Chapter 1

Limits and Continuity

1.1 A Brief Preview of Calculus

1. The slope appears to be 2.

Second point	m_{sec}
$(2, 5)$	3
$(1.1, 2.21)$	2.1
$(1.01, 2.0201)$	2.01
$(0, 1)$	1
$(0.9, 1.81)$	1.9
$(0.99, 1.9801)$	1.99

2. The slope appears to be 4.

Second point	m_{sec}
$(3, 10)$	5
$(2.1, 5.41)$	4.1
$(2.01, 5.0401)$	4.01
$(1, 2)$	3
$(1.9, 4.61)$	3.9
$(1.99, 4.9601)$	3.99

3. The slope appears to be 0.

Second point	m_{sec}
$(1, 0.5403)$	-0.4597
$(0.1, 0.995)$	-0.05
$(0.01, 0.99995)$	-0.005
$(-1, 0.5403)$	0.4597
$(-0.1, 0.995)$	0.05
$(-0.01, 0.99995)$	0.005

4. The slope appears to be 1.

Second point	m_{sec}
$(1, 0.5403)$	0.9466
$(1.5, 0.0707)$	0.9986
$(1.57, 0.0008)$	1
$(2.5, -0.8011)$	0.8621
$(2, -0.4161)$	0.9695
$(1.6, -0.0292)$	1

5. The slope appears to be 3.

Second point	m_{sec}
$(2, 10)$	7
$(1.1, 3.331)$	3.31
$(1.01, 3.030301)$	3.0301
$(0, 2)$	1
$(0.9, 2.729)$	2.71
$(0.99, 2.970299)$	2.9701

6. The slope appears to be 12.

Second point	m_{sec}
$(3, 27)$	19
$(2.1, 11.261)$	12.61
$(2.01, 10.120601)$	12.0601
$(1, 3)$	7
$(1.9, 8.859)$	11.41
$(1.99, 9.880599)$	11.9401

7. The slope appears to be $\frac{1}{2}$.

Second point	m_{sec}
$(1, \sqrt{2})$	0.4142
$(0.1, 1.0488)$	0.488
$(0.01, 1.004988)$	0.4988
$(-1, 0)$	1
$(-0.1, 0.9487)$	0.513
$(-0.01, 0.99499)$	0.501

8. The slope appears to be 0.25.

Second point	m_{sec}
$(2, 1.7321)$	0.2679
$(2.9, 1.9748)$	0.252
$(2.99, 1.9975)$	0.25
$(4, 2.2361)$	0.2361
$(3.1, 2.0248)$	0.248
$(3.01, 2.0025)$	0.25

9. The slope appears to be 1.

Second point	m_{sec}
(1, e)	1.718282
(0.1, 1.1052)	1.051709
(0.01, 1.0101)	1.005017
(-1, 0.3679)	0.632121
(-0.1, 0.9048)	0.951626
(-0.01, 0.9901)	0.995017

10. The slope appears to be 2.72.

Second point	m_{sec}
(0, 1)	1.7183
(0.9, 2.4596)	2.587
(0.99, 2.6912)	2.71
(2, 7.3891)	4.6708
(1.1, 3.0042)	2.859
(1.01, 2.7456)	2.73

11. The slope appears to be 1.

Second point	m_{sec}
(0.1, -2.3026)	2.5584
(0.9, -0.1054)	1.054
(0.99, -0.01005034)	1.005034
(2, 0.6931)	0.6931
(1.1, 0.09531)	0.9531
(1.01, 0.00995)	0.995

Note that we used 0.1 rather than 0 as an evaluation point because $\ln x$ is not defined at 0.

12. The slope appears to be 0.5.

Second point	m_{sec}
(1, 0)	0.6931
(1.9, 0.6419)	0.512
(1.99, 0.6881)	0.5
(3, 1.0986)	0.4055
(2.1, 0.7419)	0.488
(2.01, 0.6981)	0.5

13. (a)

Left	Right	Length
(0, 1)	(0.5, 1.25)	0.559
(0.5, 1.25)	(1, 2)	0.901
(1, 2)	(1.5, 3.25)	1.346
(1.5, 3.25)	(2, 5)	1.820
	Total	4.6267

(b)

Left	Right	Length
(0, 1)	(0.25, 1.063)	0.258
(0.25, 1.063)	(0.5, 1.25)	0.313
(0.5, 1.25)	(0.75, 1.563)	0.400
(0.75, 1.563)	(1, 2)	0.504
(1, 2)	(1.25, 2.563)	0.616
(1.25, 2.563)	(1.5, 3.25)	0.732
(1.5, 3.25)	(1.75, 4.063)	0.850
(1.75, 4.063)	(2, 5)	0.970
	Total	4.6417

(c) Actual length approximately 4.6468.

14. (a)

Left	Right	Length
(0, 2)	(0.25, 2.016)	0.250
(0.25, 2.016)	(0.5, 2.125)	0.273
(0.5, 2.125)	(0.75, 2.422)	0.388
(0.75, 2.422)	(1, 3)	0.6299
	Total	1.541

(b)

Left	Right	Length
(0, 2)	(0.125, 2.00)	0.125
(0.125, 2.00)	(0.25, 2.016)	0.126
(0.25, 2.016)	(0.38, 2.05)	0.130
(0.38, 2.05)	(0.5, 2.13)	0.144
(0.5, 2.13)	(0.63, 2.244)	0.173
(0.63, 2.244)	(0.75, 2.422)	0.217
(0.75, 2.422)	(0.875, 2.67)	0.278
(0.875, 2.67)	(1, 3)	0.353
	Total	1.546

(c) Actual length approximately 2.0682.

15. (a) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$,

$\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 1)	(0.393, 0.92)	0.400
(0.393, 0.92)	(0.785, 0.71)	0.449
(0.785, 0.71)	(1.18, 0.383)	0.509
(1.18, 0.383)	(1.571, 0)	0.548
	Total	1.906

(b) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{16}$, $\frac{\pi}{8}$, $\frac{3\pi}{16}$, $\frac{\pi}{4}$, $\frac{5\pi}{16}$, $\frac{3\pi}{8}$, $\frac{7\pi}{16}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 1)	(0.196, 0.98)	0.197
(0.196, 0.98)	(0.393, 0.92)	0.204
(0.393, 0.92)	(0.589, 0.83)	0.217
(0.589, 0.83)	(0.785, 0.71)	0.232
(0.785, 0.71)	(0.982, 0.56)	0.248
(0.982, 0.56)	(1.178, 0.38)	0.262
(1.178, 0.38)	(1.37, 0.195)	0.272
(1.37, 0.195)	(1.571, 0)	0.277
	Total	1.909

(c) Actual length approximately 1.9101.

16. (a) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 0)	(0.393, 0.38)	0.548
(0.393, 0.38)	(0.785, 0.71)	0.509
(0.785, 0.71)	(1.18, 0.924)	0.449
(1.18, 0.924)	(1.57, 1)	0.400
	Total	1.906

(b) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{16}$, $\frac{\pi}{8}$, $\frac{3\pi}{16}$, $\frac{\pi}{4}$, $\frac{5\pi}{16}$, $\frac{3\pi}{8}$, $\frac{7\pi}{16}$, and $\frac{\pi}{2}$.

Left	Right	Length
(0, 0)	(0.196, 0.2)	0.277
(0.196, 0.2)	(0.39, 0.38)	0.272
(0.39, 0.38)	(0.589, 0.56)	0.262
(0.589, 0.56)	(0.785, 0.71)	0.248
(0.785, 0.71)	(0.982, 0.83)	0.232
(0.982, 0.83)	(1.18, 0.924)	0.217
(1.18, 0.924)	(1.374, 0.98)	0.204
(1.374, 0.98)	(1.571, 1)	0.197
	Total	1.909

(c) Actual length approximately 1.9101.

17. (a)

Left	Right	Length
(0, 1)	(0.75, 1.323)	0.817
(0.75, 1.323)	(1.5, 1.581)	0.793
(1.5, 1.581)	(2.25, 1.803)	0.782
(2.25, 1.803)	(3, 2)	0.776
	Total	3.167

(b)

Left	Right	Length
(0, 1)	(0.375, 1.17)	0.413
(0.375, 1.17)	(0.75, 1.323)	0.404
(0.75, 1.323)	(1.125, 1.46)	0.399
(1.125, 1.46)	(1.5, 1.58)	0.395
(1.5, 1.58)	(1.88, 1.696)	0.392
(1.88, 1.696)	(2.25, 1.80)	0.390
(2.25, 1.80)	(2.63, 1.904)	0.388
(2.63, 1.904)	(3, 2)	0.387
	Total	3.168

(c) Actual length approximately 3.168.

18. (a)

Left	Right	Length
(1, 1)	(1.25, 0.8)	0.3202
(1.25, 0.8)	(1.5, 0.67)	0.2833
(1.5, 0.67)	(1.75, 0.571)	0.2675
(1.75, 0.571)	(2, 0.5)	0.2600
	Total	1.1310

(b)

Left	Right	Length
(1, 1)	(1.125, 0.89)	0.167
(1.125, 0.89)	(1.25, 0.8)	0.153
(1.25, 0.8)	(1.375, 0.73)	0.145
(1.375, 0.73)	(1.5, 0.67)	0.139
(1.5, 0.67)	(1.625, 0.62)	0.135
(1.625, 0.62)	(1.75, 0.57)	0.133
(1.75, 0.57)	(1.875, 0.53)	0.131
(1.875, 0.53)	(2, 0.5)	0.129
	Total	1.132

(c) Actual length approximately 1.1321.

19. (a)

Left	Right	Length
(-2, 5)	(-1, 2)	3.162
(-1, 2)	(0, 1)	1.414
(0, 1)	(1, 2)	1.414
(1, 2)	(2, 5)	3.162
	Total	9.153

(b)

Left	Right	Length
(-2, 5)	(-1.5, 3.25)	1.820
(-1.5, 3.25)	(-1, 2)	1.346
(-1, 2)	(-0.5, 1.25)	0.901
(-0.5, 1.25)	(0, 1)	0.559
(0, 1)	(0.5, 1.25)	0.559
(0.5, 1.25)	(1, 2)	0.901
(1, 2)	(1.5, 3.25)	1.346
(1.5, 3.25)	(2, 5)	1.820
	Total	9.253

(c) Actual length approximately 9.2936.

20. (a)

Left	Right	Length
(-1, 1)	(-0.5, 1.875)	1.0078
(-0.5, 1.875)	(0, 2)	0.5154
(0, 2)	(0.5, 2.125)	0.5154
(0.5, 2.125)	(1, 3)	1.0078
	Total	3.0463

(b)

Left	Right	Length
(-1, 1)	(-0.75, 1.58)	0.630
(-0.75, 1.58)	(-.5, 1.88)	0.388
(-.5, 1.88)	(-0.25, 1.98)	0.273
(-0.25, 1.98)	(0, 2)	0.251
(0, 2)	(0.25, 2.016)	0.251
(0.25, 2.016)	(0.5, 2.13)	0.273
(0.5, 2.13)	(0.75, 2.42)	0.388
(0.75, 2.42)	(1, 3)	0.630
	Total	3.084

(c) Actual length approximately 3.0957.

