)$
$h'(2) = f'(g(2))g'(2)$
$= f'(3)\cdot(4) = -12$

35. $f(x) = x^3+4x-1$ is a one-to-one function with $f(0) = -1$ and $f'(0) = 4$. Therefore $g(-1) = 0$ and

$$g'(-1) = \frac{1}{f'(g(-1))} = \frac{1}{f'(0)} = \frac{1}{4}.$$

36. $f(x) = x^3 + 2x + 1$ is a one-to-one function with $f(-1) = -2$ and $f'(-1) = 5$. Therefore $g(-2) = -1$ and

$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(-1)} = \frac{1}{5}.$$

37. $f(x) = x^5 + 3x^3 + x$ is a one-to-one function with $f(1) = 5$ and $f'(1) = 5+9+1 = 15$. Therefore $g(5) = 1$ and

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{15}.$$

38. $f(x) = x^5+4x-2$ is a one-to-one function with $f(0) = -2$ and $f'(0) = 4$. Therefore $g(-2) = 0$ and

$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(0)} = \frac{1}{4}.$$

39. $f(x) = \sqrt{x^3+2x+4}$ is a one-to-one function and $f(0) = 2$ so $g(2) = 0$. Meanwhile,

$f'(x) = \dfrac{1}{2\sqrt{x^3+2x+4}}(3x^2+2)$
$f'(0) = 1/2$
$g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(0)} = 2.$

40. $f(x) = \sqrt{x^5+4x^3+3x+1}$ is a one-to-one function and $f(1) = 3$ so $g(3) = 1$. Meanwhile,

$f'(x) = \dfrac{1}{2}(x^5+4x^3+3x+1)^{-1/2}(5x^4+12x^2+3)$
$f'(1) = \dfrac{20}{6} = \dfrac{10}{3}$
$g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(1)} = \dfrac{3}{10}.$

41. $f(x) = (x^2+3)^2\cdot 2x$
Recognizing the "$2x$" as the derivative of x^2+3, we guess $g(x) = c(x^2+3)^3$ where c is some constant.
$g'(x) = 3c(x^2+3)^2\cdot 2x$
which will be $f(x)$ only if $3c = 1$, so $c = 1/3$, and
$g(x) = \dfrac{(x^2+3)^3}{3}.$

42. A good initial guess is $(x^3+4)^{5/3}$, then adjust the constant to get

$$g(x) = \frac{1}{5}(x^3+4)^{5/3}.$$

43. $f(x) = \dfrac{x}{\sqrt{x^2+1}}.$
Recognizing the "x" as half the derivative of x^2+1, and knowing that differentiation throws the square root into the denominator, we guess $g(x) = c\sqrt{x^2+1}$ where c is some constant and find that

$$g'(x) = \frac{c}{2\sqrt{x^2+1}}(2x)$$

will match $f(x)$ if $c = 1$, so

$$g(x) = \sqrt{x^2+1}.$$

44. A good initial guess is $(x^2+1)^{-1}$, then adjust the constant to get

$$g(x) = -\frac{1}{2}(x^2+1)^{-1}.$$

45. As a temporary device given *any* f, set $g(x) = f(-x)$. Then by the chain rule,

$$g'(x) = f'(-x)(-1) = -f'(-x).$$

In the even case ($g = f$) this reads $f'(-x) = -f'(x)$ and shows f' is odd. In the odd case ($g = -f$–and therefore $g' = -f'$), this reads $-f'(x) = -f'(-x)$ or $f'(x) = f'(-x)$ and shows f' is even.

46. Chain rule gives $2xf'(x^2)$.

47. $$\frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\cdot\frac{d}{dx}\sqrt{x} = f'(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

48. Chain rule gives
$$\frac{1}{2\sqrt{4f(x)+1}}\cdot 4f'(x).$$

49. $$\begin{aligned}\frac{d}{dx}\left(\frac{1}{1+[f(x)]^2}\right) &= -\left(\frac{1}{1+[f(x)]^2}\right)^2\cdot\frac{d}{dx}\left(1+[f(x)]^2\right)\\ &= -\frac{1}{(1+[f(x)]^2)^2}\cdot 2f(x)\cdot f'(x)\end{aligned}$$

50. To say that $f(x)$ is symmetric about the line $x = a$ is the same as saying that $f(a+x) = f(a-x)$. Taking derivatives (using the chain rule), we have

$$\begin{aligned}\frac{d}{dx}f(a+x) &= f'(a+x)\\ \frac{d}{dx}f(a-x) &= f'(a-x)(-1)\\ &= -f'(a-x).\end{aligned}$$

Thus $f'(a+x) = -f'(a-x)$ and the graph of $f'(x)$ is symmetric through the point $(a, 0)$.

51. $f'(x) = b'(a(x))a'(x)$.
$a(2) = 0$, $b'(0) = -3$, $a'(2) = 2$, so $f'(2) = -3\cdot 2 = -6$.

52. $f'(x) = a'(b(x))b'(x)$.
$b(0) = 1$, $a'(1) = 1$, and $b'(0) = -3$, so $f'(0) = -3$.

53. $f'(x) = c'(a(x))a'(x)$.
$a(-1) = 0$, $c'(0) = -3$, $a'(-1) = -2$, so $f'(-1) = -3\cdot -2 = 6$.

54. $f'(x) = b'(c(x))c'(x)$.
$c(1) = -1$, $b'(-1) = -3$, and $c'(1) = 0$, so $f'(1) = 0$.

55. $f(x) = (x^3 - 3x^2 + 2x)^{1/3}$
$f'(x) =$
$\frac{1}{3}(x^3 - 3x^2 + 2x)^{-2/3}\cdot(3x^2 - 6x + 2)$
The derivative of f does not exist at values of x for which
$$\begin{aligned}0 &= x^3 - 3x^2 + 2x\\ &= x(x^2 - 3x + 2)\\ &= x(x-1)(x-2)\end{aligned}$$
Thus, the derivative of f does not exist for $x = 0, 1, 2$. The derivative fails to exist at these points because the tangent lines at these points are vertical.

56. We can write $f(x)$ as
$$f(x) = \begin{cases} -2x-(x-4)-(x+4) & x \le -4\\ -2x-(x-4)+(x+4) & -4 < x < 0\\ 2x-(x-4)+(x+4) & 0 \le x < 4\\ 2x+(x-4)-(x+4) & 4 \le x\end{cases}$$

so

$$f(x) = \begin{cases} -4x & x \le -4 \\ -2x+8 & -4 < x < 0 \\ 2x+8 & 0 \le x < 4 \\ 4x & 4 \le x \end{cases}$$

and therefore

$$f'(x) = \begin{cases} -4 & x < -4 \\ -2 & -4 < x < 0 \\ 2 & 0 < x < 4 \\ 4 & 4 < x \end{cases}$$

but $f'(x)$ is not defined at $x = \pm 4$ or $x = 0$. The function $f(x)$ is piecewise linear and these points correspond graphically to the places where $f(x)$ switches from one linear function to another.

2.6 Derivatives of Trigonometric Functions

1. The peaks and valleys of $\cos(x)$ (e.g., 0, π, 2π, etc.) are matched with the zeros of $\sin(x)$, and the decreasing intervals for $\cos(x)$ (e.g., $[0, \pi]$) correspond to the intervals where $\sin(x)$ is positive, hence where $-\sin(x)$ is negative. These features lend credibility to the notion that $-\sin(x)$ might be the derivative of $\cos(x)$.

2. We use the assumption that x is in radians in Lemma 6.3. The derivative of $\sin x^\circ = \sin(\frac{\pi}{180^\circ} x)$ is $\frac{\pi}{180^\circ}\cos(x^\circ)$. The factor of $\frac{\pi}{180^\circ}$ comes from applying the chain rule.

3. $f(x) = 4\sin x - x$
$f'(x) = 4\cos x - 1$

4. $f(x) = x^2 + 2\cos^2 x$

$$\begin{aligned} f'(x) &= 2x + 2\cos x(-\sin x) \\ &= 2x - 2\cos x \sin x \end{aligned}$$

5. $f(x) = \tan^3 x - \csc^4 x$

$$\begin{aligned} f'(x) &= 3\tan^2 x \sec^2 x \\ &\quad + 4\csc^3 x \csc x \cot x \\ &= 3\tan^2 x \sec^2 x + 4\csc^4 x \cot x \end{aligned}$$

6. $f(x) = 4\sec x^2 - 3\cot x$

$$\begin{aligned} f'(x) &= 4(\sec x^2 \tan x^2)(2x) \\ &\quad - 3(-\csc^2 x) \\ &= 8x \sec x^2 \tan x^2 + 3\csc^2 x \end{aligned}$$

7. $f(x) = x\cos 5x^2$

$$\begin{aligned} f'(x) &= (1)\cos 5x^2 + x(-\sin 5x^2)\cdot 10x \\ &= \cos 5x^2 - 10x^2 \sin 5x^2 \end{aligned}$$

8. $f(x) = 4x^2 - 3\tan x$
$f'(x) = 8x - 3\sec^2 x$

9. $f(x) = \sin(\tan(x^2))$
$f'(x) = \cos(\tan(x^2)) \cdot \sec^2(x^2) \cdot 2x$

10. $f(x) = \sqrt{\sin^2 x + 2}$

$$f'(x) = \frac{1}{2}(\sin^2 x + 2)^{-1/2}(2\sin x \cos x)$$

11. $f(x) = \dfrac{\sin(x^2)}{x^2}$

$$\begin{aligned} f'(x) &= \frac{x^2\cos(x^2)\cdot 2x - \sin(x^2)\cdot 2x}{x^4} \\ &= \frac{2x[x^2\cos(x^2)\cdot 2x - \sin(x^2)]}{x^4} \\ &= \frac{2[x^2\cos(x^2) - \sin(x^2)]}{x^3} \end{aligned}$$

12. $f(x) = \dfrac{x^2}{\csc^4 x}$

$$\begin{aligned} f'(x) &= \frac{2x\csc^4 x - 4x^2\csc^4 x \cot x}{\csc^8 x} \\ &= \frac{2x - 4x^2 \cot x}{\csc^4 x} \end{aligned}$$

13. $f(t) = \sin t \sec t = \tan t$
$f'(t) = \sec^2 t$

14. $f(t) = \sqrt{\cos t \cdot \dfrac{1}{\cos t}} = 1$
$f'(t) = 0$

15. $f(x) = \dfrac{1}{\sin(4x)} = \csc(4x)$
$f'(x) = -\csc(4x)\cot(4x)\cdot(4)$
$= -4\csc(4x)\cot(4x)$
$= \dfrac{-4\cos(4x)}{\sin^2(4x)}$

16. $f(x) = x^2\sec^2 3x$
$f'(x) = 2x\sec^2 3x$
$+x^2 2(\sec 3x)(\sec 3x\tan 3x)(3)$

17. $f(x) = 2\sin x\cos x$
$f'(x) = 2\cos x\cdot\cos x + 2\sin x(-\sin x)$
$= 2\cos^2 x - 2\sin^2 x$

18. $f(x) = 4\sin^2 x + 4\cos^2 x$
$= 4(\sin^2 x + \cos^2 x) \equiv 4$
$f'(x) \equiv 0$

19. $f(x) = \tan\sqrt{x^2+1}$
$f'(x) = (\sec^2\sqrt{x^2+1})\cdot$
$\left(\dfrac{1}{2}\right)(x^2+1)^{-1/2}(2x)$

20. $f(x) = 4x^2\sin x\sec 3x$
$f'(x) = 4(2x)\sin x\sec 3x$
$+ 4x^2\dfrac{d}{dx}(\sin x\sec 3x)$
$= 8x\sin x\sec 3x$
$+4x^2(\cos x\sec 3x + \sin x\sec 3x\tan 3x(3))$

21. Answers depend on CAS.

22. Answers depend on CAS.

23. Answers depend on CAS.

24. Answers depend on CAS.

25. $f(x) = \sin 4x,\ a = \dfrac{\pi}{8},$
$f\left(\dfrac{\pi}{8}\right) = \sin\dfrac{\pi}{2} = 1$
$f'(x) = 4\cos 4x$
$f'\left(\dfrac{\pi}{8}\right) = 4\cos\dfrac{\pi}{2} = 0$
So the equation of the tangent line is
$y - 1 = 0\left(x - \dfrac{\pi}{8}\right)$ or $y = 1$.

26. $f(0) = 0$. $f'(x) = 3\sec^2 3x$, so $f'(0) = 3$. The equation of the tangent line is $y = 3x$.

27. $f(x) = \cos x,\ a = \dfrac{\pi}{2},$
$f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} = 0$
$f'(x) = -\sin x$
$f'\left(\dfrac{\pi}{2}\right) = -\sin\dfrac{\pi}{2} = -1$
So the equation of the tangent line is
$y - 0 = -1\left(x - \dfrac{\pi}{2}\right)$ or $y = -x + \pi/2$.

28. $f\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$
$f'(x) = \sin x + x\cos x$, so $f'\left(\dfrac{\pi}{2}\right) = 1$.
The equation of the tangent line is $y = x$.

29. $s(t) = t^2 - \sin(2t),\ t_0 = 0$
$v(t) = s'(t) = 2t - 2\cos(2t)$
$v(0) = 0 - 2\cos(0) = 0 - 2 = -2$ ft/s

30. $s(t) = t\cos(t^2+\pi),\ t_0 = 0$
$v(t) = s'(t) = \cos(t^2+\pi) - 2t^2\sin(t^2+\pi)$
$v(0) = \cos\pi - 0 = -1$ ft/s

31. $s(t) = \dfrac{\cos t}{t},\ t_0 = \pi$
$v(t) = s'(t)$
$= \dfrac{-1}{t^2}\cos t + \dfrac{1}{t}(-\sin t)$
$v(\pi) = -\dfrac{\cos\pi}{\pi^2} - \dfrac{\sin\pi}{\pi}$
$= \dfrac{1}{\pi^2} - \dfrac{1}{\pi}(0) = \dfrac{1}{\pi^2}$ ft/s

32. $s(t) = 4 + 3\sin t,\ t_0 = \pi$
$v(t) = s'(t) = 3\cos t$
$v(\pi) = -3$ ft/s

33. $f(t) = 4\sin 3t$
$f'(t) = 12\cos 3t$
The maximum speed of 12 occurs when the vertical position is zero.

34. The velocity is 0 when the spring changes directions at the top and bottom. The velocity is $f'(t) = 12\cos 3t$, which is 0 whenever $3t = k\frac{\pi}{2}$ or $t = k\frac{\pi}{6}$ for any odd integer k. The location of the spring at these times is given (for any odd integer k) by $f\left(k\frac{\pi}{6}\right) = 4\sin\left(3k\frac{\pi}{6}\right) = 4\sin\left(k\frac{\pi}{2}\right) = \pm 4$.

35. $Q(t) = 3\sin 2t + t + 4$
$I(t) = \frac{dQ}{dt} = 6\cos 2t + 1$
At time $t = 0$, $I(0) = 7$ amps. At time $t = 1$, $I(1) = 6\cos 2 + 1 \approx -1.497$ amps.

36. The current is given by $I(t) = Q'(t) = -16\sin 4t - 3$. At $t = 0$, the current is -3 amps. At $t = 1$, the current is $I(1) \approx 9.1088$ amps.