21. The sum of the areas of the rectangles is $11/8 = 1.375$.
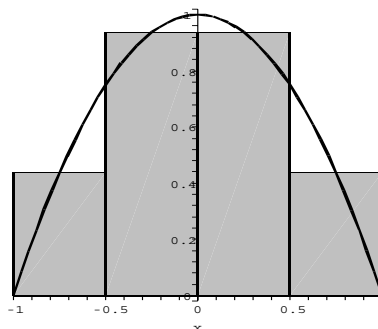
22. The sum of the areas of the rectangles is $43/32 = 1.34375$.
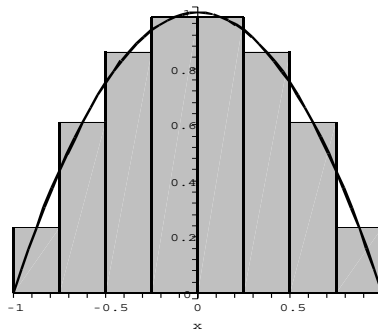
23. (a) The width of the entire region $(-1 \leq x \leq 1)$ is 2, so the width of each rectangle is $2/16 = 0.125$. The left endpoints of the rectangles are

$$-1, -1+\tfrac{2}{16}, \ldots, -1+\tfrac{28}{16}, -1+\tfrac{30}{16}$$

so the midpoints of the rectangles are

$-1 + \frac{1}{16}, -1 + \frac{3}{16}, \ldots, -1 + \frac{31}{16}.$

The heights of the rectangles are then given by the function $f(x) = 1 - x^2$ evaluated at those midpoints. We multiply each height by the width (0.125) and add them all to obtain the approximation 1.3359375 for the area.

(b) Using the same method as in (a), the width of the rectangles is now $2/32 = 0.0625$, and the midpoints are

$-1 + \frac{1}{32}, -1 + \frac{3}{32}, \ldots, -1 + \frac{63}{32}.$

The approximation is 1.333984375.

(c) Using the same method as in (a), the width of the rectangles is now $2/64 = 0.03125$, and the midpoints are

$-1 + \frac{1}{64}, -1 + \frac{3}{64}, \ldots, -1 + \frac{127}{64}.$

The approximation is 1.333496094.

The actual area is $4/3$.

24. The following is a graph with 4 rectangles:
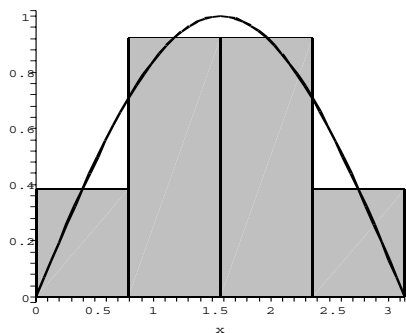
(a) Using the same method as in exercise 23, the width of the rectangles is $\pi/16$, and the midpoints are

$\frac{\pi}{16}, \frac{3\pi}{16}, \ldots, \frac{15\pi}{16}.$

The approximation is 2.003216378.

(b) Using the same method as in exercise 23, the width of the rectangles is now $\pi/32$, and the midpoints are

$\frac{\pi}{32}, \frac{3\pi}{32}, \ldots, \frac{31\pi}{32}.$

The approximation is 2.000803417.

(c) Using the same method as in exercise 23, the width of the rectangles is now $\pi/64$, and the midpoints are

$\frac{\pi}{64}, \frac{3\pi}{64}, \ldots, \frac{63\pi}{64}.$

The approximation is 2.000200812.

The actual area is 2.

25. The following is a graph with 4 rectangles:
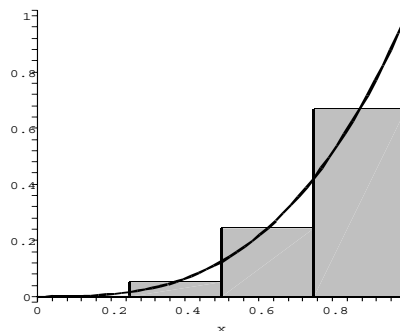
(a) Using the same method as in exercise 23, the width of the rectangles is $1/16$, and the midpoints are

$\frac{1}{16}, \frac{3}{16}, \ldots, \frac{15}{16}.$

The approximation is 0.249511719.

(b) Using the same method as in exercise 23, the width of the rectangles is now $1/32$, and the midpoints are

$\frac{1}{32}, \frac{3}{32}, \ldots, \frac{31}{32}.$

The approximation is 0.24987793.

(c) Using the same method as in exercise 23, the width of the rectangles is now $1/64$, and the midpoints are

$\frac{1}{64}, \frac{3}{64}, \cdots, \frac{63}{64}$.
The approximation is 0.249969482.

The actual area is 1/4.

26. The following is a graph with 4 rectangles:
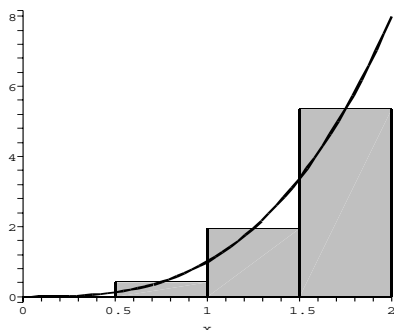
(a) Using the same method as in exercise 23, the width of the rectangles is 2/16, and the midpoints are

$\frac{1}{16}, \frac{3}{16}, \cdots, \frac{31}{16}$.
The approximation is 3.992187500.

(b) Using the same method as in exercise 23, the width of the rectangles is now 2/32, and the midpoints are

$\frac{1}{32}, \frac{3}{32}, \cdots, \frac{63}{32}$.
The approximation is 3.998046875.

(c) Using the same method as in exercise 23, the width of the rectangles is now 2/64, and the midpoints are

$\frac{1}{64}, \frac{3}{64}, \cdots, \frac{127}{64}$.
The approximation is 3.999511719.

The actual area is 4.

1.2   The Concept of Limit

1. (a) $\lim\limits_{x \to 0^-} f(x) = -2$

(b) $\lim\limits_{x \to 0^+} f(x) = 2$

(c) Does not exist.

(d) $\lim\limits_{x \to 1^-} f(x) = 1$

(e) $\lim\limits_{x \to -1} f(x) \approx 0.1$

(f) $\lim\limits_{x \to 2^-} f(x) = -1$

(g) $\lim\limits_{x \to 2^+} f(x) = 3$

(h) Does not exist.

(i) $\lim\limits_{x \to -2} f(x) \approx 1.8$

(j) $\lim\limits_{x \to 3} f(x) \approx 2.5$

2. (a) $\lim\limits_{x \to 0^-} f(x) = 3$

(b) $\lim\limits_{x \to 0^+} f(x) = 1$

(c) Does not exist.

(d) $\lim\limits_{x \to 2^-} f(x) \approx 1.5$

(e) $\lim\limits_{x \to -2} f(x) = 3$

(f) $\lim\limits_{x \to 1^-} f(x) = 2$

(g) $\lim\limits_{x \to 1^+} f(x) = 2$

(h) $\lim\limits_{x \to 1} f(x) = 2$

(i) $\lim\limits_{x \to -1} f(x) \approx 3.8$

(j) $\lim\limits_{x \to 3} f(x) = 1$

3. (a) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 2x = 4$

(b) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x^2 = 4$

(c) $\lim\limits_{x \to 2} f(x) = 4$

(d) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} 2x = 2$
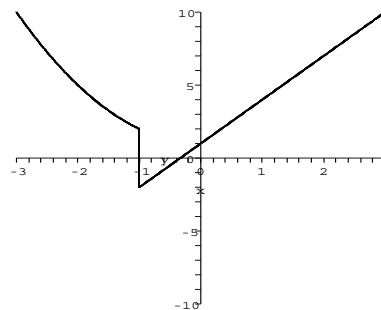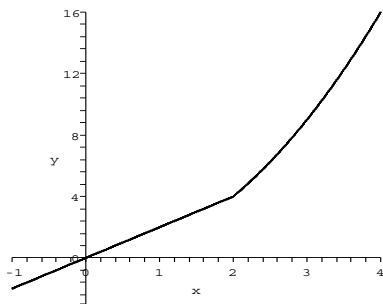
4. (a) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^3 - 1 = -1$

(b) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \sqrt{x+1} - 2$
$= -1$

(c) $\lim\limits_{x\to 0} f(x) = -1$

(d) $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} x^3 - 1 = -2$

(e) $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \sqrt{x+1} - 2 = 0$
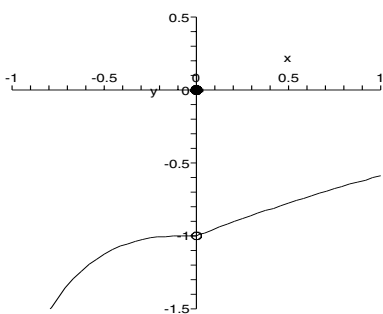
6. (a) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} 2x+1 = -1$

(b) $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} 3 = 3$

(c) $\lim\limits_{x\to -1} f(x)$ does not exist.

(d) $\lim\limits_{x\to 1} f(x) = 3$

(e) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} 3 = 3$
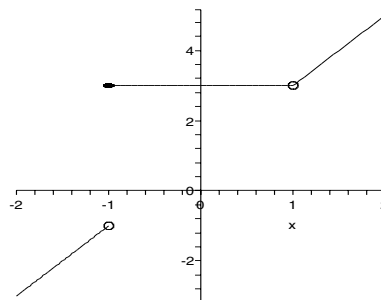
5. (a) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to 1^-} x^2 + 1 = 2$

(b) $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to 1^+} 3x + 1 = -2$

(c) $\lim\limits_{x\to -1} f(x)$ does not exist

(d) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} 3x + 1 = 4$

7. $f(1.5) = 2.22$, $f(1.1) = 2.05$,
$f(1.01) = 2.01$, $f(1.001) = 2.00$.

The values of $f(x)$ seem to be approaching 2 as x approaches 1 from the right.

$f(0.5) = 1.71$, $f(0.9) = 1.95$,
$f(0.99) = 1.99$, $f(0.999) = 2.00$.

The values of $f(x)$ seem to be approaching 2 as x approaches 1 from the left. Since the limits from the left and right exist and are the same, the limit exists.

8. $f(-1.5) = -0.4$
$f(-1.1) = -0.4762$
$f(-1.01) = -0.4975$
$f(-1.001) = -0.4998$

The values of $f(x)$ seem to be approaching -0.5 as x approaches -1 from the left.

$f(-0.5) = -0.6667$
$f(-0.9) = -0.5263$
$f(-0.99) = -0.5025$
$f(-0.999) = -0.5003$

The values of $f(x)$ seem to be approaching -0.5 as x approaches -1 from the right. Since the limits from the left and right exist and are the same, the limit exists.

9. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 2.

10. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately $\frac{1}{3}$.

11. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 1.

12. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately -1.

13. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 1.

14. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 0.

15. The numerical evidence suggests that the function the function blows up at $x = 1$. From the graph we see that the function has a vertical asymptote at $x = 1$.

16. The limit exists and equals -2.

17. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately $3/2$.

18. The limit exists and equals 4.

19. The limit does not exist because the graph oscillates wildly near $x = 0$.

20. The limit exists and equals 0.

21. The numerical evidence suggests that

$\lim\limits_{x \to 2^-} \frac{x-2}{|x-2|} = -1$ while $\lim\limits_{x \to 2^+} \frac{x-2}{|x-2|} = 1$

so $\lim\limits_{x \to 2} \frac{x-2}{|x-2|}$ does not exist. There is a break in the graph at $x = 2$.

22. The function approaches $1/2$ from the left, and $-1/2$ from the right. Since these are not equal, the limit does not exist.

23. The function $\ln x$ is not defined for $x \leq 0$ so the limit does not exist. The numerical evidence suggests that the function blows up as x approaches 0 from the right. From the graph we see that the function has a one-sided vertical asymptote at $x = 0$.

24. The limit exists and equals 0.

25. The limit exists and equals 1.

26. The limit exists and equals 1.

27. Numerical and graphical evidence show that the limits

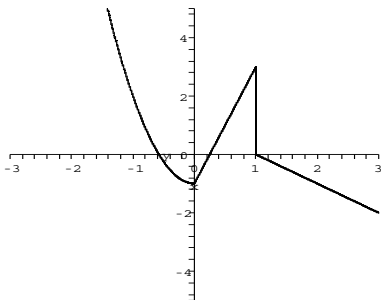$$\lim_{x \to 1} \frac{x^2 + 1}{x - 1} \text{ and } \lim_{x \to 2} \frac{x + 1}{x^2 - 4}$$

do not exist (both have vertical asymptotes). Our conjecture is that if $g(a) = 0$ and $f(a) \neq 0$, $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

28.

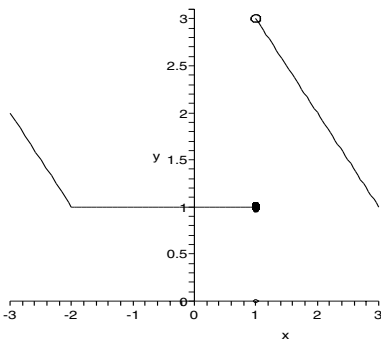$$\lim_{x \to -1} \frac{x + 1}{x^2 + 1} = 0 \text{ and } \lim_{x \to \pi} \frac{\sin x}{x} = 0.$$

If the numerator $f(a) = 0$, and the denominator $g(a) \neq 0$, then the limit $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$.

29. One possibility:

30. One possibility:

31. One possibility:
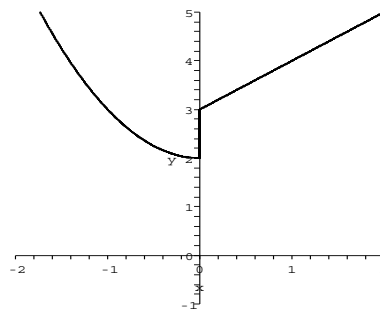
32. One possibility:
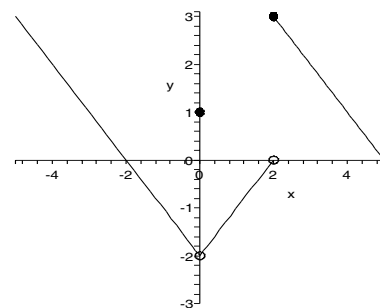
33. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 1/2.
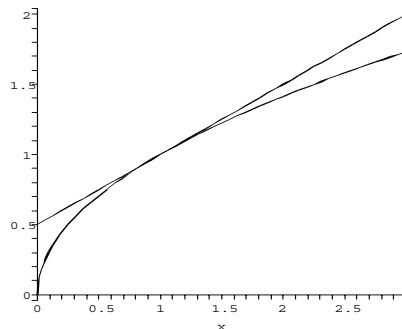
34. By inspecting the graph, and using a sequence of values (as in exercises 7 and 8), we see that the limit is approximately 1/2.

35. The first argument gives the correct value; the second argument is not valid because it looks only at certain values of x.

36. As x approaches 0 from the right, $1/x$ increases without bound, hence $-1/x$ decreases without bound, and $e^{-1/x}$ approaches 0 when x approaches 0 from the right. On the other hand, as x approaches 0 from the left, $1/x$ decreases without bound, hence $-1/x$ increases without bound, and $e^{-1/x}$ increases without bound as well. This argument shows that the limit in question **does not exist**.

37.

x	$(1+x)^{\frac{1}{x}}$	x	$(1+x)^{\frac{1}{x}}$
0.1	2.59	-0.1	2.87
0.01	2.70	-0.01	2.73
0.001	2.7169	-0.001	2.7196

$$\lim_{x \to 0}(1+x)^{1/x} \approx 2.7182818$$

38. We see that $1/x$ is increasing without bound when x is approaches 0. While it is true that 1 raised to any power is 1, numbers close to 1 raised to large enough powers may be very far from 1.

39.

x	$x^{\sec x}$
0.1	0.099
0.01	0.010
0.001	0.001

$$\lim_{x \to 0^+} x^{\sec x} = 0$$

For negative x the values of $x^{\sec x}$ are usually not real numbers, so $\lim_{x \to 0^-} x^{\sec x} = 0$ does not exist.

40. While it is true that 0 raised to any power is 0, numbers close to 0 raised to small enough powers may be very far from 0. This computation is accidentally correct because $\sec x$ is approaching 1 when x is approaches 0.

41. Possible answers:
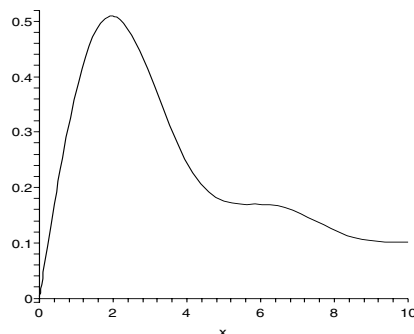
$$f(x) = \frac{x^2}{x}$$

$$g(x) = \begin{cases} 1 & \text{if } x \le 0 \\ -1 & \text{if } x > 0 \end{cases}$$

42. There are many possibilities. Here is a simple one

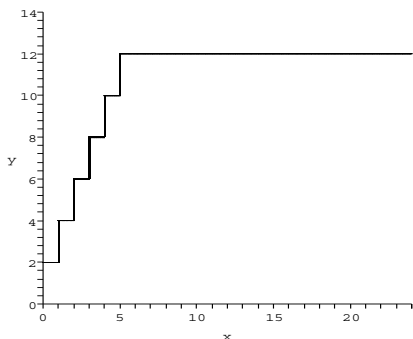$$f(x) = \begin{cases} -x & x < 0 \\ 3 & x = 0 \\ x & x > 0 \end{cases}$$

43. As x gets arbitrarily close to a, $f(x)$ gets arbitrarily close to L.

44. The limit of $h(\omega)$ as $\omega \to 0^+$ seems to be 0.

For $\omega = 0$, the ball position the batter sees at $t = 0.4$ is the same as what he tries to hit.

45. For $3 \le t \le 4$, $f(t) = 8$, so $\lim_{t \to 3.5} f(t) = 8$. Also $\lim_{t \to 4^-} f(t) = 8$. On the other hand, for $4 \le t \le 5$, $f(t) = 10$, so $\lim_{t \to 4^+} f(t) = 10$. Hence $\lim_{t \to 4} f(t)$ does not exist.

46. The limit does not exist at $t = 1$, 2, 3, 4, and 5 hours. In each case the limit from the left is two dollars less than the limit from the right. We would be in a hurry to move our car just before the hour to try to save \$2. Just after the hour, we can relax and take our time as the next price increase doesn't come until the next hour.

1.3 Computation of Limits

1. $\lim\limits_{x \to 0}(x^2 - 3x + 1) = 0^2 - 3(0) + 1 = 1$

2. $\lim\limits_{x \to 2} \sqrt[3]{2x + 1} = \sqrt[3]{2(2) + 1} = \sqrt[3]{5}.$

3. $\lim\limits_{x \to 0} \cos^{-1}(x^2) = \cos^{-1} 0 = \dfrac{\pi}{2}.$

4. $\lim\limits_{x \to 2} \dfrac{x - 5}{x^2 + 4} = \dfrac{2 - 5}{2^2 + 4} = -\dfrac{3}{8}$

5. $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$
$= \lim\limits_{x \to 3} \dfrac{(x - 3)(x + 2)}{x - 3}$
$= \lim\limits_{x \to 3}(x + 2) = 3 + 2 = 5$

6. $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 3x + 2}$
$= \lim\limits_{x \to 1} \dfrac{(x - 1)(x + 2)}{(x - 1)(x - 2)}$
$= \lim\limits_{x \to 1} \dfrac{(x + 2)}{(x - 2)} = \dfrac{3}{-1} = -3.$

7. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$
$= \lim\limits_{x \to 2} \dfrac{(x - 2)(x + 1)}{(x + 2)(x - 2)}$
$= \lim\limits_{x \to 2} \dfrac{x + 1}{x + 2} = \dfrac{2 + 1}{2 + 2} = \dfrac{3}{4}$

8. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 + 2x - 3}$
$= \lim\limits_{x \to 1} \dfrac{(x - 1)(x^2 + x + 1)}{(x + 3)(x - 1)}$
$= \lim\limits_{x \to 1} \dfrac{x^2 + x + 1}{x + 3} = \dfrac{1^2 + 1 + 1}{1 + 3} = \dfrac{3}{4}$

9. $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x} = \lim\limits_{x \to 0} \dfrac{\sin x}{\frac{\sin x}{\cos x}}$
$= \lim\limits_{x \to 0} \cos x = \cos 0 = 1$

10. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$
$= \lim\limits_{x \to 0} \dfrac{\sin x}{x \cos x}$
$= \left(\lim\limits_{x \to 0} \dfrac{\sin x}{x}\right)\left(\lim\limits_{x \to 0} \dfrac{1}{\cos x}\right) = 1.$

11. $\lim\limits_{x \to 0} \dfrac{x e^{-2x+1}}{x^2 + x}$
$= \lim\limits_{x \to 0} \dfrac{x\left(e^{-2x+1}\right)}{x(x + 1)}$
$= \lim\limits_{x \to 0} \dfrac{e^{-2x+1}}{x + 1} = \dfrac{e^{-2(0)+1}}{0 + 1} = e$

12. $\lim\limits_{x \to 0^+} x^2 \csc^2 x$
$= \lim\limits_{x \to 0^+} \dfrac{x^2}{\sin^2 x}$
$= \left(\lim\limits_{x \to 0^+} \dfrac{1}{\frac{\sin x}{x}}\right)\left(\lim\limits_{x \to 0^+} \dfrac{1}{\frac{\sin x}{x}}\right) = 1.$

13. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$
$= \lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}\left(\dfrac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2}\right)$
$= \lim\limits_{x \to 0} \dfrac{x + 4 - 4}{x(\sqrt{x + 4} + 2)}$
$= \lim\limits_{x \to 0} \dfrac{x}{x(\sqrt{x + 4} + 2)}$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2}$$
$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

14. $\lim_{x \to 0} \dfrac{2x}{3-\sqrt{x+9}}$

$$= \lim_{x \to 0} \frac{2x}{\left(3-\sqrt{x+9}\right)} \frac{\left(3+\sqrt{x+9}\right)}{\left(3+\sqrt{x+9}\right)}$$
$$= \lim_{x \to 0} \frac{2x(3+\sqrt{x+9})}{-x}$$
$$= \lim_{x \to 0} -2(3+\sqrt{x+9}) = -12$$

15. $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$

$$= \lim_{x \to 1} \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}$$
$$= \lim_{x \to 1}(\sqrt{x}+1) = \sqrt{1}+1 = 2$$

16. $\lim_{x \to 4} \dfrac{\sqrt{x}-2}{x-4}$

$$= \lim_{x \to 4} \frac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{(x-4)\left(\sqrt{x}+2\right)}$$
$$= \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x \to 4} \frac{1}{(\sqrt{x}+2)} = \frac{1}{4}.$$

17. $\lim_{x \to 1} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1} \right)$

$$= \lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{(x-1)(x+1)} \right)$$
$$= \lim_{x \to 1} \left(\frac{x+1}{(x-1)(x+1)} - \frac{2}{(x-1)(x+1)} \right)$$
$$= \lim_{x \to 1} \left(\frac{x-1}{(x-1)(x+1)} \right)$$
$$= \lim_{x \to 1} \left(\frac{1}{x+1} \right) = \frac{1}{2}$$

18. Undefined. The limit from the right is 0, but the limit from the left does not exist.

19. $\lim_{x \to 0} \dfrac{1-e^{2x}}{1-e^x}$

$$= \lim_{x \to 0} \frac{\left(1-e^x\right)\left(1+e^x\right)}{1-e^x}$$

$$= \lim_{x \to 0} \left(1+e^x\right) = 2$$

20. $\lim_{x \to 0} \sin(e^{-1/x^2}) = \lim_{x \to 0} \sin(0) = 0$

21. $\lim_{x \to 0^+} \dfrac{\sin(|x|)}{x} = \lim_{x \to 0^+} \dfrac{\sin(x)}{x} = 1$

$$\lim_{x \to 0^-} \frac{\sin(|x|)}{x}$$
$$= \lim_{x \to 0^-} \frac{\sin(-x)}{x}$$
$$= \lim_{x \to 0^-} \frac{-\sin(x)}{x} = -1$$

Since the limit from the left does not equal the limit from the right, we see that $\lim_{x \to 0} \frac{\sin(|x|)}{x}$ does not exist.