37. $f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x = f(x)$

$$\begin{aligned}\Rightarrow f^{(75)}(x) &= (f^{(72)})^{(3)}(x)\\ &= (f^{(18\cdot 4)})^{(3)}(x)\\ &= f''' = -\cos x\end{aligned}$$

$$\begin{aligned}f^{(150)}(x) &= (f^{(148)})^{(2)}(x)\\ &= (f^{(37\cdot 4)})^{(2)}(x)\\ &= f'' = -\sin x\end{aligned}$$

38. If $f(x) = \cos(x)$, then $f^{(4)}(x) = f(x)$

$$\begin{aligned}f^{(77)}(x) &= f^{(19\cdot 4+1)}(x)\\ &= f'(x) = -\sin x\end{aligned}$$

$f^{(120)}(x) = f^{(30\cdot 4)}(x) = f(x) = \cos x$

39. Since $0 \le \sin\theta \le \theta$, we have
$-\theta \le -\sin(\theta) \le 0$ which implies
$-\theta \le \sin(-\theta) \le 0$
so for $-\frac{\pi}{2} \le \theta \le 0$ we have
$\theta \le \sin\theta \le 0$.
We also know that
$\lim_{\theta\to 0^-} \theta = 0 = \lim_{\theta\to 0^-} 0$,
so the Squeeze Theorem implies that $\lim_{\theta\to 0^-} \sin\theta = 0$.

40. Since $\cos^2\theta + \sin^2\theta = 1$, we have $\cos\theta = \sqrt{1-\sin^2\theta}$. Then

$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\sqrt{1-\sin^2\theta} = \pm 1.$$

Since $\cos\theta$ is a continuous function and $\cos 0 = 1$, we conclude that $\lim_{\theta\to 0}\cos\theta = 1$.

41. If $f(x) = \cos(x)$, then

$$\begin{aligned}&\frac{f(x+h)-f(x)}{h}\\ &= \frac{\cos(x+h)-\cos(x)}{h}\\ &= \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= (\cos x)\frac{(\cos h - 1)}{h} - (\sin x)\left(\frac{\sin h}{h}\right).\end{aligned}$$

Taking the limit according to Lemma 6.1

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= (\cos x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}\\ &\quad - (\sin x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h}\\ &= \cos x\cdot 0 - \sin x\cdot 1\\ &= -\sin x.\end{aligned}$$

42.

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)\\ &= \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x}\\ &= -\frac{1}{\sin^2 x} = -\csc^2 x.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{\cos x\cdot 0 - 1(-\sin x)}{\cos^2 x}\\ &= \frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right)\\ &= \sec x\tan x.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{\sin x\cdot 0 - 1\cos x}{\sin^2 x}\\ &= -\frac{1}{\sin x}\left(\frac{\cos x}{\sin x}\right)\\ &= -\csc x\cot x.\end{aligned}$$

43. (a) $$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{x\to 0}\frac{3\sin 3x}{3x}\\ &= 3\cdot\lim_{x\to 0}\frac{\sin(3x)}{(3x)}\\ &= 3\cdot 1 = 3\end{aligned}$$

(b) $$\begin{aligned}\lim_{t\to 0}\frac{\sin t}{4t} &= \frac{1}{4}\lim_{t\to 0}\frac{\sin t}{t}\\ &= \frac{1}{4}\cdot 1 = \frac{1}{4}\end{aligned}$$

(c) $$\lim_{x\to 0}\frac{\cos x - 1}{5x} = \frac{1}{5}\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

(d) Let $u = x^2$: then $u \to 0$ as $x \to 0$, and
$$\lim_{x\to 0}\frac{\sin x^2}{x^2} = \lim_{u\to 0}\frac{\sin u}{u} = 1$$

44. (a) $$\lim_{t\to 0}\frac{2t}{\sin t} = \lim_{t\to 0}\frac{2}{\frac{\sin t}{t}} = 2$$

(b) Let $u = x^2$: then $u \to 0$ as $x \to 0$, and
$$\lim_{x\to 0}\frac{\cos x^2 - 1}{x^2} = \lim_{u\to 0}\frac{\cos u - 1}{u} = 0$$

(c) $$\lim_{x\to 0}\frac{\sin 6x}{\sin 5x} = \lim_{x\to 0}\frac{\frac{6\sin 6x}{6x}}{\frac{5\sin 5x}{5x}} = \frac{6}{5}$$

(d) $$\begin{aligned}\lim_{x\to 0}\frac{\tan 2x}{x} &= \lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}}{x}\\ &= \lim_{x\to 0}\frac{2\sin 2x}{2x}\frac{1}{\cos 2x} = 2\end{aligned}$$

45. If $x \neq 0$, then f is continuous by Theorem 4.2 in Section 1.4, and f is differentiable by the Quotient rule (Theorem 4.2 in Section 2.4). Thus, we only need to check $x = 0$. To see that f is continuous at $x = 0$:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x}{x} = 1$$

(by Lemma 6.3)
Since $\lim_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

To see that f is differentiable at $x = 0$:

$$\begin{aligned}f'(a) &= \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\\ f'(0) &= \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0}\\ &= \lim_{x\to 0}\frac{(\sin x)/x - 1}{x}\end{aligned}$$

In the proof of Lemma 6.3, equation 6.8 was derived:

$$1 > \frac{\sin x}{x} > \cos x$$

Thus,

$$0 > \frac{\sin x}{x} - 1 > \cos x - 1$$

and therefore, if $x > 0$,

$$0 > \frac{\frac{\sin x}{x} - 1}{x} > \frac{\cos x - 1}{x}$$

and if $x < 0$,

$$0 < \frac{\frac{\sin x}{x} - 1}{x} < \frac{\cos x - 1}{x}$$

By Lemma 6.4,

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

Applying the squeeze theorem to the previous two inequalities, we obtain

$$\lim_{x\to 0}\frac{\frac{\sin x}{x} - 1}{x} = 0$$

and so $f'(0) = 0$.

46. From Exercise 45 and the quotient rule, we have

$$f'(x) = \begin{cases} 0 & x = 0 \\ \frac{x\cos x - \sin x}{x^2} & x \neq 0 \end{cases}$$

Thus, to show that $f'(x)$ is continous, we need only show that $\lim_{x\to 0} f'(x) = f'(0) = 0$.

$$\begin{aligned} \lim_{x\to 0} f'(x) &= \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} \\ &= \lim_{x\to 0} \frac{x(\cos x - \frac{\sin x}{x})}{x^2} \\ &= \lim_{x\to 0} \frac{\cos x - \frac{\sin x}{x}}{x} = 0 \end{aligned}$$

since $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

47. For $x \neq 0$,

$$\begin{aligned} f'(x) &= \frac{x\cos x - \sin x}{x^2} \\ f''(x) &= \frac{x^2(\cos x - x\sin x - \cos x)}{x^4} \\ &\quad - \frac{2x(x\cos x - \sin x)}{x^4} \\ &= \frac{-x^3\sin x - 2x^2\cos x + 2x\sin x}{x^4} \\ &= \frac{(2 - x^2)\sin x - 2x\cos x}{x^3} \end{aligned}$$

Thus, $f''(x)$ exists and is continuous for all $x \neq 0$. For $x = 0$,

$$\begin{aligned} f''(0) &= \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\frac{x\cos x - \sin x}{x^2} - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} \end{aligned}$$

Applying L'Hospital's rule, one obtains

$$\begin{aligned} f''(0) &= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} \\ &= -\frac{1}{3}\lim_{x\to 0} \frac{\sin x}{x} = -\frac{1}{3} \end{aligned}$$

Finally, applying L'Hospital's rule to $f''(x)$, one obtains

$$\begin{aligned} &\lim_{x\to 0} f''(x) \\ &= \lim_{x\to 0} \frac{(2 - x^2)\sin x - 2x\cos x}{x^3} \\ &= \lim_{x\to 0} \left[\frac{(2 - x^2)\cos x - 2x\sin x}{3x^2} \right. \\ &\quad \left. + \frac{2x\sin x - 2\cos x}{3x^2}\right] \\ &= \lim_{x\to 0} \frac{-x^2\cos x}{3x^2} \\ &= -\frac{1}{3}\lim_{x\to 0} \cos x = -\frac{1}{3} \end{aligned}$$

Thus, $\lim_{x\to 0} f''(x) = f''(0)$, and so f'' is continuous at $x = 0$.

48. We first show that $f(x)$ is continuous; the only place we need to check is $x = 0$, so we consider $\lim_{x\to 0} x^3\sin(\frac{1}{x})$. We know that, for $x \neq 0$,

$$-1 \le \sin(1/x) \le 1.$$

So, for $x < 0$, we have

$$-x^3 \ge x^3\sin(1/x) \ge x^3,$$

where the inequalities have changed direction because $x^3 < 0$ when $x < 0$. Likewise, for $x > 0$, we have

$$-x^3 \le x^3\sin(1/x) \le x^3.$$

Since $\lim_{x\to 0} x^3 = 0 = \lim_{x\to 0} -x^3$, the Squeeze Theorem implies that $\lim_{x\to 0} x^3\sin(\frac{1}{x}) = 0$ and since this equals $f(0)$, we see that $f(x)$ is continuous for all x.

We now need to show that $f(x)$ is differentiable for all x. Again, we only need to check for $x = 0$. For $x \neq 0$,

$$f'(x) = 3x^2\sin(1/x) - x\cos(1/x).$$

We need to see that $f'(0)$ exists. We have

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^3\sin(1/x) - 0}{x} \\ &= \lim_{x\to 0} x^2\sin(1/x) \end{aligned}$$

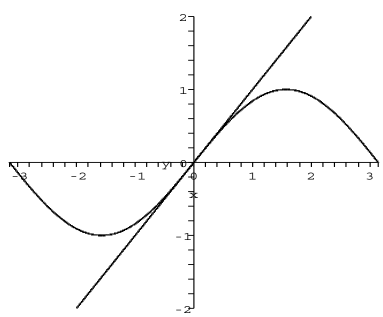
Using the fact that, for all $x \neq 0$,

$$-x^2 \le x^2\sin(1/x) \le x^2$$

and the Squeeze Theorem, we see that $f'(0)$ exists and equals 0. Thus $f(x)$ is differentiable for all x.

Finally, we need to show that $f'(x)$ is continuous for all x. For this, we need to show that $\lim_{x\to 0} f'(x) = f'(0)$, i.e., $\lim_{x\to 0} f'(x) = 0$. We have
$\lim_{x\to 0} f'(x) =$
$\lim_{x\to 0}(3x^2 \sin(1/x) - x\cos(1/x))$
$= \lim_{x\to 0} 3x^2 \sin(1/x) - \lim_{x\to 0} x\cos(1/x)$.
Using the Squeeze Theorem on each piece as before shows that $\lim_{x\to 0} f'(0) = 0$ as desired and so $f'(x)$ is continuous for all x, i.e., $f(x)$ is C^1.

49. The sketch: $y = x$ and $y = \sin(x)$



It is not possible visually to either detect or rule out intersections near $x = 0$ (other than zero itself).

We have that $f'(x) = \cos x$, which is less than 1 for $0 < x < 1$. If $\sin x \geq x$ for some x in the interval $(0, 1)$, then there would be a point on the graph of $y = \sin x$ which lies above the line $y = x$, but then (since $\sin x$ is continuous) the slope of the tangent line of $\sin x$ would have to be greater than or equal to 1 at some point in that interval, contradicting $f'(x) < 1$. Since $\sin x < x$ for $0 < x < 1$, we have $-\sin x > -x$ for $0 < x < 1$. Then $-\sin x = \sin(-x)$ so $\sin(-x) > -x$ for $0 < x < 1$ which is the same as saying $\sin x > x$ for $-1 < x < 0$.

Since $-1 \leq \sin x \leq 1$, the only interval on which $y = \sin x$ might intersect $y = x$ is $[-1, 1]$. We know they intersect at $x = 0$ and we just showed that they do not intersect on the intervals $(-1, 0)$ and $(0, 1)$. So the only other points they might intersect are $x = \pm 1$, but we know that $\sin(\pm 1) \neq \pm 1$, so these graphs intersect only at $x = 0$.

50. $0 < k \leq 1$ produces one intersection. For $1 < k < 7.8$ (roughly) there are exactly three intersections. For $k \approx 7.8$ there are 5 intersections. For $k > 7.8$ there are 7 or more intersections.

51. As seen from the graphs, changing the scale on the x-axis increases the number of oscillations or periods on the display. As the number of periods on the display increase, the graph looks more and more like a bunch of line segments. Its inflection points and concavity are no longer detectable.

2.7 Derivatives of Exponential and Logarithmic Functions

1. $f'(x) = 3x^2 \cdot e^x + x^3 \cdot e^x = e^x x^2(x + 3)$

2. $f'(x) = 2e^{2x} \cos 4x + e^{2x}(-\sin 4x)4$

3. $f'(x) = 1 + 2^x \ln 2$

4. $f'(x) = 4^{3x} + x4^{3x}(\ln 4)3$

5. $f'(x) = 2e^{4x+1} \cdot 4 = 8e^{4x+1}$

6. $f(x) = e^{-x}$, so $f'(x) = -e^{-x}$

7. $f'(x) = (1/3)^{x^2} \cdot \ln(1/3) \cdot 2x$
$= -2x\ln(3)(1/3)^{x^2}$

8. $f'(x) = 4^{-x^2}(\ln 4)(-2x)$

9. $f'(x) = 4^{-3x+1} \cdot \ln 4 \cdot (-3)$
$= -6\ln(2)4^{-3x+1}$

10. $f'(x) = (1/2)^{1-x}\ln(1/2)(-1)$

11. $f'(x) = \dfrac{x \cdot 4e^{4x} - e^{4x} \cdot 1}{x^2}$
$= \dfrac{e^{4x}(4x-1)}{x^2}$

12. $f'(x) = \dfrac{e^{6x} - 6xe^{6x}}{e^{12x}} = \dfrac{1-6x}{e^{6x}}$

13. $f'(x) = \dfrac{1}{2x} \cdot (2) = \dfrac{1}{x}$

14. $f(x) = \dfrac{1}{2}\ln 8 + \dfrac{1}{2}\ln x$, so
$f'(x) = \dfrac{1}{2x}$

15. $f'(x) = \dfrac{3x^2+3}{x^3+3x} = \dfrac{3(x^2+1)}{x(x^2+3)}$

16. $f'(x) = 3x^2\ln x + x^3\dfrac{1}{x}$

17. $f'(x) = \dfrac{1}{\cos x} \cdot -\sin x = -\tan x$

18. $f'(x) = e^{\sin 2x}(\cos 2x)(2)$

19. $f'(x) = \cos\left[\ln(\cos x^3)\right] \cdot \dfrac{1}{\cos x^3} \cdot (-\sin x^3) \cdot 3x^2$
$= -3x^2 \cdot \cos\left[\ln(\cos x^3)\right] \cdot \tan x^3$

20. $f'(x) = \dfrac{1}{\sin x^2}(\cos x^2)(2x)$

21. $f(x) = \dfrac{\sqrt{\ln x^2}}{x} = \dfrac{\sqrt{2\ln x}}{x}$, so
$f'(x) = \dfrac{x \cdot \frac{1}{2\sqrt{2\ln x}} \cdot \frac{2}{x} - \sqrt{2\ln x} \cdot 1}{x^2}$
$= \dfrac{1-2\ln x}{x^2\sqrt{2\ln x}} = \dfrac{1-\ln x^2}{x^2\sqrt{\ln x^2}}$

22. $f'(x) = \dfrac{2^x e^x - e^x 2^x \ln 2}{2^{2x}}$

23. $f'(x) = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

24. $f'(x) = \dfrac{1}{3}(e^{2x}x^3)^{-2/3}(2e^{2x}x^3 + e^{2x}3x^2)$

25. $f(1) = 3e^1 = 3e$
$f'(x) = 3e^x$
$f'(1) = 3e^1 = 3e$
So the equation of the tangent line is $y - 3e = 3e(x-1)$ or $y = 3ex$.

26. $f(1) = 2$. $f'(x) = 2e^{x-1}$, so $f'(1) = 2$.
The equation of the tangent line is $y = 2(x-1) + 2$.

27. $f(1) = 3$
$f'(x) = 3^x \ln 3$
$f'(1) = 3 \cdot \ln 3$
So the equation of the tangent line is $y = (3 \cdot \ln 3)(x-1) + 3$.

28. $f(1) = 2$. $f'(x) = 2^x \ln 2$, so $f'(1) = 2\ln 2$.
The equation of the tangent line is $y = 2\ln 2(x-1) + 2$.