22. $\lim_{x \to 0} \dfrac{\sin^2(x^2)}{x^4}$

$$= \lim_{x \to 0} \left(\frac{\sin(x^2)}{x^2} \right) \left(\frac{\sin(x^2)}{x^2} \right)$$
$$= \left(\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \right) \left(\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \right)$$
$$= (1)(1) = 1$$

23. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x = 2(2) = 4$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 2^2 = 4$$
$$\lim_{x \to 2} f(x) = 4$$

24. Undefined. The limit from the left is 2, but the limit from the right is -2.

25. $\lim_{x \to 0} f(x) = \lim_{x \to 0}(3x+1) = 3(0)+1 = 1$

26. $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2.$

27. $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (2x+1)$
$$= 2(-1)+1 = -1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 3 = 3$$
Therefore $\lim_{x \to -1} f(x)$ does not exist.

28. $\lim_{x \to 1^-} f(x) = 3,$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x+1 = 3,$$
Therefore $\lim_{x \to 1} f(x) = 3.$

29. $\displaystyle\lim_{h\to 0}\frac{(2+h)^2-4}{h}$
$\displaystyle=\lim_{h\to 0}\frac{(4+4h+h^2)-4}{h}$
$\displaystyle=\lim_{h\to 0}\frac{4h+h^2}{h}=\lim_{h\to 0}4+h=4$

30. $\displaystyle\lim_{h\to 0}\frac{(1+h)^3-1}{h}$
$\displaystyle=\lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h}$
$\displaystyle=\lim_{h\to 0}\frac{h(3+3h+h^2)}{h}$
$\displaystyle=\lim_{h\to 0}3+3h+h^2=3$

31. $\displaystyle\lim_{h\to 0}\frac{h^2}{\sqrt{h^2+h+3}-\sqrt{h+3}}$
$\displaystyle=\lim_{h\to 0}\frac{h^2(\sqrt{h^2+h+3}+\sqrt{h+3})}{(h^2+h+3)-(h+3)}$
$\displaystyle=\lim_{h\to 0}\frac{h^2(\sqrt{h^2+h+3}+\sqrt{h+3})}{h^2}$
$\displaystyle=\lim_{h\to 0}\sqrt{h^2+h+3}+\sqrt{h+3}=2\sqrt{3}$
To get from the first line to the second, we have multiplied by
$$\frac{\sqrt{h^2+h+3}+\sqrt{h+3}}{\sqrt{h^2+h+3}+\sqrt{h+3}}.$$

32. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+x+4}-2}{x^2+x}=$
$\displaystyle\lim_{x\to 0}\frac{(\sqrt{x^2+x+4}-2)}{x^2+x}\frac{(\sqrt{x^2+x+4}+2)}{(\sqrt{x^2+x+4}+2)}$
$\displaystyle=\lim_{x\to 0}\frac{x^2+x}{(x^2+x)(\sqrt{x^2+x+4}+2)}$
$\displaystyle=\lim_{x\to 0}\frac{1}{\sqrt{x^2+x+4}+2}=\frac{1}{4}$

33. $\displaystyle\lim_{t\to -2}\frac{\frac{1}{2}+\frac{1}{t}}{2+t}$
$\displaystyle=\lim_{t\to -2}\frac{\frac{t+2}{2t}}{2+t}$
$\displaystyle=\lim_{t\to -2}\frac{1}{2t}=-\frac{1}{4}$

34. $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{5x}$

$\displaystyle=\lim_{x\to 0}\frac{\sin 2x}{5x\cos 2x}$
$\displaystyle=\lim_{x\to 0}\frac{\sin 2x}{2x}\frac{2}{5\cos 2x}=\frac{2}{5}.$

35.

x^2	$x^2\sin(1/x)$
-0.1	0.0054
-0.01	5×10^{-5}
-0.001	-8×10^{-7}
0.1	-0.005
0.01	-5×10^{-5}
0.001	8×10^{-7}

Conjecture: $\displaystyle\lim_{x\to 0}x^2\sin(1/x)=0$.
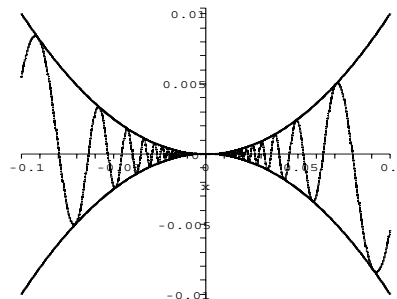Let $f(x)=-x^2, h(x)=x^2$. Then
$$f(x)\le x^2\sin\left(\frac{1}{x}\right)\le h(x)$$
$$\lim_{x\to 0}(-x^2)=0, \lim_{x\to 0}(x^2)=0$$
Therefore, by the Squeeze Theorem,
$$\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)=0.$$

36. You cannot use the Squeeze Theorem as in exercise 35 because the secant function is not bounded between -1 and 1 like the sine function is. This is difficult to investigate graphically because of the infinitely many vertical asymptotes as x approaches 0.
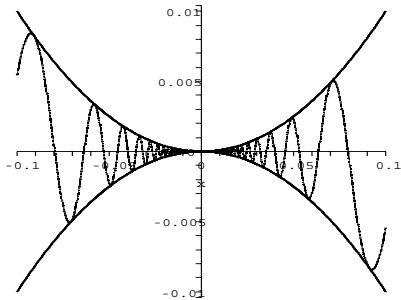
37. Let $f(x)=0$, $h(x)=\sqrt{x}$. We see that
$$f(x)\le \sqrt{x}\cos^2(1/x)\le h(x),$$

$$\lim_{x \to 0^+} 0 = 0, \ \lim_{x \to 0^+} \sqrt{x} = 0$$

Therefore, by the Squeeze Theorem,

$$\lim_{x \to 0^+} \sqrt{x} \cos^2\left(\frac{1}{x}\right) = 0.$$

38. Saying that $|f(x)| \le M$ for all x is the same as saying $-M \le f(x) \le M$ for all x. This implies that

$$-Mx^2 \le x^2 f(x) \le Mx^2.$$

Since $\pm Mx^2 \to 0$ as $x \to 0$, the Squeeze Theorem shows that $\lim_{x \to 0} x^2 f(x) = 0$.

39. $\lim_{x \to 4^+} \sqrt{16 - x^2}$ does not exist because the domain of the function is $[-4, 4]$.

40. $\lim_{x \to 4^-} \sqrt{16 - x^2} = 0$.

41. $\lim_{x \to -2^-} \sqrt{x^2 + 3x + 2} = 0$.

42. $\lim_{x \to -2^+} \sqrt{x^2 + 3x + 2}$ does not exist because the domain of the function is $(-\infty, -2) \cup (-1, \infty)$.

43. $\lim_{x \to 0^+} \dfrac{\sqrt{1 - \cos x}}{x} = \sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$

44. $\lim_{x \to 0} \dfrac{1 - \cos^2 x}{x^2} = \lim_{x \to 0} \dfrac{\sin^2 x}{x^2}$
$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{\sin x}{x}\right) = 1.$$

45. $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x) = g(a)$ because $g(x)$ is a polynomial. Similarly,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} h(x) = h(a).$$

46. Evaluate $g(a)$ and $h(a)$. If they are equal, the limit exists and is this value. If they are not equal, the limit does not exist.

47. (a) $\lim_{x \to 2}(x^2 - 3x + 1)$
$$= 2^2 - 3(2) + 1 \quad \text{(Theorem 3.2)}$$
$$= -1$$

(b) $\lim_{x \to 0} \dfrac{x - 2}{x^2 + 1}$
$$= \frac{\lim_{x \to 0}(x - 2)}{\lim_{x \to 0}(x^2 + 1)}$$
(Theorem 3.1(iv))
$$= \frac{\lim_{x \to 0} x - \lim_{x \to 0} 2}{\lim_{x \to 0} x^2 + \lim_{x \to 0} 1}$$
(Theorem 3.1(ii))
$$= \frac{0 - 2}{0 + 1}$$
(Equations 3.1, 3.2, and 3.5)
$$= -2$$

48. (a) $\lim_{x \to -1}(x + 1)\sin x$
$$= \lim_{x \to -1}(x + 1) \lim_{x \to -1} \sin x$$
(Theorem 3.1).
Using Theorems 3.2 and 3.4 we get that this is equal to $(-1 + 1)\sin(-1) = 0$.

(b) By Theorem 3.1,

$$\lim_{x \to 1} \frac{xe^x}{\tan x} = \frac{(\lim_{x \to 1} x)(\lim_{x \to 1} e^x)}{\lim_{x \to 1} \tan x}.$$

Using Theorem 3.2 and Theorem 3.4 we see that this equals $\dfrac{e}{\tan 1}$.

49. Velocity is given by the limit
$$\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 + 2 - (2^2 + 2)}{h}$$
$$= \lim_{h \to 0} \frac{4h + h^2}{h}$$
$$= \lim_{h \to 0} 4 + h = 4.$$

50. Velocity is given by the limit
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{h^2 + 2 - 2}{h}$$
$$= \lim_{h \to 0} h = 0.$$

51. Velocity is given by the limit
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{(0+h)^3 - (0)^3}{h}$$
$$= \lim_{h \to 0} \frac{h^3}{h}$$
$$= \lim_{h \to 0} h^2 = 0.$$

52. Velocity is given by the limit
$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{(1+h)^3 - 1}{h} = 3$$
(see exercise 30).

53. $m = \lim_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h} \dfrac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$
$$= \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}$$
$$= \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}.$$
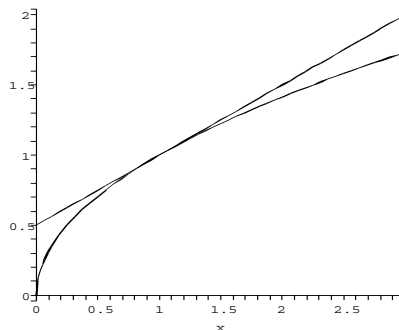
54. $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$
$$= \lim_{x \to 1} \frac{(\sqrt{x} - 1)}{(x - 1)} \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)}$$
$$= \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)}$$
$$= \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

55. $\lim_{x \to 0^+} (1 + x)^{1/x} = e \approx 2.71828$

56. $\lim_{x \to 0^+} e^{1/x}$ does not exist.