29. $f(1) = 0$
$f'(x) = 2x\ln x + x^2 \cdot \dfrac{1}{x} = 2x\ln x + x$
$f'(1) = 2 \cdot 1\ln 1 + 1 = 2 \cdot 0 + 1 = 1$
So the equation of the tangent line is $y = 1(x-1) + 0$ or $y = x - 1$.

30. $f(1) = 0$. $f'(x) = \frac{6}{x}$, so $f'(1) = 6$.
The equation of the tangent line is $y = 6(x-1)$.

31. $v'(t) = 100 \cdot 3^t \ln 3$
$\dfrac{v'(t)}{v(t)} = \dfrac{100 \cdot 3^t \ln 3}{100 \cdot 3^t} = \ln 3 \approx 1.10$
So the percentage change is about 110%.

32. $v'(t) = 1004^t(\ln 4)$
$\dfrac{v'(t)}{v(t)} = \ln 4 \approx 1.3863$

The instantaneous percentage rate of change is 138.6%.

33. $v(t) = 100e^t$
$v'(t) = 100e^t$
$\dfrac{v'(t)}{v(t)} = \dfrac{100e^t}{100e^t} = 1$
So the percentage change is 100%.

34. $v'(t) = -100e^{-t}$
$\dfrac{v'(t)}{v(t)} = -1$
The instantaneous percentage rate of change is -100%.

35. $p(t) = 200 \cdot 3^t$
$\ln(p(t)) = \ln(200) + t\ln(3)$
$\dfrac{p'(t)}{p(t)} = \dfrac{d}{dt}[\ln(p(t)] = \ln 3 \approx 1.099$,
so the rate of change of population is about 110% per unit of time.

36. The population after t days will be $p(t) = 500 \cdot 2^{t/4}$. The rate of change is
$p'(t) = 500 \cdot 2^{t/4}(\ln 2)(1/4)$,
so the relative rate of change is
$\dfrac{\ln 2}{4} \approx 0.1733$. Therefore the percentage rate of change is about 17.3%.

37. $f(t) = Ae^{rt}$
$APY = \dfrac{f(1) - A}{A} = \dfrac{Ae^r - A}{A} = e^r - 1$

(a) $APY = e^{0.05} - 1 \approx .05127$ (5.1%)

(b) $APY = e^{0.1} - 1 \approx .10517$ (10.5%)

(c) $APY = e^{0.2} - 1 \approx .22140$ (22.1%)

(d) $APY = e^{\ln 2} - 1 = 1$ (100%)

(e) $APY = e^1 - 1 \approx 1.71828$ (172%)

38. From exercise 37 we have
$APY = \frac{f(1)-A}{A} = e^r - 1$

(a) To obtain 100% APY, we need $1 = e^r - 1 \Rightarrow e^r = 2 \Rightarrow r = \ln 2$.

(b) To obtain 10% APY, we need $0.1 = e^r - 1 \Rightarrow e^r = 1.1 \Rightarrow$
$r = \ln 1.1 \approx 0.09531$.

39. $f(x) = x^{\sin x}$
$\ln f(x) = \sin x \cdot \ln x$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(\sin x \cdot \ln x)$
$= \cos x \cdot \ln x + \dfrac{\sin x}{x}$
$f'(x) = x^{\sin x}\left(\dfrac{x\cos x \cdot \ln x + \sin x}{x}\right)$

40. $f(x) = x^{4-x^2}$
$\ln f(x) = (4 - x^2)\ln x$
$\dfrac{f'(x)}{f(x)} = -2x\ln x + (4 - x^2)\dfrac{1}{x}$
$f'(x) = x^{4-x^2}\left(-2x\ln x + (4 - x^2)\dfrac{1}{x}\right)$

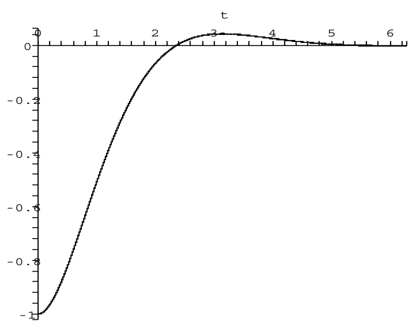
41. $f(x) = (\sin x)^x$
$\ln f(x) = x \cdot \ln(\sin x)$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(x \cdot \ln(\sin x))$
$= \dfrac{x\cos x}{\sin x} + \ln(\sin x)$
$= x\cot x + \ln(\sin x)$
$f'(x) = (\sin x)^x \cdot (x\cot x + \ln(\sin x))$

42. $f(x) = (x^2)^{4x}$
$\ln f(x) = 8x\ln x$
$\dfrac{f'(x)}{f(x)} = 8\ln x + 8x\dfrac{1}{x}$
$f'(x) = (x^2)^{4x}(8\ln x + 8)$

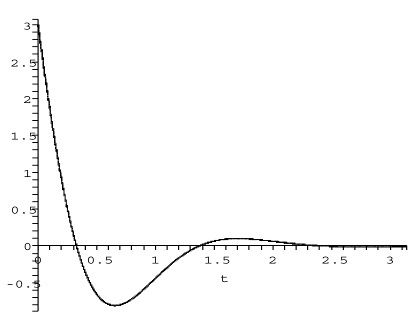
43. $f(x) = x^{\ln x}$
$\ln f(x) = \ln x \cdot \ln x = \ln^2 x$
$\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(\ln^2 x) = \dfrac{2\ln x}{x}$
$f'(x) = x^{\ln x}\left[\dfrac{2\ln x}{x}\right] = 2x^{[(\ln x)-1]}\ln x$

44. $f(x) = x^{\sqrt{x}}$
$\ln f(x) = \sqrt{x} \ln x$
$\dfrac{f'(x)}{f(x)} = \dfrac{1}{2\sqrt{x}} \ln x + \sqrt{x}\dfrac{1}{x}$
$f'(x) = x^{\sqrt{x}} \left(\dfrac{1}{2\sqrt{x}} \ln x + \dfrac{1}{\sqrt{x}} \right)$

45. $f(t) = e^{-t} \cos t$
$v(t) = f'(t) = -e^{-t} \cos t + e^{-t}(-\sin t)$
$= -e^{-t}(\cos t + \sin t)$
If the velocity is zero, it is because $\cos t = -\sin t$, so
$t = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots, \dfrac{(3+4n)\pi}{4}, \ldots$
Position when velocity is zero:
$f(3\pi/4) = e^{-3\pi/4} \cos(3\pi/4)$
$= e^{-3\pi/4}(-1/\sqrt{2}) \approx -.067020$
$f(7\pi/4) = e^{-7\pi/4} \cos(7\pi/4)$
$= e^{-7\pi/4}(1/\sqrt{2}) \approx .002896$
Graph of the velocity function:



46. $f'(t) = -2e^{-2t} \sin 3t + 3e^{-2t} \cos 3t$
$= e^{-2t}(-2 \sin 3t + 3 \cos 3t)$



The velocity of the spring is zero when it is changing direction at the top and bottom of the motion. This occurs when $3\cos 3t = 2\sin 3t$ or $\tan 3t = 3/2$, i.e., at $t = \frac{1}{3}\tan^{-1}(3/2) \approx 0.3276$. The position of the spring at this time is approximately $f(0.3276) \approx 0.4321$.

47. Graphically, the maximum velocity seems to occur at $t = \pi$.

48. Graphically, the maximum velocity seems to occur at $t = 0$; the maximum velocity is not reached on $t > 0$.

49. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$
$f'(x) = \dfrac{e^x + e^{-x}}{2} = \cosh x$
$g(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$
$g'(x) = \dfrac{e^x - e^{-x}}{2} = \sinh x$

50. $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}$
$f'(x) = \dfrac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x}$
$= \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$
$= \text{sech}^2 x = \dfrac{4}{(e^x + e^{-x})^2}$

51. If $f(x) = \sinh x$, then $f'(x) = \cosh x$ and $f''(x) = \sinh x = f(x)$.

If $f(x) = \cosh x$, then $f'(x) = \sinh x$ and $f''(x) = \cosh x = f(x)$.

52. (a) $f'(x) = -\cosh(\cos x) \sin x$

(b) $f'(x) = \sinh(x^2)2x - \cosh(x^2)2x$
$= 2x(\sinh(x^2) - \cosh(x^2))$

53. Let $(a, \ln a)$ be the point of intersection of the tangent line and the graph of $y = f(x)$.
$f(x) = \ln x$
$f'(x) = \dfrac{1}{x}$
$m = f'(a) = \dfrac{1}{a}$

Since the tangent line passes through the origin, the equation of the tangent line is

$$y = mx = \frac{1}{a}x$$

Since $(a, \ln a)$ is a point on the tangent line,

$$\ln a = \frac{1}{a}a = 1$$

so $a = e$.

54. Let (a, e^a) be the point of intersection of the tangent line and the graph of $y = f(x)$.
$f(x) = e^x$
$f'(x) = e^x$
$m = f'(a) = e^a$
Since the tangent line passes through the origin, the equation of the tangent line is

$$y = mx = e^a x$$

Since (a, e^a) is a point on the tangent line,

$$e^a = e^a a$$

so $a = 1$. The slope of the tangent line in 53 is $1/e$ while the slope of the tangent line here is e.

55. $f(x) = e^{\ln x^2}$

$$f'(x) = e^{\ln x^2} \cdot \frac{d}{dx} \ln x^2$$
$$= e^{\ln x^2} \cdot \frac{2}{x} = 2x$$

Much easier if one noticed at the outset that $f(x) = x^2$.

56. The derivative does not exist because the function is not defined for any values of x! We know $-x^2 \leq 0$ for all x and the natural logarithm is not defined for $x \leq 0$.

57. $f(x) = \ln \sqrt{4e^{3x}} = \frac{1}{2}\left[\ln\left(4 \cdot e^{3x}\right)\right]$

$$= \frac{1}{2}\left[\ln 4 + \ln e^{3x}\right] = \frac{\ln 4 + 3x}{2}$$

$$f'(x) = \frac{3}{2}$$

58. $f(x) = \ln e^{4x} - 2\ln x = 4x - 2\ln x$, so $f'(x) = 4 - \frac{2}{x}$.

59. We approximate $\lim_{h\to 0} \frac{a^h-1}{h}$ for $a = 3$.

h	$\frac{a^h-1}{h}$
0.01	1.10466919
0.001	1.09921598
0.0001	1.09867264
0.00001	1.09861832
−0.01	1.09259958
−0.001	1.09800903
−0.0001	1.09855194

The limit seems to be approaching approximately 1.0986, which is very close to $\ln 3 \approx 1.09861$.

60. We approximate $\lim_{h\to 0} \frac{a^h-1}{h}$ for $a = \frac{1}{3}$.

h	$\frac{a^h-1}{h}$
0.01	−1.09259958
0.001	−1.09800904
0.0001	−1.09855194
0.00001	−1.09860625
−0.01	−1.10466919
−0.001	−1.09921598
−0.0001	−1.09867264

The limit seems to be approaching approximately -1.0986, which is very close to $\ln \frac{1}{3} \approx -1.09861228867$.

61. $x(t) = \frac{6}{2e^{-8t} + 1} = 6(2e^{-8t} + 1)^{-1}$

$$x'(t) = -6(2e^{-8t} + 1)^{-2} \cdot (-16e^{-8t})$$
$$= \frac{96e^{-8t}}{(2e^{-8t} + 1)^2}$$

Since $e^{-8t} > 0$ for any t, both numerator and denominator are positive, so that $x'(t) > 0$. Then, since $x(t)$ is an increasing function with a limiting value of 6 (as t goes to infinity), the concentration never exceeds (indeed, never reaches) the value of 6.

62. $x'(t) = -10(9e^{-10t}+2)^{-2}(-90e^{-10t})$
$= \dfrac{900e^{-10t}}{(9e^{-10t}+2)^2}$
Since $e^{-10t} > 0$ for all t, $x'(t) > 0$ for all t, and $x(t)$ is increasing for all t. This forces $x(t) < \lim_{t\to\infty} x(t) = 5$.

63. If $g(x) = e^x$, then
$g'(x) = e^x$ and $g''(x) = e^x$ so
$g(0) = g'(0) = g''(0) = e^0 = 1$.

If $f(x) = \dfrac{a+bx}{1+cx}$, then $f(0) = a$,
$f'(x) = \dfrac{b(1+cx)-(a+bx)(c)}{(1+cx)^2}$
$= \dfrac{b-ac}{(1+cx)^2} = (b-ac)(1+cx)^{-2}$
$f'(0) = b - ac$
$f''(x) = (b-ac)(-2)(1+cx)^{-3}c$
$= \dfrac{-2c(b-ac)}{(1+cx)^3}$
$f''(0) = -2c(b-ac)$
$1 = g(0) = f(0) = a$ so $a = 1$.
$1 = g'(0) = f'(0) = b - ac = b - c$
$1 = g''(0) = f''(0) = -2c(b-ac) = -2c$
so $c = -1/2$ and $b = 1 + c = 1 - 1/2 = 1/2$
so $b = 1/2$.
In summary, $a = 1$, $b = 1/2$, $c = -1/2$ and
$g(x) = \dfrac{1+(x/2)}{1-(x/2)} = \dfrac{2+x}{2-x}$.

64. Answers very depending on source. Linear growth corrseponds to constant slope. In other words the population changes by the same fixed amount per year. In exponential growth, the size of the change depends on the size of the population. The percantage change is the same, though, from year to year.

65. $f(x) = e^{-x^2/2}$
$f'(x) = e^{-x^2/2}\cdot(-2x/2)$
$= -xe^{-x^2/2}$
$f''(x) = -\left[x(-xe^{-x^2/2}) + 1\cdot e^{-x^2/2}\right]$
$= e^{-x^2/2}(x^2-1)$
This will be zero only when $x = \pm 1$.

66. $f(x) = e^{-x^2/8}$, $f'(x) = (-x/4)e^{-x^2/8}$, and
$f''(x) = (-1/4)e^{-x^2/8} + (x^2/16)e^{-x^2/8}$
$= e^{-x^2/8}((-1/4)+x^2/16)$.
This is zero when $x = \pm 2$. The graph is flatter in the middle, but the tails are thicker.

67. It helps immensely to leave the name f as it was in #65, and give a new name g to the new function here, so that

$$g(x) = e^{-(x-m)^2/2c^2} = f(u)$$

in which $u = \frac{x-m}{c}$. Then
$g'(x) = f'(u)\dfrac{du}{dx} = \dfrac{f'(u)}{c} = \dfrac{-uf(u)}{c}$
$= \dfrac{-(x-m)e^{-(x-m)^2/2c^2}}{c^2}$,
$g''(x) = \dfrac{d}{dx}\left(\dfrac{f'(u)}{c}\right) = \dfrac{f''(u)\frac{du}{dx}}{c}$
$= \dfrac{f''(u)}{c^2} = \dfrac{(u^2-1)f(u)}{c^2}$
$= \dfrac{((x-m)^2-c^2)e^{-(x-m)^2/2c^2}}{c^4}$
This will be zero only when $x = m \pm c$.

68. $f(x) = e^{-(x-m)^2/2c^2}$,
$f'(x) = \dfrac{-(x-m)}{c^2}e^{-(x-m)^2/2c^2}$,
and this is equal to zero when $x = m$.

2.8 Implicit Differentiation and Inverse Trigonometric Functions

1. Explicitly:
$4y^2 = 8 - x^2$
$y^2 = \frac{8-x^2}{4}$
$y = \pm\frac{\sqrt{8-x^2}}{2}$ (choose plus to fit (2,1))

For $y = \dfrac{\sqrt{8-x^2}}{2}$,
$y' = \dfrac{1}{2}\dfrac{(-2x)}{2\sqrt{8-x^2}} = \dfrac{-x}{2\sqrt{8-x^2}}$,
$y'(2) = -1/2$.

Implicitly:
$\dfrac{d}{dx}(x^2 + 4y^2) = \dfrac{d}{dx}(8)$
$2x + 8y \cdot y' = 0$
$y' = \dfrac{-2x}{8y} = \dfrac{-x}{4y}$
at $(2, 1)$: $y' = \dfrac{-2}{4 \cdot 1} = -\dfrac{1}{2}$

2. Explicitly:
$y = \dfrac{4\sqrt{x}}{x^3 - x^2}$
$y' = \dfrac{(x^3 - x^2)\frac{2}{\sqrt{x}} - 4\sqrt{x}(3x^2 - 2x)}{(x^3 - x^2)^2}$.
Implicitly differentiating:
$3x^2y + x^3y' - \dfrac{2}{\sqrt{x}} = 2xy + x^2y'$,
and we solve for y' to get
$y' = \dfrac{2xy + \frac{2}{\sqrt{x}} - 3x^2y}{x^3 - x^2}$.
Substitute $x = 2$ into the first expression, and $(x, y) = (2, \sqrt{2})$ into the second to get
$y' = -\dfrac{7\sqrt{2}}{4}$

3. Explicitly:
$y(1 - 3x^2) = \cos x$
$y = \dfrac{\cos x}{1 - 3x^2}$
$y'(x) = \dfrac{(1 - 3x^2)(-\sin x) - \cos x(-6x)}{(1 - 3x^2)^2}$
$= \dfrac{-\sin x + 3x^2 \sin x + 6x \cos x}{(1 - 3x^2)^2}$
$y'(0) = 0$

Implicitly:
$\dfrac{d}{dx}(y - 3x^2y) = \dfrac{d}{dx}(\cos x)$
$y' - (6xy + 3x^2y') = -\sin x$
$y'(1 - 3x^2) = 6xy - \sin x$
$y' = \dfrac{6xy - \sin x}{1 - 3x^2}$
at $(0, 1)$: $y' = 0$ (again).

4. Explicitly:
$y = -x \pm \sqrt{x^2 - 4}$. At the point $(-2, 2)$, the sign is irrelevant, so we choose $y = -x + \sqrt{x^2 - 4}$.
$y' = -1 + \dfrac{1}{2\sqrt{x^2 - 4}}2x$
$= -1 + \dfrac{x}{\sqrt{x^2 - 4}}$.
Implicitly differentiating:
$2yy' + 2y + 2xy' = 0$,
and we solve for y':
$y' = \dfrac{-2y}{2x + 2y}$
Substitute $x = -2$ into the first expression, and $(x, y) = (-2, 2)$ into the second expression to see that y' is undefined. There is a vertical tangent at this point.

5. $\dfrac{d}{dx}(x^2y^2 + 3y) = \dfrac{d}{dx}(4x)$
$2xy^2 + x^2 2y \cdot y' + 3y' = 4$
$y'(2x^2y + 3) = 4 - 2xy^2$
$y' = \dfrac{4 - 2xy^2}{2x^2y + 3}$

6. $3y^3 + 3x(3y^2)y' - 4 = 20yy'$
$y' = \dfrac{3y^3 - 4}{20y - 9xy^2}$

7. $\frac{d}{dx}(\sqrt{xy}-4y^2)=\frac{d}{dx}(12)$
$\frac{1}{2\sqrt{xy}}\cdot\frac{d}{dx}(xy)-8y\cdot y'=0$
$\frac{1}{2\sqrt{xy}}\cdot(xy'+y)-8y\cdot y'=0$
$(xy'+y)-16y\cdot y'\sqrt{xy}=0$
$y'\left(x-16y\sqrt{xy}\right)=-y$
$y'=\frac{-y}{\left(x-16y\sqrt{xy}\right)}=\frac{y}{16y\sqrt{xy}-x}$

8. $\cos(xy)(y+xy')=2x$
$y'=\frac{2x-y\cos(xy)}{x\cos(xy)}$

9. $x+3=4xy+y^3$
$1=\frac{d}{dx}\left(4xy+y^3\right)=4(xy'+y)+3y^2y'$
$1-4y=y'(3y^2+4x)$
$y'=\frac{1-4y}{3y^2+4x}$

10. $3+3y^2y'-4y'=20x$
$y'=\frac{20x-3}{3y^2-4}$

11. $\frac{d}{dx}(e^{x^2y}-e^y)=\frac{d}{dx}(x)$
$e^{x^2y}\frac{d}{dx}(x^2y)-e^yy'=1$
$e^{x^2y}(2xy+x^2y')-e^yy'=1$
$y'(x^2e^{x^2y}-e^y)=1-2xye^{x^2y}$
$y'=\frac{1-2xye^{x^2y}}{x^2e^{x^2y}-e^y}$

12. $e^y+xe^yy'-3y'\sin x-3y\cos x=0$
$y'=\frac{3y\cos x-e^y}{xe^y-3\sin x}$

13. $\frac{d}{dx}\left(\sqrt{x+y}-4x^2\right)=\frac{d}{dx}(y)$
$\frac{1}{2\sqrt{x+y}}\cdot(1+y')-8x=y'$
$y'\left(\frac{1}{2\sqrt{x+y}}-1\right)=\frac{-1}{2\sqrt{x+y}}+8x$
$y'\left(\frac{1-2\sqrt{x+y}}{2\sqrt{x+y}}\right)=\frac{16x\sqrt{x+y}-1}{2\sqrt{x+y}}$
$y'=\frac{16x\sqrt{x+y}-1}{1-2\sqrt{x+y}}$

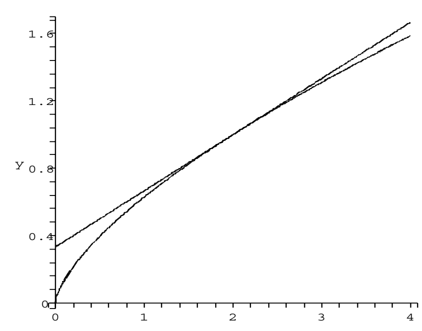
14. $(\sin y)y'-2yy'=0$
$y'=0$

15. $\frac{d}{dx}\left(e^{4y}-\ln y\right)=\frac{d}{dx}(2x)$
$e^{4y}\cdot 4y'-\frac{1}{y}\cdot y'=2$
$y'\left(4e^{4y}-\frac{1}{y}\right)=2$
$y'\left(\frac{4ye^{4y}-1}{y}\right)=2$
$y'=\frac{2y}{4ye^{4y}-1}$

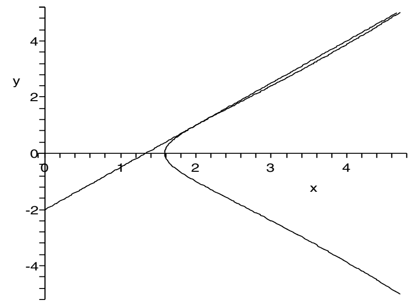
16. $2xe^{x^2}y+e^{x^2}y'-3y'=2x$
$y'=\frac{2x-2xe^{x^2}y}{e^{x^2}-3}$

17. Rewrite: $x^2=4y^3$

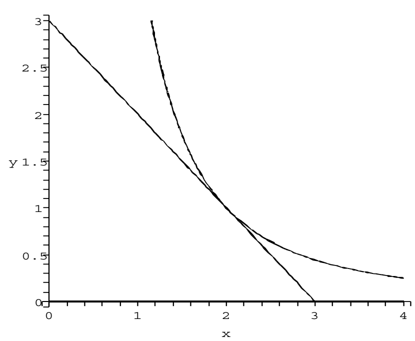
Differentiate by x: $2x=12y^2\cdot y'$
$y'=\frac{2x}{12y^2}=\frac{x}{6y^2}$
at $(2,1)$: $y'=\frac{2}{6\cdot 1^2}=\frac{1}{3}$
The equation of the tangent line is
$y-1=\frac{1}{3}(x-2)$ or $y=\frac{1}{3}(x+1)$.



18. $2xy^2+x^22yy'=4$, so $y'=\frac{4-2xy^2}{2x^2y}$.
y' at $(1,2)$ is -1, and the equation of the line is $y=-1(x-1)+2$.



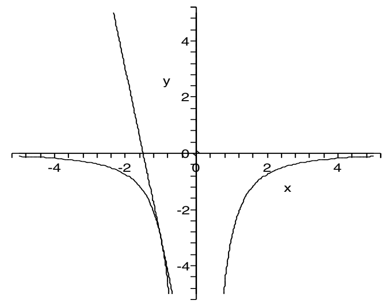
19. This one has $y = 0$ as part of the curve(s), but our point of reference is not on that part, so we can assume y is not zero, cancel it, and come to
$x^2y = 4$
$\frac{d}{dx}(x^2y) = \frac{d}{dx}(4)$
$2xy + x^2 \cdot y' = 0$
$y' = \frac{-2y}{x}$
at $(2, 1)$: $y' = -2/2 = -1$.
The equation of the tangent line is
$y - 1 = (-1)(x - 2)$ or $y = -x + 3$.



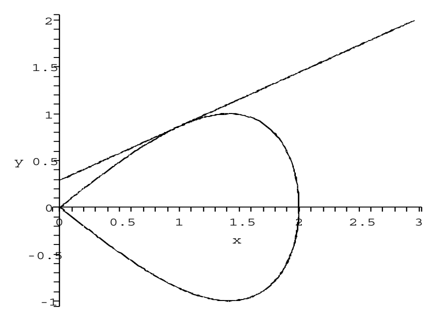
20. $3x^2y^2 + x^3 2yy' = -3y - 3xy'$, so

$$y' = \frac{-3y - 3x^2y^2}{2x^3y + 3x}.$$

y' at $(-1, -3)$ is -6, and the equation of the line is $y = -6(x + 1) - 3$

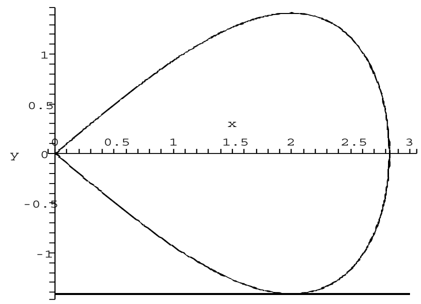


21. $4y^2 = 4x^2 - x^4$
$8yy' = 8x - 4x^3$
$y' = \frac{x(2 - x^2)}{2y}$
The slope of the tangent line at $(1, \sqrt{3}/2)$ is
$m = \frac{(1)(2 - 1^2)}{2\left(\frac{\sqrt{3}}{2}\right)}$
$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$
The equation of the tangent line is
$y - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}(x - 1)$
$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3}$
$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{6}.$



22. $x^4 - 8x^2 = -8y^2$
$4x^3 - 16x = -16yy'$
$y' = \frac{4x^3 - 16x}{-16y} = \frac{x^3 - 4x}{-4y}$
The slope of the tangent line at $(2, -\sqrt{2})$ is

$m = \dfrac{2^3 - 4(2)}{-4(-\sqrt{2})} = 0.$
The equation of the tangent line is $y = -\sqrt{2}$.



23. $\dfrac{d}{dx}(x^2 + y^3 - 3y) = \dfrac{d}{dx}(4)$
$2x + 3y^2y' - 3y' = 0$
$y'(3y^2 - 3) = -2x$
$y' = \dfrac{2x}{3 - 3y^2}$

Horizontal tangents:
From the formula, $y' = 0$ only when $x = 0$. When $x = 0$, we have $0^2 + y^3 - 3y = 4$. Using a CAS to solve this, we find that
$y = \left(2 - \sqrt{3}\right)^{1/3} + \left(2 + \sqrt{3}\right)^{1/3} \approx 2.2$
is a horizontal tangent line, tangent to the curve at the (approximate) point $(0, 2.2)$.

Vertical tangents: the denominator in y' must be zero.
$3 - 3y^2 = 0$
$y^2 = 1 \quad \text{or} \quad y = \pm 1.$
When $y = 1$ we have
$x^2 + (1)^3 - 3(1) = 4$
$x^2 = 6 \quad \text{or} \quad x = \pm\sqrt{6} \approx \pm 2.4.$
Also, when $y = -1$, we have
$x^2 + (-1)^3 - 3(-1) = 4$
$x^2 = 2$
$x = \pm\sqrt{2} \approx \pm 1.4.$
Thus, we find 4 vertical tangent lines: $x = -\sqrt{6}$, $x = -\sqrt{2}$, $x = \sqrt{2}$, $x = \sqrt{6}$, tangent to the curve (respectively) at the points $\left(-\sqrt{6}, 1\right)$, $\left(-\sqrt{2}, -1\right)$, $\left(\sqrt{2}, -1\right)$, and $\left(\sqrt{6}, 1\right)$.

24. $\dfrac{d}{dx}(xy^2 - 2y) = \dfrac{d}{dx}(2)$
$y^2 + 2xyy' - 2y' = 0$
$y' = \dfrac{y^2}{2 - 2xy}$
The curve has horizontal tangents where $y' = 0$. There are no points on the curve where this is true because $y = 0$ has no solutions on the original curve $xy^2 - 2y = 2$. The curve can have vertical tangents where y' is undefined. The only such point on the curve is $(-\frac{1}{2}, -2)$.

25. $\dfrac{d}{dx}(x^2y^2 + 3x - 4y) = \dfrac{d}{dx}(5)$
$x^2 2yy' + 2xy^2 + 3 - 4y' = 0$
Differentiate both sides of this with respect to x:
$\dfrac{d}{dx}(x^2 2yy' + 2xy^2 + 3 - 4y') = \dfrac{d}{dx}(0)$
$2(2xyy' + x^2(y')^2 + x^2yy'')$
$\quad + 2(2xyy' + y^2) - 4y'' = 0$
$2xyy' + x^2(y')^2 + x^2yy''$
$\quad + 2xyy' + y^2 - 2y'' = 0$
$4xyy' + x^2(y')^2 + y^2 = y''(2 - x^2y)$
$y'' = \dfrac{4xyy' + x^2(y')^2 + y^2}{2 - x^2y}$

26. $\dfrac{d}{dx}(x^{2/3} + y^{2/3}) = \dfrac{d}{dx}(4)$
$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0,$
multiply by $\frac{3}{2}$ and implicitly differentiate again:
$\dfrac{-1}{3}x^{-4/3} + \dfrac{-1}{3}y^{-4/3}y'y' + y^{-1/3}y'' = 0,$
so
$y'' = \dfrac{x^{-4/3} + y^{-4/3}(y')^2}{3y^{-1/3}}$

27. $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(x^3 - 6x + 4\cos y)$
$2yy' = 3x^2 - 6 - 4\sin y \cdot y'$

Differentiating again with respect to x: $2[yy'' + (y')^2]$
$= 6x - 4[\sin y \cdot y'' + \cos y \cdot (y')^2]$,
$yy'' + (y')^2$
$= 3x - 2\sin y \cdot y'' - 2\cos y \cdot (y')^2$,
$y''(y + 2\sin y) = 3x - [2\cos y + 1](y')^2$
$$y'' = \frac{3x - [2\cos y + 1](y')^2}{y + 2\sin y}$$

28. $\frac{d}{dx}(e^{xy} + 2y - 3x) = \frac{d}{dx}(\sin y)$
$e^{xy}(y + xy') + 2y' - 3 = \cos y \cdot y'$,
$e^{xy}(y + xy')^2 + e^{xy}(y' + y' + xy'') + 2y''$
$= -\sin y(y')^2 + \cos y \cdot y''$, and
$$y'' = \frac{e^{xy}(y + xy')^2 + 2e^{xy}y' + \sin y(y')^2}{\cos y - xe^{xy} - 2}.$$

29. $f'(x) = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx}\sqrt{x}$
$= \frac{1}{2(1 + x)\sqrt{x}}$

30. $f'(x) = \frac{1}{\sqrt{1 - (x^3 + 1)^2}}(3x^2)$

31. $f'(x) = \frac{1}{1 + (\cos x)^2} \cdot \frac{d}{dx}\cos x$
$= \frac{-\sin x}{1 + (\cos x)^2}$

32. $f'(x) = 4\frac{1}{x^4\sqrt{x^8 - 1}}4x^3$
$= \frac{16}{x\sqrt{x^8 - 1}}$

33. $f'(x) = 4\sec(x^4)\tan(x^4) \cdot 4x^3$

34. $f'(x) = \frac{1}{2}(2 + \tan^{-1} x)^{-1/2}\frac{1}{1 + x^2}$

35. $f'(x) = e^{\tan^{-1} x}\frac{d}{dx}\tan^{-1} x$
$= \frac{e^{\tan^{-1} x}}{1 + x^2}$

36. $f'(x) = \frac{\cot^{-1} x \cdot 2x - x^2 \cdot \frac{-1}{1+x^2}}{(\cot^{-1} x)^2}$
$= \frac{x\left(\cot^{-1} x \cdot 2(1 + x^2) + x\right)}{(\cot^{-1} x)^2(1 + x^2)}$

37. $f'(x) = \frac{(x^2 + 1)\frac{1}{x^2+1} - \tan^{-1} x(2x)}{(x^2 + 1)^2}$
$= \frac{1 - 2x\tan^{-1} x}{(x^2 + 1)^2}$

38. $f'(x) = \frac{1}{\sqrt{1 - \sin^2 x}}\cos x = 1$
Note that $\sin^{-1}(\sin x) = x$.