57. $\lim_{x \to 0^+} x^{-x^2} = 1$

58. $\lim_{x \to 0^+} x^{\ln x}$ does not exist.

59. As x gets close to 0, $1/x$ gets larger and larger in absolute value, so $\sin(1/x)$ oscillates more and more rapidly between 1 and -1, so the limit does not exist.

60. $\lim_{x \to 0} e^{1/x}$ does not exist.

61. When x is small and positive, $1/x$ is large and positive, so $\tan^{-1}(1/x)$ approaches $\pi/2$. But when x is small and negative, $1/x$ is large and negative, so $\tan^{-1}(1/x)$ approaches $-\pi/2$. So the limit does not exist.

62. $\lim_{x \to 0} \ln |\frac{1}{x}|$ does not exist.

63. $\lim\limits_{x \to a} [2f(x) - 3g(x)]$

$= 2 \lim\limits_{x \to a} f(x) - 3 \lim\limits_{x \to a} g(x)$

$= 2(2) - 3(-3) = 13$

64. $\lim\limits_{x \to a} [3f(x)g(x)]$

$= 3(\lim\limits_{x \to a} f(x))(\lim\limits_{x \to a} g(x))$

$= 3(2)(-3) = -18$

65. $\lim\limits_{x \to a} \left[\dfrac{f(x) + g(x)}{h(x)} \right]$

does not exist, because

$\lim\limits_{x \to a} [f(x) + g(x)]$

$= \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

$= 2 - 3 = -1$

and $\lim_{x \to a} h(x) = 0$.

66. $\lim\limits_{x \to a} \left[\dfrac{3f(x) + 2g(x)}{h(x)} \right]$ cannot be determined from the given information since

$\lim\limits_{x \to a} [3f(x) + 2g(x)]$

$= 3 \lim\limits_{x \to a} f(x) + 2 \lim\limits_{x \to a} g(x)$

$= 3(2) + 2(-3) = 0$

and $\lim_{x \to a} h(x) = 0$.

67. $\lim\limits_{x \to a} [f(x)]^3$

$= \left[\lim\limits_{x \to a} f(x) \right] \left[\lim\limits_{x \to a} f(x) \right] \left[\lim\limits_{x \to a} f(x) \right]$

$= L \cdot L \cdot L = L^3$

$\lim\limits_{x \to a} [f(x)]^4 = \left[\lim\limits_{x \to a} f(x) \right] \left[\lim\limits_{x \to a} [f(x)]^3 \right]$

$= L \cdot L^3 = L^4$

68. Since we have a starting place, and we have shown that we can always get from one step to the next, the theorem must be true for any positive integer.

Given that $\lim\limits_{x \to a} f(x) = L$.

Assume that $\lim\limits_{x \to a} [f(x)]^k = L^k$. Now

$\lim\limits_{x \to a} [f(x)]^{k+1} = \lim\limits_{x \to a} [f(x)]^k f(x)$

$= \lim\limits_{x \to a} [f(x)]^k \lim\limits_{x \to a} f(x) = L^k L = L^{k+1}$.

Therefore $\lim\limits_{x \to a} [f(x)]^n = L^n$ for any positive integer n.

69. We can't split the limit of a product into a product of limits unless we know that both limits exist; the limit of the product of a term tending toward 0 and a term with an unknown limit is not necessarily 0 but instead is unknown.

70. The limit of a quotient is not the quotient of the limits if the denominator is 0. The fraction $\frac{0}{0}$ is indeterminate, and can equal any finite value or be undefined.

71. One possibility is

$$f(x) = \frac{1}{x}, g(x) = -\frac{1}{x}.$$

72. $f(x) = x$, $g(x) = \frac{1}{x}$. $\lim\limits_{x \to 0} f(x)g(x) = 1$, but $\lim\limits_{x \to 0} g(x)$ does not exist.

73. Yes. If $\lim\limits_{x \to a} [f(x) + g(x)]$ exists, then, it would also be true that

$$\lim\limits_{x \to a} [f(x) + g(x)] - \lim\limits_{x \to a} f(x)$$

exists. But by Theorem 3.1 (ii)

$\lim\limits_{x \to a} [f(x) + g(x)] - \lim\limits_{x \to a} f(x)$
$= \lim\limits_{x \to a} [[f(x) + g(x)] - [f(x)]]$
$= \lim\limits_{x \to a} g(x)$

so $\lim\limits_{x \to a} g(x)$ would exist, but we are given that $\lim\limits_{x \to a} g(x)$ does not exist.

74. False. For example, let $f(x) = 1/x$. $\lim\limits_{x \to 0} f(x)$ does not exist, but

$$\lim\limits_{x \to 0} \frac{1}{f(x)} = \lim\limits_{x \to 0} x = 0.$$

75. $\lim\limits_{x \to 0^+} T(x) = \lim\limits_{x \to 0^+} (0.14x) = 0 = T(0).$

$\lim\limits_{x \to 10,000^-} T(x) = 0.14(10,000) = 1400$

$\lim\limits_{x \to 10,000^+} T(x)$
$= 1500 + 0.21(10,000) = 3600$

Therefore $\lim\limits_{x \to 10,000} T(x)$ does not exist. A small change in income should result in a small change in tax liability. This is true near $x = 0$ but is not true near $x = 10,000$. As your income grows past \$10,000 your tax liability jumps enormously.

76. If $\lim\limits_{x \to 0^+} T(x) = 0$, then $a = 0$. If $\lim\limits_{x \to 20,000}$ exists, then b must be 2400. These limits should exist so that \$0 income corresponds to \$0 tax, and so that the tax function doesn't have sudden jumps.

77. $\lim\limits_{x \to 3^-} [x] = 2$; $\lim\limits_{x \to 3^+} [x] = 3$
Therefore $\lim\limits_{x \to 3} [x]$ does not exist.

78. (a) $\lim\limits_{x \to 1} [x]$ does not exist. Approaches 0 from left, 1 from right.

(b) $\lim\limits_{x \to 1.5} [x] = 1$.

(c) $\lim\limits_{x \to 1.5} [2x]$ does not exist. Approaches 2 from left, 3 from right.

(d) $\lim\limits_{x \to 1} x - [x]$ does not exist. Approaches 1 from left, 0 from right.

1.4 Continuity and its Consequences

1. Discontinuous at $x = -2$ (limit does not exist), and at $x = 2$ (function undefined).

2. Discontinuous at $x = -4$ (function undefined), at $x = -2$ (limit not equal to function value), and at $x = 3$ (limit does not exist).

3. Discontinuous at $x = -2$ (function undefined), at $x = 1$ (function undefined), and at $x = 4$ (limit does not exist).

4. Discontinuous at $x = -2$, $x = 0$, and $x = 4$ (function undefined).

5. Discontinuous at $x = -2$ (limit does not exist), at $x = 2$ (function undefined), and at $x = 4$ (limit does not exist).

6. Discontinuous at $x = -2$ (limit does not exist), at $x = 0$ (limit not equal to function value), and at $x = 2$ (limit does not exist).

7. $f(1)$ is not defined and $\lim\limits_{x \to 1} f(x)$ does not exist.

8. Discontinuous because function is not defined at $x = 1$.

9. $f(0)$ is not defined and $\lim\limits_{x \to 0} f(x)$ does not exist.

10. Discontinuous because function is not defined at $x = 0$.

11. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2) = 4$
$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3x - 2) = 4$
$\lim\limits_{x \to 2} f(x) = 4; f(2) = 3$
$\lim\limits_{x \to 2} f(x) \neq f(2)$

12. Discontinuous because function is not defined at $x = 2$.

13. $f(x) = \dfrac{x - 1}{(x + 1)(x - 1)}$ has a removable discontinuity at $x = 1$ and a non-removable discontinuity at $x =$

−1; the removable discontinuity is removed by

$$g(x) = \frac{1}{x+1}.$$

14. $f(x)$ is discontinuous where the denominator is 0. The function is not defined at $x = -2$ and $x = 1$. (Not removable.)

15. No discontinuities.

16. $f(x)$ is discontinuous where the denominator is 0. The function is not defined at $x = 1 \pm \sqrt{5}$. (Not removable.)

17. $f(x) = \dfrac{x^2 \sin x}{\cos x}$ has non-removable discontinuities at $x = \frac{\pi}{2} + k\pi$ for any integer k.

18. Discontinuous wherever $\sin x = 0$. That is $x = k\pi$ for any integer k. (Not removable.)

19. By sketching the graph, or numerically, one can see that $\lim\limits_{x \to 0} x \ln x^2 = 0$. Thus, one can remove the discontinuity at $x = 0$ by defining

$$g(x) = \begin{cases} x \ln x^2 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

20. Discontinuous wherever $-4/x^2$ is undefined. That is, at $x = 0$. (Not removable.)

21. $f(x)$ has a non-removable discontinuity at $x = 1$.

22. Continuous everywhere since $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, and $f(0) = 1$.

23. $f(x)$ has a non-removable discontinuity at $x = 1$:
$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (3x - 1) = -4$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 + 5x) = -4$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 5x) = 6$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x^3) = 3$$

24. $f(x)$ is undefined at $x = 0$, and therefore discontinuous there. If $f(0)$ is defined to be 0, the function is continuous everywhere.

25. Continuous where $x + 3 > 0$, i.e. on $(-3, \infty)$

26. Continuous where $x^2 - 4 > 0$, i.e. on $(\infty, -2)$ and $(2, \infty)$.

27. Continuous everywhere, i.e. on $(-\infty, \infty)$.

28. Continuous where $x - 1 > 0$, i.e. on $(1, \infty)$.

29. Continuous everywhere, i.e. on $(-\infty, \infty)$.

30. Continuous for all $x \neq 0$ ($f(x)$ is undefined at $x = 0$).

31. Continuous where $x + 1 > 0$, i.e. on $(-1, \infty)$.

32. Continuous where $4 - x^2 > 0$, i.e. on $(-2, 2)$.

33.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2\frac{\sin x}{x}$$
$$= 2 \lim_{x \to 0^-} \frac{\sin x}{x} = 2$$

Hence a must equal 2 if f is continuous.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} b \cos x$$
$$= b \lim_{x \to 0^-} \cos x = b,$$

so b and a must equal 2 if f is continuous.

34. We need $ae^0 + 1 = \sin^{-1} 0$, so $a = -1$. We need $2^2 - 2 + b = \sin^{-1} 1$, so $b = \frac{\pi}{2} - 2$.

35. First note that
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \ln(x-2) + x^2$$
$$= \ln(3-2) + 3^2 = 9.$$

Also $f(3) = 2e^{3b} + 1$, so if f is continuous, $2e^{3b} + 1$ must equal 9; that is $e^{3b} = 4$, so $b = \frac{\ln 4}{3}$. Then note that

$$f(0) = 2e^{(b)(0)} + 1 = 3.$$

Also,
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} a(\tan^{-1} x + 2)$$
$$= a(\tan^{-1} 0 + 2)$$
$$= a(0 + 2) = 2a,$$

so a must equal $3/2$ if f is continuous.