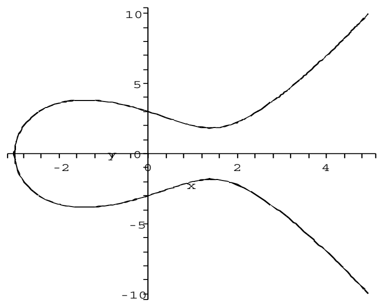
39. $x^2 + y^3 - 2y = 3$
$y' = \frac{-2x}{3y^2 - 2}$
If $x = 1.9$, solving for y requires solving the equation $y^3 - 2y + 0.61 = 0$. Using the equation of the tangent line found in Example 8.1, $y = -4x + 9$, $y(1.9) \approx 1.4$.

If $x = 2.1$, solving for y requires solving the equation $y^3 - 2y + 1.41 = 0$. Using the equation of the tangent line found in Example 8.1, $y = -4x + 9$, $y(2.1) \approx 0.6$.

40. Using the tangent line
$y = \frac{7}{6}(x - 2) - 2$,
we find approximate points $(1.9, -2.1167)$ and $(2.1, -1.8833)$.

41. Both of the points $(-3, 0)$ and $(0, 3)$ are on the curve:
$0^2 = (-3)^3 - 6(-3) + 9 = -27 + 18 + 9$
$3^2 = (0)^3 - 6(0) + 9 = 9$
The equation of the line through these points has slope
$\frac{0 - 3}{-3 - 0} = \frac{-3}{-3} = 1$
and y-intercept 3, so $y = x + 3$. This line intersects the curve at:
$y^2 = x^3 - 6x + 9$
$(x + 3)^2 = x^3 - 6x + 9$
$x^2 + 6x + 9 = x^3 - 6x + 9$
$x^3 - 12x - x^2 = 0$
$x(x^2 - x - 12) = 0$
Therefore, $x = 0$, -3 or 4 and so the

third point is $(4, 7)$.



42. $3^2 = (-1)^3 - 6(-1) + 4$ is true.

$2yy' = 3x^2 - 6$, so $y' = \frac{3x^2-6}{2y}$, and at $(-1, 3)$ the slope is $-\frac{1}{2}$. The line is $y = -\frac{1}{2}(x+1) + 3$.

To find the other point of intersection, substitute the equation of the line into the equation for the elliptic curve and simplify: $(-\frac{1}{2}x+\frac{5}{2})^2 = x^3-6x+4$
$x^2 - 10x + 25 = 4x^3 - 24x + 16$
$4x^3 - x^2 - 14x - 9 = 0$
We know already that $x = -1$ is a solution (actually a double solution), so we can factor out $(x+1)$. Long division yields $(x+1)^2(4x-9)$. The second point has x-coordinate $9/4$, which can be substituted into the equation for the line to get $y = 11/8$.

43. For the inverse hyperbolic tangent function,
$y = \tanh^{-1} x \Longleftrightarrow x = \tanh y$
Differentiating both sides of $x = \tanh y$ implicitly, we obtain

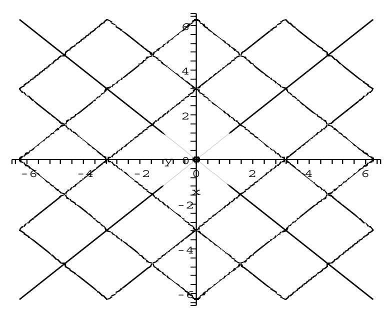
$$1 = \frac{\left(e^y + e^{-y}\right)^2 - \left(e^y - e^{-y}\right)^2}{\left(e^y + e^{-y}\right)^2} y'$$
$$= \left(1 - \frac{\left(e^y - e^{-y}\right)^2}{\left(e^y + e^{-y}\right)^2}\right) y'$$
$$= \left(1 - \left[\frac{e^y - e^{-y}}{e^y + e^{-y}}\right]^2\right) y'$$
$$= \left(1 - [\tanh y]^2\right) y'$$
$$= \left(1 - x^2\right) y'$$
$$y' = \frac{1}{1-x^2}$$

For the inverse hyperbolic cotangent function,
$y = \coth^{-1} x \Longleftrightarrow x = \coth y$
Differentiating both sides of $x = \coth y$ implicitly, we obtain

$$1 = \frac{\left(e^y - e^{-y}\right)^2 - \left(e^y + e^{-y}\right)^2}{\left(e^y - e^{-y}\right)^2} y'$$
$$= \left(1 - \frac{\left(e^y + e^{-y}\right)^2}{\left(e^y - e^{-y}\right)^2}\right) y'$$
$$= \left(1 - \left[\frac{e^y + e^{-y}}{e^y - e^{-y}}\right]^2\right) y'$$
$$= \left(1 - [\coth y]^2\right) y'$$
$$= \left(1 - x^2\right) y'$$
$$y' = \frac{1}{1-x^2}$$

The derivative formulas are not identical because their domains are different. The domain of the inverse hyperbolic tangent function and its derivative is $|x| < 1$, and the domain of the inverse hyperbolic cotangent function and its derivative is $|x| > 1$.

44. Graph of points satisfying $(\cos x)^2 + (\sin y)^2 = 1$:



Notice that for a point (x, y) to satisfy this equation, it must be on a line of the form $y = \pm x + k\pi$ for some integer k. Taking the derivative implicitly gives

$$\frac{dy}{dx} = \frac{\sin x \cos x}{\sin y \cos y}.$$

For any point (x, y) satisfying the original equation, this derivative is always ± 1, indicating that these line segments are indeed straight.

45. $y = \sin^{-1} x + \cos^{-1} x$
$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} + \dfrac{-1}{\sqrt{1-x^2}} = 0$
Therefore, $y = c$, where c is a constant. To determine c, substitute any convenient value of x, such as $x = 0$.
$\sin^{-1} x + \cos^{-1} x = c$
$\sin^{-1} 0 + \cos^{-1} 0 = c$
$0 + \dfrac{\pi}{2} = c$
Thus,
$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$.

46. $y = \sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+1}}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{\sqrt{1-\frac{x^2}{x^2+1}}} \cdot \left(\frac{\sqrt{x^2+1} - x(1/2)(x^2+1)^{-1/2}(2x)}{x^2+1}\right)$$
$$= \frac{1-\frac{x^2}{x^2+1}}{\sqrt{1-\frac{x^2}{x^2+1}}} \cdot \frac{\sqrt{x^2+1}}{x^2+1}$$
$$= \frac{\sqrt{1-\frac{x^2}{x^2+1}}}{\sqrt{x^2+1}} \cdot \left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{1+x^2} = \frac{d}{dx}\tan^{-1} x$$

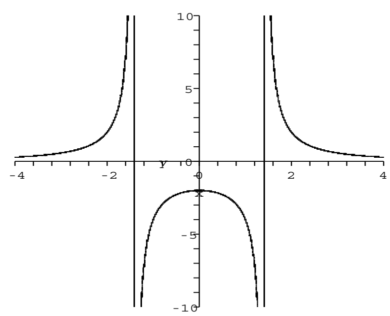
Thus, if we set
$y = \sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right) - \tan^{-1} x$,
then $\frac{dy}{dx} = 0$ so $y = c$ for some constant c. Substitute $x = 0$ into the above expression to find $c = 0$ and so
$\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right) = \tan^{-1} x$.

47. $\dfrac{d}{dx}(x^2 y - 2y) = \dfrac{d}{dx}(4)$
$2xy + x^2 y' - 2y' = 0$
$y'(x^2 - 2) = -2xy$
$y' = \dfrac{-2xy}{x^2-2}$

The derivative is undefined at $x = \pm\sqrt{2}$, suggesting that there might be vertical tangent lines at these points. Similarly, $y' = 0$ at $y = 0$, suggesting that there might be a horizontal tangent line at this point.

However, plugging $x = \pm\sqrt{2}$ into the original equation gives $0 = 4$, a contradiction which shows that there are no points on this curve with x value $\pm\sqrt{2}$. Likewise, plugging $y = 0$ into the original equation gives $0 = 4$. Again, this is a contradiction which shows that there are no points on the graph with y value of 4.

Sketching the graph, we see that there is a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm\sqrt{2}$.



48. For the first type of curve, $y + xy' = 0$, and $y' = -y/x$.

For the second type of curve, $2x - 2yy' = 0$, and $y' = x/y$.

At any point of intersection, the tangent line to the first curve is perpendicular to the tangent line to the second.

49. If $y_1 = c/x$, then $y_1' = -c/x^2 = -y_1/x$. If $y_2^2 = x^2 + k$, then $2y_2(y_2') =$

$2x$ and $y_2' = x/y_2$. If we are at a particular point (x_0, y_0) on both graphs, this means $y_1(x_0) = y_0 = y_2(x_0)$ and
$y_1' \cdot y_2' = \left(\frac{-y_0}{x_0}\right) \cdot \left(\frac{x_0}{y_0}\right) = -1$
This means that the slopes are negative reciprocals and the curves are orthogonal.

50. For the first type of curve, $2x + 2yy' = c$, and
$y' = \frac{c - 2x}{2y}$.
For the second type of curve, $2x + 2yy' = ky'$, and
$y' = \frac{2x}{k - 2y}$.
Multiply the first y' by x/x and the second by y/y. This gives
$y' = \frac{cx - 2x^2}{2xy} = \frac{y^2 - x^2}{2xy}$, and
$y' = \frac{2xy}{ky - 2y^2} = \frac{2xy}{x^2 - y^2}$.
These are negative reciprocals of each other, so the families of curves are orthogonal.

51. For the first type of curve, $y' = 3cx^2$.
For the second type of curve, $2x + 6yy' = 0$, and
$y' = \frac{-2x}{6y} = \frac{-x}{3y}$
$= \frac{-x}{3cx^3} = \frac{-1}{3cx^2}$.
These are negative reciprocals of each other, so the families of curves are orthogonal.

52. For the first type of curve, $y' = 4cx^3$.
For the second type of curve, $2x + 8yy' = 0$, and
$y' = \frac{-2x}{8y} = \frac{-x}{4y}$
$= \frac{-x}{4cx^4} = \frac{-1}{4cx^3}$.
These are negative reciprocals of each other, so the families of curves are orthogonal.

53. Conjecture: The family of functions $\{y_1 = cx^n\}$ is orthogonal to the family of functions $\{x^2 + ny_2^2 = k\}$ whenever $n \neq 0$.
If $y_1 = cx^n$, then $y_1' = cnx^{n-1} = ny_1/x$. If $ny_2^2 = -x^2 + k$, then $2ny_2(y_2') = -2x$ and $y_2' = -x/ny_2$. If we are at a particular point (x_0, y_0) on both graphs, this means $y_1(x_0) = y_0 = y_2(x_0)$ and
$y_1' \cdot y_2' = \left(\frac{ny_0}{x_0}\right) \cdot \left(-\frac{x_0}{ny_0}\right) = -1.$
This means that the slopes are negative reciprocals and the curves are orthogonal.

54. The equation for the circle is
$x^2 + (y - c)^2 = r^2$.
Differentiating implicitly gives
$2x + 2(y - c) \cdot y' = 0$ so
$y' = \frac{-x}{y - c}$.
At the point of tangency the derivatives must be the same. Since the derivative of $y = x^2$ is $2x$, we must solve the equation
$2x = \frac{-x}{y - c}$.
This gives $y = c - 1/2$, as desired. Since $y = x^2$, plugging $y = c - 1/2$ into the equation for the circle gives
$c - 1/2 + (c - 1/2 - c)^2 = r^2$
$c - 1/2 + 1/4 = r^2$
$c = r^2 + 1/4$.

55. In example 8.6, we are given
$\theta'(d) = \frac{2(-130)}{4 + d^2}$.
Setting this equal to -3 and solving for d gives $d^2 = 82 \Rightarrow d \approx 9$ft. The batter can track the ball after they would have to start swinging (when the ball is 30 feet away), but not all the way to home plate.

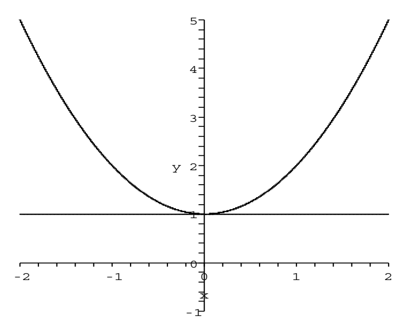
56. From Example 8.6, the rate of change of the angle is
$\theta'(t) = \dfrac{1}{1+\left[\frac{d(t)}{2}\right]^2} \cdot \dfrac{d'(t)}{2}$.
Given a maximum rotational rate of $\theta'(t) = -3$ (radians/second), the distance from the plate at which a player can track the ball can be obtained by solving the equation
$-3 = \dfrac{2d'(t)}{4+[d(t)]^2}$
for $d(t)$ in terms of $d'(t)$. This leads to
$d(t) = \dfrac{\sqrt{-6 \cdot d'(t) - 36}}{3}$,
if $d'(t) \le -6$ which may be reasonable since the distance is decreasing as the ball approaches the plate. We get $d(t) = 4$ for $d'(t) = -30$ ft/sec and $d(t) = 9.45$ for $d'(t) = -140$ ft/sec. This would mean a player can track the ball to within 4 feet from the plate in slowpitch, but only to within 9.45 feet from the plate in the major leagues.

57. The viewing angle is given by the formula
$\theta(x) = \tan^{-1}(3/x) - \tan^{-1}(1/x)$.
This will be maximum where the derivative is zero.
$\theta'(x) = \dfrac{1}{1+(3/x)^2} \cdot \dfrac{-3}{x^2} - \dfrac{1}{1+1/x^2} \cdot \dfrac{-1}{x^2}$
$= \dfrac{1}{1+x^2} - \dfrac{3}{9+x^2}$.
This is zero when
$$\frac{1}{1+x^2} = \frac{3}{9+x^2} \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}.$$

58. If A is the viewing angle formed between the rays from the person's eye to the top of the frame and to the bottom of the frame, and if x is the distance between the person and the wall, then since the frame extends from 6 to 8 feet, we have $\tan A = \dfrac{2}{x}$, or $A = \arctan(2/x)$. Then,
$\dfrac{dA}{dx} = \dfrac{1}{1+\left(\frac{2}{x}\right)^2} \cdot \left(\dfrac{-2}{x^2}\right) = \dfrac{-2}{x^2+4}$.
Since the derivative is negative, the angle is a decreasing function of x. Strictly speaking, $\arctan(2/x)$ is undefined at $x = 0$ but $\arctan(2/x) \to \pi/2$ as $x \to 0$. The angle A continues to enlarge (up to a right angle) as x decreases to zero. In this case the maximal viewing angle is not a feasible one.

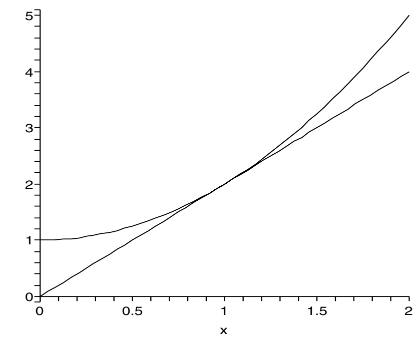
2.9 The Mean Value Theorem

1. $f(x) = x^2 + 1, [-2, 2]$
$f(-2) = 5 = f(2)$
As a polynomial, $f(x)$ is continuous on $[-2, 2]$, differentiable on $(-2, 2)$, and the conditions of Rolle's Theorem hold. There exists $c \in (-2, 2)$ such that $f'(c) = 0$. But $f'(c) = 2c$, $\Rightarrow c = 0$.



2. $f(x) = x^2 + 1, [0, 2]$
$f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the conditions of the Mean Value Theorem hold. We need to find c so that
$f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5-1}{2-0} = 2$.

$f'(x) = 2x = 2$ when $x = 1$, so $c = 1$.



3. $f(x) = x^3 + x^2$, on $[0, 1]$, with $f(0) = 0$, $f(1) = 2$. As a polynomial $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Since the conditions of the Mean Value Theorem hold there exists a number $c \in (0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{2 - 0}{1 - 0} = 2.$$

But $f'(c) = 3c^2 + 2c$.

$\Rightarrow 3c^2 + 2c = 2$,

$3c^2 + 2c - 2 = 0$.

By the quadratic formula

$$c = \frac{-2 \pm \sqrt{2^2 - 4(3)(-2)}}{2(3)}$$

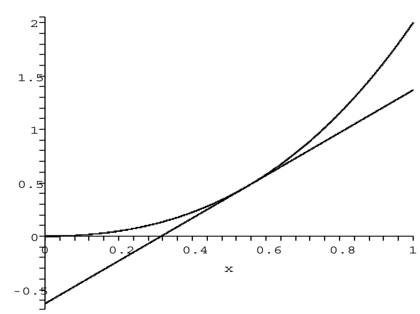
$$= \frac{-2 \pm \sqrt{28}}{6}$$

$$= \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}$$

$\Rightarrow c \approx -1.22$ or $c \approx 0.55$

But since $-1.22 \notin (0, 1)$ we accept only the other alternative:

$$c = \frac{-1 + \sqrt{7}}{3} \approx 0.55$$

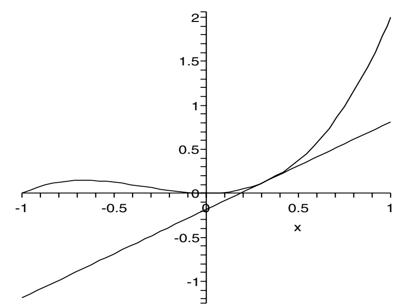


4. $f(x) = x^3 + x^2, [-1, 1]$

$f(x)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ so the conditions of the Mean Value Theorem hold. We need to find c so that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1.$$

$f'(x) = 3x^2 + 2x = 1$ when $x = -1$ and $x = \frac{1}{3}$, so $c = \frac{1}{3}$.



5. $f(x) = \sin x$, $[0, \pi/2]$,

$f(0) = 0$, $f(\pi/2) = 1$.

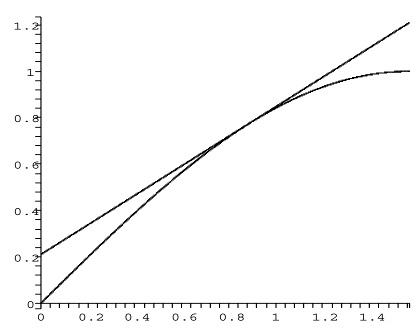
As a trig function, $f(x)$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. The conditions of the Mean Value Theorem hold, and there exists $c \in (0, \pi/2)$ such that

$$f'(c) = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0}$$

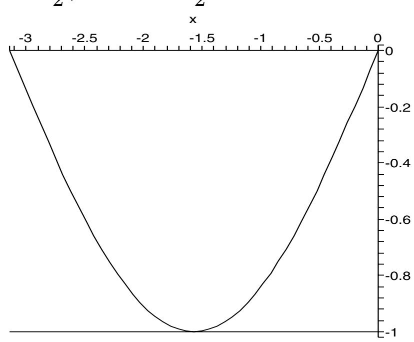
$$= \frac{1 - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}.$$

But $f'(c) = \cos(c)$ and c is to be in the first quadrant, therefore

$$c = \cos^{-1}\left(\frac{2}{\pi}\right) \approx .88$$



6. $f(x) = \sin x, [-\pi, 0]$
$f(x)$ is continuous on $[-\pi, 0]$ and differentiable on $(-\pi, 0)$. Also,
$\sin(-\pi) = 0 = \sin(0)$
so the conditions of Rolle's Theorem hold. We need to find c so that
$f'(c) = 0$.
$f'(x) = \cos x = 0$ on $(-\pi, 0)$ when $x = -\frac{\pi}{2}$, so $c = \frac{-\pi}{2}$.



7. If $f'(x) > 0$ for all x then for each (a, b) with $a < b$ we know there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) > 0.$$

$a < b$ makes the denominator positive, and so we must have the numerator also positive, which implies $f(a) < f(b)$.

8. Let $a < b$. f is differentiable on (a, b) and continuous on $[a, b]$, since it is differentiable for all x. This means that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$. Therefore $f(b) - f(a) = f'(c)(b - a)$ is negative, and $f(a) > f(b)$.

9. $f'(x) = 3x^2 + 5$. This is positive for all x, so $f(x)$ is increasing.

10. $f'(x) = 5x^4 + 9x^2 \geq 0$ for all x. $f' = 0$ only at $x = 0$, so $f(x)$ is increasing.

11. $f'(x) = -3x^2 - 3$. This is negative for all x, so $f(x)$ is decreasing.

12. $f'(x) = 4x^3 + 4x$ is negative for negative x, and positive for positive x, so $f(x)$ is neither an increasing function nor a decreasing function.

13. $f'(x) = e^x$. This is positive for all x, so $f(x)$ is increasing.

14. $f'(x) = -e^{-x} < 0$ for all x, so $f(x)$ is a decreasing function.

15. $f'(x) = \frac{1}{x}$

$f'(x) > 0$ for $x > 0$, that is, for all x in the domain of f. So $f(x)$ is increasing.

16. $f'(x) = \frac{1}{x^2} \cdot 2x = 2/x$ is negative for negative x, and positive for positive x, so $f(x)$ is neither an increasing function nor a decreasing function.

17. Let $f(x) = x^3 + 5x + 1$. As a polynomial, $f(x)$ is continuous and differentiable for all x, with $f'(x) = 3x^2 + 5$, which is positive for all x so $f(x)$ is strictly increasing for all x. Therefore the equation can have at most one solution.

Since $f(x)$ is negative at $x = -1$ and positive at $x = 1$, and $f(x)$ is continuous, there must be a solution to $f(x) = 0$.

18. The derivative is $3x^2 + 4 > 0$ for all x. Therefore the function is strictly increasing, and so the equation can have at most one solution. Because the function is negative at $x = 0$ and positive at $x = 1$, and continuous, we know the equation has exactly one solution.

19. Let $f(x) = x^4 + 3x^2 - 2$. The derivative is $f'(x) = 4x^3 + 6x$. This is negative for negative x, and positive for positive x so $f(x)$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Since $f(0) = -2 \neq 0$, $f(x)$ can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere and $f(-1) = 2 = f(1)$, therefore $f(x) = 0$ has exactly one solution between $x = -1$ and $x = 0$, exactly one solution between $x = 0$ and $x = 1$, and no other solutions.

20. Let $f(x) = x^4 + 6x^2 - 1$. The derivative is $4x^3 + 12x$. This is negative for negative x, and positive for positive x so $f(x)$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Since $f(0) = -1 \neq 0$, $f(x)$ can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere and $f(-1) = 6 = f(1)$, therefore $f(x) = 0$ has exactly one solution between $x = -1$ and $x = 0$, exactly one solution between $x = 0$ and $x = 1$, and no other solutions.

21. $f(x) = x^3 + ax + b$, $a > 0$. Any cubic (actually any *odd degree*) polynomial heads in opposite directions ($\pm\infty$) as x goes to the oppositely signed infinities, and therefore by the Intermediate Value Theorem has at least one root. For the uniqueness, we look at the derivative, in this case $3x^2 + a$. Because $a > 0$ by assumption, this expression is strictly positive. The function is strictly increasing and can have at most one root.

22. The derivative is $4x^3 + 2ax$. This is negative for negative x, and positive for positive x so $f(x)$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$, and so can have at most one zero for $x < 0$ and one zero for $x > 0$. The function is continuous everywhere, $f(0) = -b$, and $\lim_{x\to\pm\infty} f(x) = \infty$, therefore $f(x)$ has exactly one solution for $x < 0$, exactly one solution for $x > 0$, and no other solutions.

23. $f(x) = x^5 + ax^3 + bx + c$, $a > 0$, $b > 0$

Here is another odd degree polynomial (see #21) with at least one root.

$f'(x) = 5x^4 + 3ax^2 + b$ is evidently strictly positive because of our assumptions about a, b. Exactly as in #21, there can be at most one root.

24. A third degree polynomial $p(x)$ has at least one zero because
$\lim_{x\to-\infty} p(x) = -\lim_{x\to\infty} p(x) = \pm\infty$,
and it is continuous. Say this zero is at $x = c$. Then we know $p(x)$ factors into $p(x) = (x - c)q(x)$, where $q(x)$ is a quadratic polynomial. Quadratic polynomials have at most two zeros, so $p(x)$ has at most three zeros.

25. The average velocity on $[a, b]$ is
$\dfrac{s(b) - s(a)}{b - a}$
By the Mean Value theorem, there exists a $c \in (a, b)$ such that
$s'(c) = \dfrac{s(b) - s(a)}{b - a}$
Thus, the instantaneous velocity at $t = c$ is equal to the average velocity between times $t = a$ and $t = b$.

26. Let $f(t)$ be the distance the first runner has gone after time t and let $g(t)$ be the distance the second runner has gone after time t. The functions $f(t)$ and $g(t)$ will be continuous and differentiable. Let $h(t) = f(t) - g(t)$.

At $t = 0$, $f(0) = 0$ and $g(0) = 0$ so $h(0) = 0$. At $t = a$, $f(a) > g(a)$ so $h(a) > 0$. Similarly, at $t = b$, $f(b) < g(b)$ so $h(b) < 0$. Thus, by the Intermediate Value Theorem, there is a time $t = t_0$ for $t_0 \in (a, b)$ where $h(t_0) = 0$. Rolle's Theorem then says that there is a time $t = c$ where $c \in (0, t_0)$ such that $h'(c) = 0$. But $h'(t) = f'(t) - g'(t)$, so $h'(c) = f'(c) - g'(c) = 0$ implies that $f'(c) = g'(c)$, i.e., at time $t = c$ the runners are going exactly the same speed.

27. Define $h(x) = f(x) - g(x)$. Then h is differentiable because f and g are, and $h(a) = h(b) = 0$. Apply Rolle's theorem to h on $[a, b]$ to conclude that there exists $c \in (a, b)$ such that $h'(c) = 0$. Thus, $f'(c) = g'(c)$, and so f and g have parallel tangent lines at $x = c$.

28. As in #27, let $h(x) = f(x) - g(x)$. Again, h is continuous and differentiable on the appropriate intervals because f and g are. Since $f(a) - f(b) = g(a) - g(b)$ (by assumption), we have $f(a) = g(a) - g(b) + f(b)$.
Then

$$\begin{aligned} h(a) &= f(a) - g(a) \\ &= g(a) - g(b) + f(b) - g(a) \\ &= f(b) - g(b) = h(b). \end{aligned}$$

Rolle's Theorem then tells us that there exists $c \in (a, b)$ such that $h'(c) = 0$ or $f'(c) = g'(c)$ so that f and g have parallel tangent lines at $x = c$.

29. $f(x) = x^2$
One candidate: $g_0(x) = kx^3$
Because we require $x^2 = g_0'(x) = 3kx^2$, we must have $3k = 1$, $k = 1/3$.

Most general solution:
$g(x) = g_0(x) + c = x^3/3 + c$
where c is an arbitrary constant.

30. If $g'(x) = 9x^4$, then $g(x) = \frac{9}{5}x^5 + c$ for any constant c.

31. Although the obvious first candidate is $g_0(x) = -1/x$,due to the disconnection of the domain by the discontinuity at $x = 0$, we could add *different* constants, one for negative x, another for positive x. Thus the most general solution is:

$$g(x) = \begin{cases} -1/x + a & \text{for } x > 0 \\ -1/x + b & \text{for } x < 0. \end{cases}$$

32. If $g'(x) = \sqrt{x}$, then $g(x) = \frac{2}{3}x^{3/2} + c$ for any constant c.

33. If $g'(x) = \sin x$, then $g(x) = -\cos x + c$ for any constant c.

34. If $g'(x) = \cos x$, then $g(x) = \sin x + c$ for any constant c.

35. $f(x) = 1/x$ on $[-1, 1]$. We easily see that $f(1) = 1$, $f(-1) = -1$, and $f'(x) = -1/x^2$. If we try to find the c in the interval $(-1, 1)$ for which

$$f'(c) = \frac{f(1) - f(1)}{1 - (-1)} = \frac{1 - (-1)}{1 - (-1)} = 1,$$

the equation would be $-1/c^2 = 1$ or $c^2 = -1$. There is of course no such c, and the explanation is that the function is not defined for $x = 0 \in (-1, 1)$ and so the function is not continuous.

The hypotheses for the Mean Value Theorem are not fulfilled.

36. $f(x)$ is not continuous on $[-1, 2]$, and not differentiable on $(-1, 2)$. Can we find $c \in (-1, 2)$ with

$$\begin{aligned} f'(c) &= \frac{f(2) - f(-1)}{2 - (-1)} \\ &= \frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}? \end{aligned}$$

$f'(x) = -\dfrac{2}{x^3} = -\dfrac{1}{4}$ when $x = 2$.
This is not in $(-1, 2)$, so no c makes

the conclusion of the Mean Value Theorem true.

37. $f(x) = \tan x$ on $[0, \pi]$, $f'(x) = \sec^2(x)$. We know the tangent has a massive discontinuity at $x = \pi/2$, so as in #35, we should not be surprised if the Mean Value Theorem does not apply. As applied to the interval $[0, \pi]$ it would say
$$\sec^2(c) = f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{\tan\pi - \tan 0}{\pi - 0} = 0.$$
But secant $= 1/\text{cosine}$ is never 0 in the interval $(-1, 1)$, so no such c exists.

38. $f(x)$ is not differentiable on $(-1, 1)$. Can we find c with
$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1?$$
$f'(x) = \dfrac{1}{3}x^{-2/3} = 1$ when $x = \pm(\frac{1}{3})^{3/2}$.
These are both in $(-1, 1)$, so we can use either of these as c to make the conclusion of the Mean Value Theorem true.

39. If a derivative g' is positive at a single point $x = b$, then $g(x)$ is an increasing function for x sufficiently near b, i.e., $g(x) > g(b)$ for $x > b$ but sufficiently near b. In this problem, we will apply that remark to f' at $x = 0$, and conclude from $f''(0) > 0$ that $f'(x) > f'(0) = 0$ for $x > 0$ but sufficiently small. This being true about the derivative f', it tells us that f itself is increasing on some interval $(0, a)$ and in particular that $f(x) > f(0) = 0$ for $0 < x < a$. On the other side (the negative side) f' is negative, f is decreasing (to zero) and therefore likewise positive. In summary, $x = 0$ is a genuine relative minimum.