36. Corollary 4.1: Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

Proof: Note that f is continuous at $g(a)$, and $\lim_{x \to a} g(x) = g(a)$ Therefore, Theorem 4.3 tells us that

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

This is equal to $f(g(a))$ since g is continuous at a. Since $f(g(a)) = \lim_{x \to a} f(g(x))$, $f \circ g$ is continuous at $x = a$.

37. $\lim_{x \to 10000^-} T(x) = \lim_{x \to 10000^-} 0.14x$
$$= 0.14(10,000) = 1400$$
$$\lim_{x \to 10000^+} T(x) = \lim_{x \to 10000^+} (c + 0.21x)$$
$$= c + 0.21(10,000)$$
$$= c + 2100$$
$$c + 2100 = 1400$$
$$c = -700$$

A small change in income should not result in a big change in tax, so the tax function should be continuous.

38. If $\lim_{x \to 0^+} T(x) = 0$, then $a = 0$. If $\lim_{x \to 20,000}$ exists, then b must be 2400.

39. For $T(x)$ to be continuous at $x = 141{,}250$ we must have

$$\lim_{x \to 141{,}250^-} T(x) = \lim_{x \to 141{,}250^+} T(x).$$

Now
$$\lim_{x \to 141{,}250^-} T(x) = \lim_{x \to 141{,}250^-} (.30)(x)a$$
$$= (.30)(141{,}250) - 5685$$
$$= 36690.$$
On the other hand,
$$\lim_{x \to 141{,}250^+} T(x) = \lim_{x \to 141{,}250^+} (.35)(x) - b$$
$$= (.35)(141{,}250) - b$$
$$= 49437.50 - b.$$

Hence
$b = 49437.50 - 36690 = 12{,}747.50.$

For $T(x)$ to be continuous at $x = 307{,}050$ we must have

$$\lim_{x \to 307{,}050^-} T(x) = \lim_{x \to 307{,}050^+} T(x).$$

Now
$$\lim_{x \to 307{,}050^-} T(x)$$
$$= \lim_{x \to 307{,}050^-} (.35)(x) - b$$
$$= (.35)(307{,}050) - 12{,}747.5$$
$$= 94{,}720.$$

On the other hand,
$$\lim_{x \to 307{,}050^+} T(x)$$
$$= \lim_{x \to 307{,}050^+} (.386)(x) - c$$
$$= (.386)(307{,}050) - c$$
$$= 118521.3 - c.$$

Hence
$c = 118{,}521.3 - 94720 = 23801.3.$

40. $\lim\limits_{x \to 6,000^-} T(x) = \lim\limits_{x \to 6,000^-} 0.10x$

$= \$600.$

$\lim\limits_{x \to 6,000^+} T(x) = \lim\limits_{x \to 6,000^+} 0.15x - 300$

$= \$600.$

So $T(6,000) = \$600 = \lim\limits_{x \to 6,000} T(x)$,

and $T(x)$ is continuous at $x = 6,000$.

41. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[2, 3]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the left half, and when $f(x)$ is negative, we use the right half of the interval. (Because the function goes from negative to positive. If the function went from positive to negative, the intervals would be reversed.)

x	$f(x)$
2	-3
3	2
2.5	-0.75
2.75	0.5625
2.625	-0.109375
2.6875	0.223
2.65625	0.557

The zero is in the interval $[2.625, 2.65625]$.

42. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[2, 3]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the left half, and when $f(x)$ is negative, we use the right half of the interval. (Because the function goes from negative to positive. If the function went

from positive to negative, the intervals would be reversed.)

x	$f(x)$
2	-2
3	13
2.5	3.625
2.25	0.3906
2.125	-0.9043
2.1875	-0.2825
2.21875	0.4758

The zero is in the interval $(2.1875, 2.21875)$.

43. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[2, 3]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the right half, and when $f(x)$ is negative, we use the left half of the interval.

x	$f(x)$
-1	1
0	-2
-0.5	-0.125
-0.625	0.256
-0.5625	0.072
-0.53125	-0.025

The zero is in the interval $[-0.5625, -0.53125]$.

44. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[-2, -1]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the left half, and when $f(x)$ is negative, we use the right half of the interval.

x	$f(x)$
-2	-2
-1	1
-1.5	0.625
-1.75	-0.3594
-1.625	0.2090
-1.6875	-0.0554
-1.65625	0.0816

The zero is in the interval $(-1.6875, -1.65625)$.

45. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[-2, -1]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the right half, and when $f(x)$ is negative, we use the left half of the interval.

x	$f(x)$
0	1
1	-0.46
0.5	0.378
0.75	-0.018
0.625	0.186
0.6875	0.085
0.71875	0.034

The zero is in the interval $[0.71875, 0.75]$.

46. The first two rows of the following table (together with the Intermediate Value Theorem) show that $f(x)$ has a root in $[-2, -1]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the left half, and when $f(x)$ is negative, we use the right half of the interval.

x	$f(x)$
-1	-0.6321
0	1
-0.5	0.1065
-0.75	-0.2776
-0.625	-0.0897
-0.5625	0.0073
-0.59375	-0.0415

The zero is in the interval $(-0.59375, -0.5625)$.

47. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 1) = 5$

$f(2) = 3(2)\text{-}1 = 5$

Thus $f(x)$ is continuous from the right at $x = 2$.

48. Yes, $f(x)$ is continuous from the right at $x = 2$, because
$\lim_{x \to 2^+} f(x) = f(2) = 3$.

49. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 3) = 3$

$f(2) = 2^2 = 4$

Thus $f(x)$ is not continuous from the right at $x = 2$.

50. No. $f(x)$ is not defined at $x = 2$.

51. A function is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

(a) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$
$f(2) = 5$
Thus $f(x)$ is not continuous from the left at $x = 2$.

(b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$
$f(2) = 3$
Thus $f(x)$ is not continuous from the left at $x = 2$.

(c) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$
$f(2) = 4$
Thus $f(x)$ is continuous from the left at $x = 2$.

(d) $f(x)$ is not continuous from the left at $x = 2$ because $f(2)$ is undefined.

52. (a) Limit might exist if $g(a)$ is also 0.

   (b) $f(x)$ is definitely discontinuous because $f(a)$ does not exist.

53. Need $g(30) = 100$ and $g(34) = 0$. We may take $g(T)$ to be linear.
$$m = \frac{0 - 100}{34 - 30} = -25$$
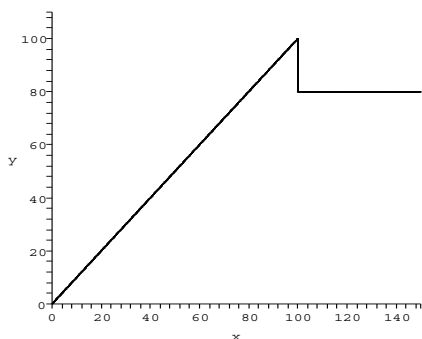$$y = -25(x - 34)$$
$$g(T) = -25(T - 34)$$

54. $\displaystyle\lim_{x \to 0} f(g(x)) = \lim_{x \to 0}(2x)^2 = 0$.

   $f(\displaystyle\lim_{x \to 0} g(x)) = f(\lim_{x \to 0} 2x) = f(0) = 4$.

   $\displaystyle\lim_{x \to 0} f(g(x)) \neq f(\lim_{x \to 0} g(x))$.

55.

The graph is discontinuous at $x = 100$. This is when the box starts moving.

56. The Intermediate Value Theorem does not apply because the function is not continuous over the interval $[-1, 2]$ (it is undefined at $x = 0$). The method of bisections converges to the discontinuity at $x = 0$.

57. Let $f(t)$ be her distance from home as a function of time on Monday. Let $g(t)$ be her distance from home as a function of time on Tuesday. Let t be given in minutes, with $t = 0$ corresponding to 7:13 a.m. Then she leaves home at $t = 0$ and arrives at her destination at $t = 410$. Let $h(t) = f(t) - g(t)$. If $h(t) = 0$ for some t, then the saleswoman was at exactly the same place at the same time on both Monday and Tuesday. $h(0) = f(0) - g(0) = -g(0) < 0$ and $h(410) = f(410) - g(410) = f(410) > 0$. By the Intermediate Value Theorem, there is a t in the interval $[0, 410]$ such that $h(t) = 0$.

58. My car was going forward as I approached the stop sign, rolled backward for a moment, then proceded forward again, so my car's velocity was positive, then negative, then positive again. Because my car's velocity is continuous, the Intermediate Value Theorem guarantees that the velocity must have been 0 in between changing from positive to negative, and again 0 between changing from negative to positive. This stopping is instantaneous; the police officer wanted to see me stop for long enough to look both ways and determine if it was safe to proceed.

59.
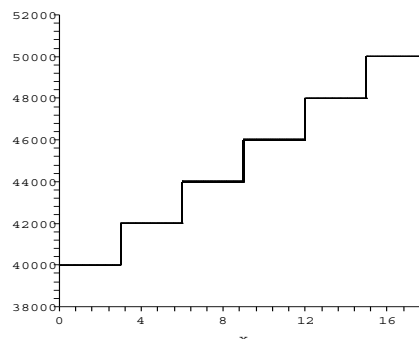
The function $s(t)$ has jump discontinuities every three months when the

salary suddenly increases by $2000. In the function $f(t)$, the $2000 increase occurs gradually over the 3 month period, so $f(t)$ is continuous. It might be easier to do calculations with $f(t)$ because it is continuous and because it is given by a simpler formula.

60. Theorem 4.2: Suppose that f and g are continuous at $x = a$. Then (ii) $(f \cdot g)$ is continuous at $x = a$ and (iii) (f/g) is continuous at $x = a$.

Proof: (ii) $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ by Theorem 3.1. This equals $f(a) \cdot g(a) = (f \cdot g)(a)$ since f and g are continuous at $x = a$.

(iii) $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x)/\lim_{x \to a} g(x)$ by Theorem 3.1. This equals $f(a)/g(a) = (f/g)(a)$ since f and g are continuous at $x = a$ and $g(a) \neq 0$.

61. We already know $f(x) \neq 0$ for $a < x < b$. Suppose $f(d) < 0$ for some d, $a < d < b$. Then by the Intermediate Value Theorem, there is an e in the interval $[c, d]$ such that $f(e) = 0$. But this e would also be between a and b, which is impossible. Thus, $f(x) > 0$ for all $a < x < b$.

62. Using the method of bisections starting with interval $[-3, -2]$ yields

x	$f(x)$
-3	-177
-2	5
-2.5	-47.16
-2.25	-14.17
-2.125	-3.14
-2.0625	1.256
-2.09375	-0.858

The root is in $(-2.09375, -2.0625)$. The actual root is approximately -2.08136.

The other root, approximately 1.15538, is found similarly.

63. $\lim_{x \to 0} x f(x) = \lim_{x \to 0} x \lim_{x \to 0} f(x)$
$= 0 f(0) = 0$

64. The function

$$f(x) = \begin{cases} \text{-1} & x \leq 0 \\ 1 & 0 < x \end{cases}$$

is not continuous at $x = 0$, but $x f(x)$ equals $|x|$ and $\lim_{x \to 0} x f(x) = 0$.

65. $\lim_{x \to a} g(x) = \lim_{x \to a} |f(x)| = \left| \lim_{x \to a} f(x) \right|$
$= |f(a)| = g(a).$

66. It is not true. The function $f(x)$ from the solution to exercise 64 is a counter-example. $|f(x)| = 1$ for all x, and so $|f(x)|$ is continuous, but $f(x)$ is not.