40. The funtion $\cos x$ is continuous and differentiable everywhere, so for any u and v we can apply the Mean Value Theorem to get $\frac{\cos u - \cos v}{u - v} = \sin c$ for some c between u and v. We know $-1 \le \sin x \le 1$, so taking absolute values, we get $|\frac{\cos u - \cos v}{u - v}| \le 1$, or $|\cos u - \cos v| \le |u - v|$.

41. Consider the function $g(x) = x - \sin(x)$, obviously with $g(0) = 0$ and $g'(x) = 1 - \cos(x)$. If there was ever a point $a > 0$ with $\sin(a) \ge a$, ($g(a) \le 0$), then by the MVT applied to g on the interval $[0, a]$, there would be a point c ($0 < c < a$) with
$$g'(c) = \frac{g(a) - g(0)}{a - 0} = \frac{g(a)}{a} \le 0.$$
This would read $1 - \cos(c) = g'(c) \le 0$ or $\cos(c) \ge 1$. The latter condition is possible only if $\cos(c) = 1$ and $\sin(c) = 0$, in which case c (being positive) would be *at minimum* π. But even in this unlikely case we still would have $\sin(a) \le 1 < \pi \le c < a$.

Since $\sin a < a$ for all $a > 0$, we have $-\sin a > -a$ for all $a > 0$, but $-\sin a = \sin(-a)$ so we have $\sin(-a) > -a$ for all $a > 0$. This is the same as saying $\sin a > a$ for all $a < 0$ so in absolute value we have $|\sin a| < |a|$ for all $a \ne 0$.

Thus the only possible solution to the equation $\sin x = x$ is $x = 0$, which we know to be true.

42. The function $\tan^{-1} x$ is continuous and differentiable everywhere, so for any $a \ne 0$ we can apply the Mean Value Theorem to get $\frac{\tan^{-1} a - \tan^{-1} 0}{a - 0} = \frac{1}{1 + c^2}$ for some c between 0 and a. Taking absolute values, we get $|\frac{\tan^{-1} a}{a}| =$

$|\frac{1}{1+c^2}| < 1$, so $|\tan^{-1} a| < |a|$ for $a \neq 0$. This means that the only solution to $\tan^{-1} x = x$ is $x = 0$.

43. Since the inverse sine function is increasing on the interval $[0, 1)$ (it has a positive derivative) we start from the previously proven inequality $\sin(x) < x$ for $0 < x$. If indeed $0 < x < 1$, we can apply the inverse sine and conclude
$x = \sin^{-1}(\sin(x)) < \sin^{-1}(x)$.

44. The function $\tan x$ is continuous and differentiable for $|x| < \pi/2$, so for any $a \neq 0$ in $(-\pi/2, \pi/2)$ we can apply the Mean Value Theorem to get $\frac{\tan a - \tan 0}{a-0} = \sec^2 c$ for some c between 0 and a. Taking absolute values, we get $|\frac{\tan a}{a}| = |\sec^2 c| > 1$, so $|\tan a| > |a|$ for $a \neq 0$. Of course $\tan 0 = 0$, so $|\tan a| \geq |a|$ for all $|a| < \pi/2$.

45. $f(x) = \begin{cases} 2x & x \leq 0 \\ 2x - 4 & x > 0 \end{cases}$
$f(x) = 2x - 4$ is continuous and differentiable on $(0, 2)$. Also, $f(0) = 0 = f(2)$. But $f'(x) \equiv 2$ on $(0, 2)$, so there is no c such that $f'(c) = 0$. Rolle's Theorem requires that $f(x)$ be continuous on the closed interval, but we have a jump discontinuity at $x = 0$, which is enough to preclude the applicability of Rolle's.

46. $f(x) = x^2$ is a counter-example. The flaw in the proof is that we do not have $f'(c) = 0$.

Ch. 2 Review Exercises

1. $\dfrac{3.4 - 2.6}{1.5 - 0.5} = \dfrac{0.8}{1} = 0.8$

2. C (large negative), B (small negative), A (small positive), and D (large positive)

3.
$$\begin{aligned} f'(2) &= \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2 - 2(2+h) - (0)}{h} \\ &= \lim_{h\to 0} \frac{4 + 4h + h^2 - 4 - 2h}{h} \\ &= \lim_{h\to 0} \frac{2h + h^2}{h} \\ &= \lim_{h\to 0} 2 + h = 2 \end{aligned}$$

4.
$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{1 + \frac{1}{x} - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{\frac{-(x-1)}{x}}{x - 1} \\ &= \lim_{x\to 1} \frac{-1}{x} = -1 \end{aligned}$$

5.
$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h\to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2} \end{aligned}$$

6.
$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^3 - 2x}{x} \\ &= \lim_{x\to 0} x^2 - 2 = -2 \end{aligned}$$

7.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 + (x+h) - (x^3 + x)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h} \end{aligned}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1$$
$$= 3x^2 + 1$$

8. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$
$$= \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3x-3(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-3h}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-3}{x(x+h)} = \frac{-3}{x^2}$$

9. The point is $(1, 0)$. $y' = 4x^3 - 2$ so the slope at $x = 1$ is 2, and the equation of the tangent line is $y - 0 = 2(x - 1)$ or $y = 2x - 2$.

10. The point is $(0, 0)$. $y' = 2\cos 2x$, so the slope at $x = 0$ is 2, and the equation of the tangent line is $y = 2x$.

11. The point is $(0, 3)$. $y' = 6e^{2x}$, so the slope at $x = 0$ is 6, and the equation of the tangent line is $y - 3 = 6(x - 0)$ or $y = 6x + 3$.

12. The point is $(0, 1)$. $y' = \dfrac{2x}{2\sqrt{x^2+1}}$, so the slope at $x = 0$ is 0, and the equation of the tangent line is $y = 1$.

13. Find the slope to $y - x^2y^2 = x - 1$ at $(1, 1)$.
$$\frac{d}{dx}(y - x^2y^2) = \frac{d}{dx}(x - 1)$$
$$y' - 2xy^2 - x^2 2y \cdot y' = 1$$
$$y'(1 - x^2 2y) = 1 + 2xy^2$$
$$y' = \frac{1 + 2xy^2}{1 - 2x^2y}$$
At $(1, 1)$:
$$y' = \frac{1 + 2(1)(1)^2}{1 - 2(1)^2(1)} = \frac{3}{-1} = -3$$
The equation of the tangent line is $y - 1 = -3(x - 1)$ or $y = -3x + 4$.

14. Implicitly differentiating:
$2yy' + e^y + xe^y y' = -1$, and
$y' = \dfrac{-1 - e^y}{2y + xe^y}$.
At $(2, 0)$ the slope is -1, and the equation of the tangent line is $y = -(x - 2)$.

15. $s(t) = -16t^2 + 40t + 10$
$v(t) = s'(t) = -32t + 40$
$a(t) = v'(t) = -32$

16. $s(t) = -9.8t^2 - 22t + 6$
$v(t) = s'(t) = -19.6t - 22$
$a(t) = s''(t) = -19.6$

17. $s(t) = 10e^{-2t}\sin 4t$
$$v(t) = s'(t)$$
$$= 10\left(-2e^{-2t}\sin 4t + 4e^{-2t}\cos 4t\right)$$
$$a(t) = v'(t)$$
$$= 10\cdot(-2)\left[-2e^{-2t}\sin 4t + e^{-2t}4\cos 4t\right]$$
$$+10(4)\cdot\left[-2e^{-2t}\cos 4t - e^{-2t}4\sin 4t\right]$$
$$= 160e^{-2t}\cos 4t - 120e^{-2t}\sin 4t$$

18. $s(t) = \sqrt{4t+16} - 4$
$$v(t) = s'(t) = \frac{4}{2\sqrt{4t+16}}$$
$$= \frac{2}{\sqrt{4t+16}}$$
$$a(t) = s''(t)$$
$$= \frac{-2\cdot 4}{2(4t+16)^{3/2}} = \frac{-4}{(4t+16)^{3/2}}$$

19. $v(t) = s'(t) = -32t + 40$
$v(1) = -32(1) + 40 = 8$
The ball is rising.
$v(2) = -32(2) + 40 = -24$
The ball is falling.

20. $v(t) = s'(t) = 20e^{-2t}(2\cos 4t - \sin 4t)$
$v(0) = 40$ and $v(\pi) = 40e^{-2\pi} \approx 0.075$.
The spring is moving in the same direction, much faster at $t = 0$.

21. (a) $m_{\text{sec}} = \dfrac{f(2) - f(1)}{2 - 1}$
$$= \frac{\sqrt{3} - \sqrt{2}}{1} \approx .318$$

(b) $m_{\text{sec}} = \dfrac{f(1.5) - f(1)}{1.5 - 1}$
$= \dfrac{\sqrt{2.5} - \sqrt{2}}{.5} \approx .334$

(c) $m_{\text{sec}} = \dfrac{f(1.1) - f(1)}{1.1 - 1}$
$= \dfrac{\sqrt{2.1} - \sqrt{2}}{.1} \approx .349$

Best estimate for the slope of the tangent line: (c) (approximately .349).

22. Point at $x = 1$ is $(1, 7.3891)$.

(a) $m_{\text{sec}} = \dfrac{f(2) - f(1)}{2 - 1}$
$= \dfrac{e^4 - e^2}{1} \approx 47.2091$

(b) $m_{\text{sec}} = \dfrac{f(1.5) - f(1)}{1.5 - 1}$
$= \dfrac{e^3 - e^2}{.5} \approx 25.3928$

(c) $m_{\text{sec}} = \dfrac{f(1.1) - f(1)}{1.1 - 1}$
$= \dfrac{e^{2.2} - e^2}{.1} \approx 16.3590$

Best estimate for the slope of the tangent line: (c) (approximately 16.3590).

23. $f'(x) = 4x^3 - 9x^2 + 2$

24. $f'(x) = \dfrac{2}{3}x^{-1/3} - 8x$

25. $f'(x) = -\dfrac{3}{2}x^{-3/2} - 10x^{-3}$
$= \dfrac{-3}{2x\sqrt{x}} - \dfrac{10}{x^3}$

26. $f'(x) = \dfrac{\sqrt{x}(-3 + 2x)}{x} - \dfrac{(2 - 3x + x^2)\frac{1}{2\sqrt{x}}}{x}$

27. $f'(t) = 2t(t+2)^3 + t^2 \cdot 3(t+2)^2 \cdot 1$
$= 2t(t+2)^3 + 3t^2(t+2)^2$
$= t(t+2)^2(5t+4)$

28. $f'(t) = 2t(t^3 - 3t + 2) + (t^2 + 1)(3t^2 - 3)$

29. $g'(x) = \dfrac{(3x^2 - 1) \cdot 1 - x(6x)}{(3x^2 - 1)^2}$
$= \dfrac{3x^2 - 1 - 6x^2}{(3x^2 - 1)^2}$
$= -\dfrac{3x^2 + 1}{(3x^2 - 1)^2}$

30. $g(x) = 3x - \dfrac{1}{x}$
$g'(x) = 3 + \frac{1}{x^2}$

31. $f'(x) = 2x \sin x + x^2 \cos x$

32. $f'(x) = 2x \cos x^2$

33. $f'(x) = \sec^2 \sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}$

34. $f'(x) = \dfrac{1}{2\sqrt{\tan x}} \sec^2 x$

35. $f'(t) = \csc t \cdot 1 + t \cdot (-\csc t \cdot \cot t)$
$= \csc t - t \csc t \cot t$

36. $f'(t) = 3 \cos 3t \cos 4t - 4 \sin 3t \sin 4t$

37. $u'(x) = 2e^{-x^2}(-2x) = -4xe^{-x^2}$

38. $u'(x) = 2(2e^{-x})(-2e^{-x}) = -8e^{-2x}$

39. $f'(x) = 1 \cdot \ln x^2 + x \cdot \frac{1}{x^2} \cdot 2x$
$= \ln x^2 + 2$

40. $f'(x) = \dfrac{1}{2\sqrt{\ln x + 1}} \cdot \dfrac{1}{x}$

41. $f'(x) = \dfrac{1}{2\sqrt{\sin 4x}} \cdot \cos 4x \cdot 4$

42. $f'(x) = 2 \cos 3x(-3 \sin 3x)$

43. $f'(x) = 2\left(\dfrac{x+1}{x-1}\right) \dfrac{d}{dx}\left(\dfrac{x+1}{x-1}\right)$
$= 2\left(\dfrac{x+1}{x-1}\right) \dfrac{(x-1) - (x+1)}{(x-1)^2}$
$= 2\left(\dfrac{x+1}{x-1}\right) \dfrac{-2}{(x-1)^2}$
$= \dfrac{-4(x+1)}{(x-1)^3}$

44. $f'(x) = \dfrac{3}{2\sqrt{3x}} e^{\sqrt{3x}}$

45. $f'(t) = e^{4t} \cdot 1 + te^{4t} \cdot 4 = (1+4t)e^{4t}$

46. $f'(x) = \dfrac{(x-1)^2 6 - 6x \cdot 2(x-1)}{(x-1)^4}$

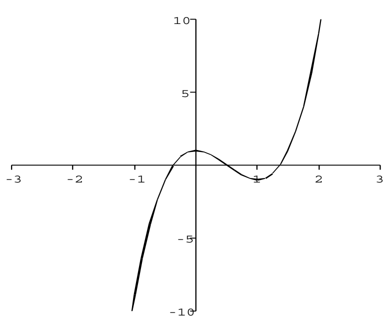
47. $\dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2$

48. $\dfrac{-1}{\sqrt{1-(x^2)^2}} \cdot 2x$

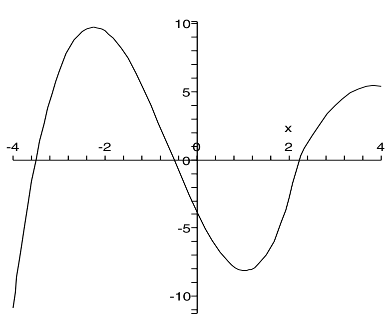
49. $\dfrac{1}{1+(\cos 2x)^2} \cdot (-2\sin 2x)$

50. $\dfrac{1}{3x^2\sqrt{(3x^2)^2-1}} \cdot 6x$

51. The derivative should look roughly like:



52. The derivative should look roughly like:



53. $f(x) = x^4 - 3x^3 + 2x^2 - x - 1$
$f'(x) = 4x^3 - 9x^2 + 4x - 1$
$f''(x) = 12x^2 - 18x + 4$

54. $f(x) = (x+1)^{1/2}$
$f'(x) = \dfrac{1}{2}(x+1)^{-1/2}$
$f''(x) = \dfrac{-1}{4}(x+1)^{-3/2}$
$f'''(x) = \dfrac{3}{8}(x+1)^{-5/2}$

55. $f(x) = xe^{2x}$
$f'(x) = 1 \cdot e^{2x} + xe^{2x} \cdot 2 = e^{2x} + 2xe^{2x}$
$f''(x) = e^{2x} \cdot 2 + 2 \cdot (e^{2x} + 2xe^{2x})$
$= 4e^{2x} + 4xe^{2x}$
$f'''(x) = 4e^{2x} \cdot 2 + 4\left(e^{2x} + 2xe^{2x}\right)$
$= 12e^{2x} + 8xe^{2x}$

56. $f(x) = 4(x+1)^{-1}$
$f'(x) = -4(x+1)^{-2}$
$f''(x) = 8(x+1)^{-3}$

57. $f(x) = \tan x$
$f'(x) = \sec^2 x$
$f''(x) = 2\sec x \cdot \sec x \tan x$
$= 2\sec^2 x \tan x$

58. $f(x) = x^6 - 3x^4 + 2x^3 - 7x + 1$
$f'(x) = 6x^5 - 12x^3 + 6x^2 - 7$
$f''(x) = 30x^4 - 36x^2 + 12x$
$f'''(x) = 120x^3 - 72x + 12$
$f^{(4)}(x) = 360x^2 - 72$

59. $f(x) = \sin 3x$
$f'(x) = \cos 3x \cdot 3 = 3\cos 3x$
$f''(x) = 3(-\sin 3x \cdot 3) = -9\sin 3x$
$f'''(x) = -9\cos 3x \cdot 3 = -27\cos 3x$
$f^{(26)}(x) = -3^{26}\sin 3x$

60. For $f(x) = e^{-2x}$, each derivative multiplies by a factor of -2, so $f^{(31)}(x) = (-2)^{31}e^{-2x}$.

61. $R(t) = P(t)Q(t)$
$R'(t) = Q'(t) \cdot P(t) + Q(t) \cdot P'(t)$
$P(0) = 2.4(\$)$
$Q(0) = 12$ (thousands)
$Q'(t) = -1.5$ (thousands per year)
$P'(t) = 0.1$ ($ per year)
$R'(0) = (-1.5) \cdot (2.4) + 12 \cdot (0.1)$
$= -2.4$ (thousand $ per year)

Revenue is decreasing at a rate of \$2400 per year.

62. The relative rate of change is $\frac{v'(t)}{v(t)}$. $v'(t) = 200(\frac{3}{2})^t \ln\frac{3}{2}$, so the relative rate of change is $\ln\frac{3}{2} \approx 0.4055$, giving an instantaneous percentage rate of change of 40.55%.

63. $f(t) = 4\cos 2t$
$v(t) = f'(t) = 4(-\sin 2t)\cdot 2$
$= -8\sin 2t$

(a) The velocity is zero when $v(t) = -8\sin 2t = 0$, i.e., when $2t = 0, \pi, 2\pi, \ldots$ so when $t = 0, \pi/2, \pi, 3\pi/2, \ldots$
$f(t) = 4$ for $t = 0, \pi, 2\pi, \ldots$
$f(t) = 4\cos 2t = -4$ for $t = \pi/2, 3\pi/2, \ldots$
The position of the spring when the velocity is zero is 4 or -4.

(b) The velocity is a maximum when $v(t) = -8\sin 2t = 8$, i.e., when $2t = 3\pi/2, 7\pi/2, \ldots$ so $t = 3\pi/4, 7\pi/4, \ldots$
$f(t) = 4\cos 2t = 0$ for $t = 3\pi/4, 7\pi/4, \ldots$
The position of the spring when the velocity is at a maximum is zero.

(c) Velocity is at a minimum when $v(t) = -8\sin 2t = -8$, i.e., when $2t = \pi/2, 5\pi/2, \ldots$ so $t = \pi/4, 5\pi/4, \ldots$
$f(t) = 4\cos 2t = 0$ for $t = \pi/4, 5\pi/4, \ldots$
The position of the spring when the velocity is at a minimum is also zero.

64. The velocity is given by
$f'(t) = -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t$.

65. $\frac{d}{dx}(x^2y - 3y^3) = \frac{d}{dx}(x^2+1)$
$2xy + x^2y' - 3\cdot 3y^2\cdot y' = 2x$
$y'(x^2 - 9y^2) = 2x - 2xy$
$$y' = \frac{2x(1-y)}{x^2 - 9y^2}$$

66. $\frac{d}{dx}(\sin(xy) + x^2) = \frac{d}{dx}(x - y)$
$\cos(xy)(y + xy') + 2x = 1 - y'$
$$y' = \frac{1 - 2x - y\cos(xy)}{x\cos(xy) + 1}.$$

67. $\frac{d}{dx}\left(\frac{y}{x+1} - 3y\right) = \frac{d}{dx}\tan x$
$$\frac{(x+1)y' - y\cdot(1)}{(x+1)^2} - 3y' = \sec^2 x$$
$y'(x+1) - y = (x+1)^2(3y' + \sec^2 x)$
$$y' = \frac{\sec^2 x(x+1)^2 + y}{(x+1)[1 - 3(x+1)]}$$

68. $\frac{d}{dx}(x - 2y^2) = \frac{d}{dx}(3e^{x/y})$
$$1 - 2yy' = 3e^{x/y}\cdot\frac{y - xy'}{y^2}$$
$$1 - 2yy' = \frac{3e^{x/y}}{y} - \frac{3e^{x/y}xy'}{y^2}$$
$$y' = \frac{\frac{3e^{x/y}}{y} - 1}{\frac{3xe^{x/y}}{y^2} - 2y}$$

69. When $x = 0$, $-3y^3 = 1$, $y = \frac{-1}{\sqrt[3]{3}}$ (call this a).

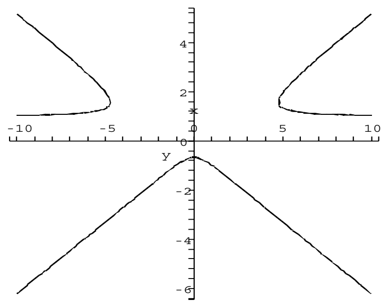
From our formula (#65), we find $y' = 0$ at this point. To find y'', implicitly differentiate the first derivative (second line in #65):

$$2(xy' + y) + (2xy' + x^2y'') - 9\left[2y(y')^2 + y^2y''\right] = 2$$

At $(0, a)$ with $y' = 0$, we find

$2a - 9a^2y'' = 2$,
$$y'' = \frac{-2\sqrt[3]{3}}{9}\left(\sqrt[3]{3} + 1\right)$$

Below is a sketch of the graph of $x^2y - 3y^3 = x^2 + 1$.



70. Plugging in $x = 0$ gives $-2y = 0$ so $y = 0$. Plugging $(0, 0)$ into the formula for y' gives a slope of $-1/2$. Implicitly differentiating the third line of the solution to #37 gives

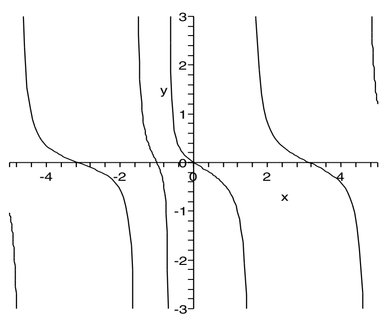
$$y''(x+1) + y' - y' \\ = 2(x+1)(3y' + \sec^2 x) \\ + (x+1)^2(3y'' + 2\sec x \cdot \sec x \tan x)$$

Plugging in $x = 0$, $y = 0$ and $y' = -1/2$ gives

$$y'' = 2(-3/2 + \sec^2(0)) \\ + (1)^2(3y'' + 2\sec^2(0)\tan(0))$$
$$y'' = 1 + 3y''.$$

So at $x = 0$, $y'' = -1/2$.

The graph is:



71. $y' = 3x^2 - 12x = 3x(x-4)$

(a) $y' = 0$ for $x = 0$ $(y = 1)$, and $x = 4$ $(y = -31)$ so there are horizontal tangent lines at $(0, 1)$ and $(4, -31)$.

(b) y' is defined for all x, so there are no vertical tangent lines.

72. $y' = \dfrac{2}{3}x^{-1/3}$

(a) The derivative is never 0, so the tangent line is never horizontal.

(b) The derivative is undefined at $x = 0$ and the tangent is vertical there.

73. $\dfrac{d}{dx}(x^2y - 4y) = \dfrac{d}{dx}x^2$
$2xy + x^2y' - 4y' = 2x$
$y'(x^2 - 4) = 2x - 2xy$
$y' = \dfrac{2x - 2xy}{x^2 - 4} = \dfrac{2x(1-y)}{x^2-4}$

(a) $y' = 0$ when $x = 0$ or $y = 1$.
At $y = 1$, $x^2 \cdot 1 - 4 \cdot 1 = x^2$
$x^2 - 4 = x^2$
This is impossible, so there is no x for which $y = 1$.
At $x = 0$, $0^2 \cdot y - 4y = 0^2$, so $y = 0$.
Therefore, there is a horizontal tangent line at $(0, 0)$.

(b) y' is not defined when $x^2 - 4 = 0$, or $x = \pm 2$. At $x = \pm 2$, $4y - 4y = 4$ so the function is not defined at $x = \pm 2$. There are no vertical tangent lines.

74. $y' = 4x^3 - 2x = 2x(2x^2 - 1)$.

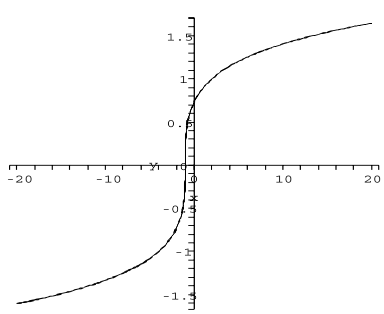
(a) The derivative is 0 at $x = 0$ and $x = \pm\sqrt{\frac{1}{2}}$, and the tangent line is horizontal at those points.

(b) The tangent line is never vertical.

75. $f(x)$ is continuous and differentiable for all x, and $f'(x) = 3x^2 + 7$, which is positive for all x. By Theorem 9.2, if the equation $f(x) = 0$ has two solutions, then $f'(x) = 0$ would have at least one solution, but it has none. We discussed at length (Section 2.9) why every odd degree polynomial has at least one root, so in this case there is exactly one root.

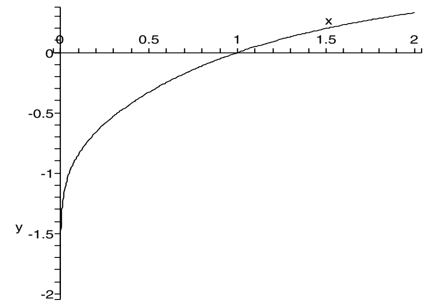
76. The derivative is $4x^3 + 4x$. This is negative for negative x, and positive for positive x. $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, so can have at most one zero for $x < 0$ and one zero for $x > 0$. Since $f(-1) = 0$, $f(1) = 0$ and $f(0) = -3$, $f(x)$ has exactly one solution for $x < 0$, exactly one solution for $x > 0$, and no other solutions.

77. $f(x) = x^5 + 2x^3 - 1$ is a one-to-one function with $f(1) = 2$, $f'(1) = 11$. If g is the name of the inverse, then $g(2) = 1$ and
$g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{11}$.



78. Since $e^{0^3+2\cdot 0} = 1$, the derivative of the inverse at $x = 1$ will be one over the derivative of e^{x^3+2x} at $x = 0$. The derivative of e^{x^3+2x} is $(3x^2+2)e^{x^3+2x}$ and this is 2 when $x = 0$. Therefore the derivative of the inverse to e^{x^3+2x} at $x = 1$ is $1/2$.

The graph is the graph of e^{x^3+2x} reflected across the line $y = x$.



79. Let $a > 0$. We know that $f(x) = \cos x - 1$ is continuous and differentiable on the interval $(0, a)$. Also $f'(x) = \sin x \leq 1$ for all x. The Mean Value Theorem implies that there exists some c in the interval $(0, a)$ such that $f'(c) = \sin c$. But

$$\begin{aligned} f'(c) &= \frac{\cos a - 1 - (\cos 0 - 1)}{a - 0} \\ &= \frac{\cos a - 1}{a}. \end{aligned}$$

Since this is equal to $\sin c$ and $\sin c \leq 1$ for any c, we get that

$$\cos a - 1 \leq a$$

as desired. This works for all positive a, but since $\cos x - 1$ is symmetric about the y axis, we get

$$|\cos x - 1| \leq |x|.$$

They are actually equal at $x = 0$.

80. This is an example of a Taylor polynomial. Later, Taylor's theorem will be used to prove such inequalities. For now, one can use multiple derivatives and argue that the rate of the rate of the rate of change (etc.) increases as one moves left to right through the inequalities.

81. To show that $g(x)$ is continuous at $x = a$, we need to show that the limit as x approaches a of $g(x)$ exists and is equal to $g(a)$. But

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

which is the definition of the derivative of $f(x)$ at $x = a$. Since $f(x)$ is differentiable at $x = a$, we know this limit exists and is equal to $f'(a)$, which, in turn, is equal to $g(a)$. Thus $g(x)$ is continuous at $x = a$.

82. We have

$$\begin{aligned} &f(x) - T(x) \\ &\quad = f(x) - f(a) - f'(a)(x-a) \\ &\quad = \left(\frac{f(x)-f(a)}{x-a} - f'(a)\right)(x-a) \end{aligned}$$

Letting $e(x) = \dfrac{f(x)-f(a)}{x-a} - f'(a)$, we obtain the desired form. Since $f(x)$ is differentiable at $x = a$, we know that

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

so

$$\begin{aligned} \lim_{x\to a} e(x) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a) \\ &= 0. \end{aligned}$$

83. $f(x) = x^2 - 2x$ on $[0, 2]$
$f(2) = 0 = f(0)$
If $f'(c) = \dfrac{f(2)-f(0)}{2-0} = \dfrac{0-0}{2} = 0$
then $2c - 2 = f'(c) = 0$ so $c = 1$.

84. $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the Mean Value Theorem applies. We need to find c so that
$f'(c) = \dfrac{f(2)-f(0)}{2-0} = \dfrac{6-0}{2-0} = 3.$
$f'(x) = 3x^2 - 1 = 3$ when $x = \sqrt{4/3}$,
so $c = 2\sqrt{3}/3$.

85. $f(x) = 3x^2 - \cos x$
One trial: $g_o(x) = kx^3 - \sin x$
$g_o'(x) = 3kx^2 - \cos x$
Need $3k = 3$, $k = 1$, and the general solution is
$g(x) = g_o(x) + c = x^3 - \sin x + c$
for c an arbitrary constant.

86. If $g'(x) = x^3 - e^{2x}$, then $g(x)$ must be
$\dfrac{1}{4}x^4 - \dfrac{1}{2}e^{2x} + c$,
for any constant c.

87. $x = 1$ is to be double root of

$$\begin{aligned} f(x) &= (x^3+1) - [m(x-1)+2] \\ &= (x^3+1-2) - m(x-1) \\ &= (x^3-1) - m(x-1) \\ &= (x-1)\left[x^2+x+1-m\right] \end{aligned}$$

Let $g(x) = x^2 + x + 1 - m$. Then $x = 1$ is a *double* root of f only if $(x - 1)$ is a *factor* of g, in which case $g(1) = 0$. Therefore we require $0 = g(1) = 3 - m$ or $m = 3$. Now
$g(x) = x^2 + x - 2 = (x-1)(x+2)$,
$f(x) = (x-1)g(x) = (x-1)^2(x+2)$
and $x = 1$ is a double root.

The line tangent to the curve $y = x^3+1$ at the point $(1, 2)$ has slope $y' = 3x^2 = 3(1) = 3(= m)$. The equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1(= m(x-1) + 2)$.

88. We are asked to find m so that
$x^3 + 2x - [m(x-2) + 12]$
$= x^3 + (2-m)x + (2m-12)$
has a double root. A cubic with a double root factors as
$(x-a)^2(x-b)$
$= x^3 - (2a+b)x^2 + (2ab+a^2)x - a^2b.$
Equating like coefficients gives a system of equations
$2a + b = 0$,
$2ab + a^2 = 2 - m$, and
$-a^2b = 2m - 12$.
The first equation gives $b = -2a$. Substituting this into the second

equation gives $m = 2 + 3a^2$. Substituting these results into the third equation gives a cubic polynomial in a with zeros $a = -1$ and $a = 2$. This gives two solutions: $m = 5$ and $m = 14$.

$f'(x) = 3x^2 + 2$, so $f'(2) = 14$. The tangent line at $(2, 12)$ is $y = 14(x - 2) + 12$.

The second solution corresponds to the tangent line to $f(x)$ at $x = -1$, which happens to pass through the point $(2, 12)$.