67. Let $b \geq a$. Then

$$\lim_{x \to b} h(x) = \lim_{x \to b} \left(\max_{a \leq t \leq b} f(t) \right)$$
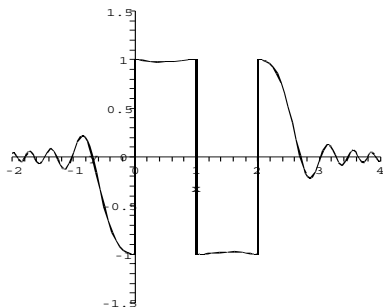$$= \max_{a \leq t \leq b} \left(\lim_{t \to b} f(t) \right)$$
$$= h(b)$$

since f is continuous. Thus, h is continuous for $x \geq a$.

No, the property would not be true if f were not assumed to be continuous. A counterexample is

$$f(x) = \begin{cases} 1 & \text{if} \quad a \leq x < b \\ 2 & \text{if} \quad b \leq x \end{cases}$$

Then $h(x) = 1$ for $a \leq x < b$, and $h(x) = 2$ for $x \geq b$. Thus, h is not continuous at $x = b$.

68. The function $f(x)$ is discontinuous where the denominator is 0, that is, at $x = 0$, $x = 1$ and $x = 2$.

1.5 Limits Involving Infinity

1. (a) $\displaystyle\lim_{x \to 1^-} \frac{1 - 2x}{x^2 - 1} = \infty$.

   (b) $\displaystyle\lim_{x \to 1^+} \frac{1 - 2x}{x^2 - 1} = -\infty$.

   (c) Does not exist.

2. (a) $\displaystyle\lim_{x \to -1^-} \frac{1 - 2x}{x^2 - 1} = \infty$.

   (b) $\displaystyle\lim_{x \to -1^+} \frac{1 - 2x}{x^2 - 1} = -\infty$.

   (c) Does not exist.

3. (a) $\displaystyle\lim_{x \to 2^-} \frac{x - 4}{x^2 - 4x + 4} = -\infty$

   (b) $\displaystyle\lim_{x \to 2^+} \frac{x - 4}{x^2 - 4x + 4} = -\infty$

   (c) $\displaystyle\lim_{x \to 2} \frac{x - 4}{x^2 - 4x + 4} = -\infty$

4. (a) $\displaystyle\lim_{x \to -1^-} \frac{1 - x}{(x + 1)^2} = \infty$

   (b) $\displaystyle\lim_{x \to -1^+} \frac{1 - x}{(x + 1)^2} = \infty$

   (c) $\displaystyle\lim_{x \to -1} \frac{1 - x}{(x + 1)^2} = \infty$

5. $\displaystyle\lim_{x \to 2^-} \frac{-x}{\sqrt{4 - x^2}} = -\infty$.

As x approaches 2 from below, the numerator is near -2 and the denominator is small and positive, so the faction goes to $-\infty$.

6. $\displaystyle\lim_{x \to -1^-} (x^2 - 2x - 3)^{-2/3} = \infty$.

As x approaches -1, $x^2 - 2x - 3$ is small, so $(x^2 - 2x - 3)^{2/3}$ is small and positive, so $(x^2 - 2x - 3)^{-2/3}$ is large and positive, so the limit is ∞.

7. $\displaystyle\lim_{x \to -\infty} \frac{-x}{\sqrt{4 + x^2}}$

$= \displaystyle\lim_{x \to -\infty} \frac{-x}{-x\sqrt{\frac{4}{x^2} + 1}}$

$= \displaystyle\lim_{x \to -\infty} \frac{1}{\sqrt{\frac{4}{x^2} + 1}}$

$= \dfrac{1}{\sqrt{1}} = 1$

8. $\displaystyle\lim_{x \to \infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1}$

$= \displaystyle\lim_{x \to \infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1} \left(\frac{1/x^2}{1/x^2} \right)$

$= \displaystyle\lim_{x \to \infty} \frac{2 - 1/x + 1/x^2}{4 - 3/x - 1/x^2} = \frac{1}{2}$.

9. $\displaystyle\lim_{x \to \infty} \frac{x^3 - 2\cos x}{3x^2 + 4x - 1}$

$= \displaystyle\lim_{x \to \infty} \frac{x^2 \left(x - \frac{2\cos x}{x^2} \right)}{x^2 \left(3 + \frac{4}{x} - \frac{1}{x^2} \right)}$

$= \displaystyle\lim_{x \to \infty} \frac{\left(x - \frac{2\cos x}{x^2} \right)}{3 + \frac{4}{x} - \frac{1}{x^2}} = \infty$

10. $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 1}{4x^3 - 5x - 1} = 0$.

11. $\displaystyle\lim_{x \to \infty} \ln 2x = \infty$.

Note that $\ln 2x = \ln 2 + \ln x$, so it is enough to show that $\ln x$ goes to ∞ as x goes to ∞. This can be seen from the graph of the function $\ln x$ on page 51.

12. $\displaystyle\lim_{x \to 0^+} \ln 2x = -\infty$.

13. $\lim\limits_{x \to 0^+} e^{-2/x} = 0.$

When x is small and positive, $-2/x$ is large and negative, and e raised to a large negative power is very small.

14. $\lim\limits_{x \to \infty} e^{-2/x} = 1.$

15. $\lim\limits_{x \to \infty} \cot^{-1} x = 0.$

(Compare Example 5.8) We are looking for the angle that θ must approach as $\cot \theta$ goes to ∞. Look at the graph of $\cot \theta$. To define the inverse cotangent, you must pick one branch of this graph, and the standard choice is the branch immediately to the right of the y-axis. Then as $\cot \theta$ goes to ∞, the angle goes to 0.

16. $\lim\limits_{x \to \infty} \sec^{-1} x = \dfrac{\pi}{2}.$

17. $\lim\limits_{x \to \infty} e^{2x-1} = \infty.$

As x gets large, $2x - 1$ gets large, and e raised to a large positive power is large and positive.

18. $\lim\limits_{x \to 0} e^{1/x^2} = \infty.$

When x is small, x^2 is small and positive, so $1/x^2$ is large and positive, and e raised to a large positive power is large and positive.

19. $\lim\limits_{x \to \infty} \sin 2x$ does not exist. As x gets larger and larger, the values of $\sin 2x$ oscillate between 1 and -1.

20. $\lim\limits_{x \to \infty} \left(e^{-3x} \cos 2x \right) = 0.$

The function $e^{-3x} \cos 2x$ is squeezed between $-e^{-3x}$ and e^{-3x}, both of which go to 0 as x goes to ∞. By the squeeze theorem, $\lim\limits_{x \to \infty} \left(e^{-3x} \cos 2x \right) = 0.$

21. As x goes to ∞, both e^{3x} and e^x go to ∞ as well. Furthermore, as x goes to ∞, so does $\ln x$. Thus it looks like

$$\lim_{x \to \infty} \left(\frac{\ln(2 + e^{3x})}{\ln(1 + e^x)} \right) = \frac{\infty}{\infty}.$$

This is an indeterminate form, i.e., we don't know from this analysis what happens in this limit. Looking at numerical and/or graphing evidence, we guess that the limit is 3.

22. $\lim\limits_{x \to \infty} \sin(\tan^{-1} x) = \lim\limits_{x \to \frac{\pi}{2}} (\sin x) = 1.$

23.
$$\lim_{x \to \frac{\pi}{2}^-} e^{-\tan x} = \lim_{x \to \infty} e^{-x}$$
$$= \lim_{x \to -\infty} e^x = 0, \text{ but}$$
$$\lim_{x \to \frac{\pi}{2}^+} e^{-\tan x} = \lim_{x \to -\infty} e^{-x}$$
$$= \lim_{x \to \infty} e^x = \infty,$$

so the limit does not exist.

24.
$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = \lim_{x \to -\infty} \tan^{-1} x$$
$$= -\frac{\pi}{2}.$$

25. Since $4 + x^2$ is never 0, there are no vertical asymptotes. We have
$$\lim_{x \to \infty} \frac{x}{\sqrt{4 + x^2}}$$
$$= \lim_{x \to \infty} \frac{x}{x\sqrt{\frac{4}{x^2} + 1}}$$
$$= \lim_{x \to \infty} \frac{1}{\sqrt{\frac{4}{x^2} + 1}}$$
$$= \frac{1}{\sqrt{1}} = 1$$

and
$$\lim_{x \to -\infty} \frac{x}{\sqrt{4 + x^2}}$$
$$= \lim_{x \to -\infty} \frac{x}{-x\sqrt{\frac{4}{x^2} + 1}}$$
$$= \lim_{x \to -\infty} \frac{-1}{\sqrt{\frac{4}{x^2} + 1}}$$

$$= \frac{-1}{\sqrt{1}} = -1,$$

so there are horizontal asymptotes at $y = 1$ and $y = -1$.

26. The function is only defined in $(-2, 2)$. Two one-sided vertical asymptotes at $x = \pm 2$. $f(x) \to \infty$ as $x \to 2^-$, and $f(x) \to -\infty$ as $x \to -2^+$. No horizontal asymptotes.

27. $4 - x^2 = 0 \Rightarrow 4 = x^2$ so we have vertical asymptotes at $x = \pm 2$. We have

$$\lim_{x \to \pm\infty} \frac{x}{4 - x^2}$$

$$= \lim_{x \to \pm\infty} \frac{x}{x^2\left(\frac{4}{x^2} - 1\right)}$$

$$= \lim_{x \to \pm\infty} \frac{1}{x\left(\frac{4}{x^2} - 1\right)} = 0.$$

So there is a horizontal asymptote at $y = 0$.

28. Vertical asymptotes at $x = \pm 2$.

$f(x) \to \infty$ as $x \to 2^-$ and $x \to -2^+$.

$f(x) \to -\infty$ as $x \to 2^+$ and $x \to -2^-$.

Horizontal asymptote at $y = -1$.

29. The denominator factors: $x^2 - 2x - 3 = (x - 3)(x + 1)$. Since neither $x = 3$ nor $x = -1$ are zeros of the numerator, we see that $f(x)$ has vertical asymptotes at $x = 3$ and $x = -1$.

$f(x) \to -\infty$ as $x \to 3^-$,
$f(x) \to \infty$ as $x \to 3^+$,
$f(x) \to \infty$ as $x \to -1^-$, and
$f(x) \to -\infty$ as $x \to -1^+$.

We have

$$\lim_{x \to \pm\infty} \frac{3x^2 + 1}{x^2 - 2x - 3}$$

$$\lim_{x \to \pm\infty} \frac{3 + 1/x^2}{1 - 2/x - 3/x^2} = 3.$$

So there is a horizontal asymptote at $y = 3$.

30. Vertical asymptote at $x = -2$.

$f(x) \to \infty$ as $x \to -2^-$.
$f(x) \to -\infty$ as $x \to -2^+$ and $x \to -2^-$.

No horizontal asymptotes.

31. The function $\ln x$ has a one-sided vertical asymptote at $x = 0$, so $f(x) = \ln(1 - \cos x)$ will have a vertical asymptote whenever $1 - \cos x = 0$, i.e., whenever $\cos x = 1$. This happens when $x = 2k\pi$ for any integer k. Since $1 - \cos x \geq 0$ for all x, $f(x)$ is defined at all points except for these vertical asymptotes. Thus as $f(x)$ approaches any of these asymptotes (from either side), it behaves like $\ln x$ approaching 0 from the right, so $f(x) \to -\infty$ as x approaches any of these asymptotes from either side.

32. One-sided vertical asymptote at $x = -2$. $f(x) \to -\infty$ as $x \to -2^+$. Horizontal asymptote at $y = \frac{1}{2}$.

33. The function is continuous for all x, so no vertical asymptotes. We have

$$\lim_{x \to \infty} 4\tan^{-1} x - 1 = 4(\lim_{x \to \infty} \tan^{-1} x) - 1$$
$$= 4(\pi/2) - 1$$
$$= 2\pi - 1$$

and

$$\lim_{x \to -\infty} 4\tan^{-1} x - 1$$
$$= 4(\lim_{x \to -\infty} \tan^{-1} x) - 1$$
$$= 4(-\pi/2) - 1$$
$$= -2\pi - 1,$$

so there are horizontal asymptotes at $y = 2\pi - 1$ and $y = -2\pi - 1$.

34. One-sided vertical asymptote at $x = 0$. $f(x) \to \infty$ as $x \to 0^-$. Horizontal asymptote at $y = 3$.

35. Vertical asymptotes at $x = \pm 2$. The slant asymptote is $y = -x$.

36. Vertical asymptote at $x = 2$. The slant asymptote is $y = x + 2$.

37. Vertical asymptotes at

$$x = \frac{-1 \pm \sqrt{17}}{2}.$$

The slant asymptote is $y = x - 1$.

38. Vertical asymptote at $x = -\sqrt[3]{2}$. The slant asymptote is $y = x$.

39. When x is large, the value of the fraction is close to 0.

40. $\lim\limits_{x \to \infty} \dfrac{2^x}{x^2} = \infty$.

41. When x is large, the value of the fraction is very close to $\frac{1}{2}$.

42. When x is large and negative, the value of the fraction is very close to 2.

43. $\lim\limits_{x \to \infty} \dfrac{x^3 + 4x + 5}{e^{x/2}} = 0$.

44. $\lim\limits_{x \to \infty} \left(e^{x/3} - x^4 \right) = \infty$.

45. When x is close to -1, the value of the fraction is close to 1.

46. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$.

47. When x is close to 0, the value of the fraction is large and negative, so the limit appears to be $-\infty$.

48. $\lim\limits_{x \to 0} \dfrac{\ln(x^2)}{x^2} = -\infty$.

49. We multiply by

$$\frac{\sqrt{4x^2 - 2x + 1} + 2x}{\sqrt{4x^2 - 2x + 1} + 2x}$$

to get:

$$\lim_{x \to \infty} \left(\sqrt{4x^2 - 2x + 1} - 2x \right)$$

$$= \lim_{x \to \infty} \frac{-2x + 1}{\sqrt{4x^2 - 2x + 1} + 2x} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x \to \infty} \frac{-2 + 1/x}{\sqrt{4 - 2/x + 1/x^2} + 2}$$

$$= \frac{-2}{\sqrt{4} + 2} = -\frac{1}{2}.$$

50. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3} - x \right) \cdot \dfrac{\sqrt{x^2 + 3} + x}{\sqrt{x^2 + 3} + x}$

$$= \lim_{x \to \infty} \frac{3}{\sqrt{x^2 + 3} + x} = 0.$$

51. $\lim\limits_{x \to \infty} \left(\sqrt{5x^2 + 4x + 7} - \sqrt{5x^2 + x + 3} \right)$

If we multiply by

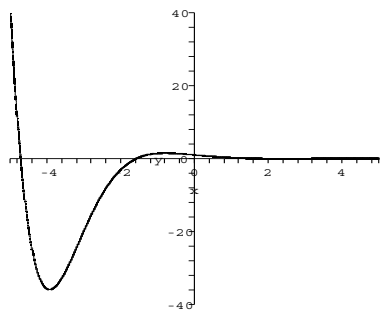$$\frac{\sqrt{5x^2 + 4x + 7} + \sqrt{5x^2 + x + 3}}{\sqrt{5x^2 + 4x + 7} + \sqrt{5x^2 + x + 3}},$$

we get

$$\lim_{x \to \infty} \frac{(5x^2 + 4x + 7) - (5x^2 + x + 3)}{\sqrt{5x^2 + 4x + 7} + \sqrt{5x^2 + x + 3})}$$

$$= \lim_{x \to \infty} \frac{3x + 4}{\sqrt{5x^2 + 4x + 7} + \sqrt{5x^2 + x + 3}}$$

$$= \lim_{x \to \infty} \frac{3 + \frac{4}{x}}{\sqrt{5 + \frac{4}{x} + \frac{7}{x^2}} + \sqrt{5 + \frac{1}{x} + \frac{3}{x^2}}}$$

$$= \frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{10}$$

52. $\lim\limits_{x \to \infty} \left(1 + \dfrac{3}{x} \right)^{2x}$

$$= \lim_{x \to \infty} \left[\left(1 + \frac{3}{x} \right)^{\frac{x}{3}} \right]^6$$

$$= \left[\lim_{x \to \infty} \left(1 + \frac{1}{\frac{x}{3}} \right)^{\frac{x}{3}} \right]^6 = e^6.$$

53.

on $[-10, 10]$ by $[-100, 100]$

The horizontal asymptote is $y = 0$ approached only as $x \to \infty$. The graph crosses the horizontal asymptote an infinite number of times.

54. $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to 0^+} f(1/x)$ because $1/x \to \infty$ as $x \to 0^+$.

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to 0^-} f(1/x)$ because $1/x \to -\infty$ as $x \to 0^-$.

55. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = \lim\limits_{x \to 0^+} (1 + x)^{1/x}$

$= \lim\limits_{x \to 0^-} (1 + x)^{1/x} = \lim\limits_{x \to -\infty} \left(1 + \dfrac{1}{x}\right)^x$

56. One possible pair: $f(x) = 1/x^2$, $g(x) = 1/x$. This pair would violate Theorem 3.1 because $\infty - \infty$ and $\frac{\infty}{\infty}$ do not make sense outside of a limit.

57. $h(0) = \dfrac{300}{1 + 9(.8^0)} = \dfrac{300}{10} = 30$ mm

$\lim\limits_{t \to \infty} \dfrac{300}{1 + 9(.8^t)} = 300$ mm

58. Length at $t = 0$ is $h(0) = 20$ mm. Length eventually is $\lim\limits_{t \to \infty} h(t) = 50$ mm.

59. $\lim\limits_{x \to 0^+} \dfrac{80x^{-.3} + 60}{2x^{-.3} + 5} \left(\dfrac{x^{.3}}{x^{.3}}\right)$

$= \lim\limits_{x \to 0^+} \dfrac{80 + 60x^{.3}}{2 + 5x^{.3}}$

$= \dfrac{80}{2} = 40$ mm

$\lim\limits_{x \to \infty} \dfrac{80x^{-.3} + 60}{2x^{-.3} + 5} = \dfrac{60}{5} = 12$ mm

60. Re-write the function as

$$f(x) = \frac{80 + 60x^{0.3}}{8 + 15x^{0.3}}$$

to see that the size with no light is $f(0) = 10$ mm, and the size with infinite light is $\lim\limits_{x \to \infty} f(x) = 4$ mm.

61. $f(x) = \dfrac{80x^{-0.3} + 60}{10x^{-0.3} + 30}$

62. $f(t) \to 0$ as $t \to 0$ and $t \to \infty$. This makes sense because the drug will require some time to reach the muscles, and should wear off over time.

63. $\lim\limits_{t \to \infty} v_N = \lim\limits_{t \to \infty} \dfrac{Ft}{m} = \infty$

$\lim\limits_{t \to \infty} v_E = \lim\limits_{t \to \infty} \dfrac{Fct}{\sqrt{m^2c^2 + F^2t^2}}$

$= \lim\limits_{t \to \infty} \dfrac{Fct}{t\sqrt{\frac{m^2c^2}{t^2} + F^2}}$

$= \lim\limits_{t \to \infty} \dfrac{Fc}{\sqrt{\frac{m^2c^2}{t^2} + F^2}}$

$= \dfrac{Fc}{\sqrt{F^2}} = c$

64. $\lim\limits_{v \to 0} m = m_0$ is the mass when the velocity is zero, or the rest mass. $\lim\limits_{v \to c^-} m = \infty$, so as velocity approaches the speed of light, mass increases without bound. At a speed of 9000ft/s, the masses increase by a factor of $\dfrac{1}{\sqrt{1 - 9000^2/9.8^2 \times 10^{16}}} \approx$ 1.0000000000421699292. The actual increase for rest mass $m_0 = 6$ is 2.53×10^{-10}.

65. As in Example 5.10, the terminal velocity is $-\sqrt{\dfrac{32}{k}}$. When $k = 0.00064$,

the terminal velocity is $-\sqrt{\frac{32}{.00064}} \approx -224$. When $k = 0.00128$, the terminal velocity is $-\sqrt{\frac{32}{.00128}} \approx -158$.

Solve $\sqrt{\frac{32}{ak}} = \frac{1}{2}\sqrt{\frac{32}{k}}$. Squaring both sides, $\frac{32}{ak} = \frac{1}{4} \cdot \frac{32}{k}$ so $a = 4$.

66. Looking at the graph, we estimate the time to 90% of terminal velocity is about 20 seconds.
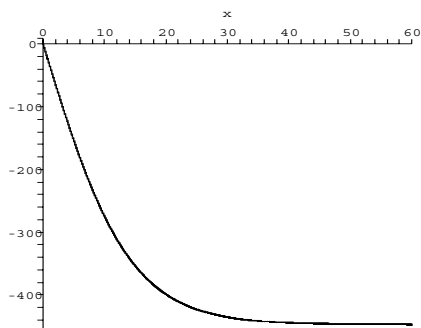

The terminal velocity when $k = 0.001$ is 178.9, and 90% of terminal velocity is 161.0. From the graph we see that it takes about 8.2s to reach 90% of terminal velocity.
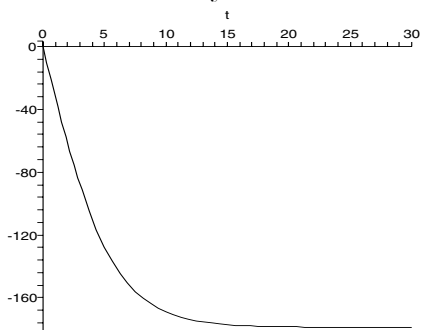
67. We must restrict the domain to $v_0 \geq 0$ because the formula makes sense only if the rocket is launched upward. To find v_e, set $19.6R - v_0^2 = 0$. Using $R \approx 6{,}378{,}000$ meters, we get $v_0 = \sqrt{19.6R} \approx 11{,}180\text{m/s}$. If the rocket is launched with initial velocity $\geq v_e$, it will never return to earth;

hence v_e is called the escape velocity.
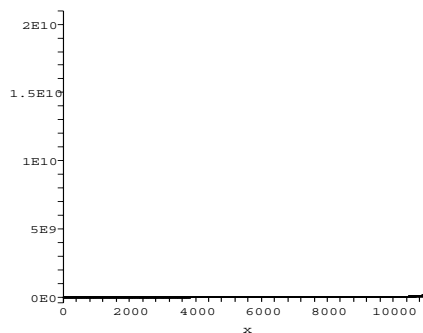
68. If the degree of the polynomial in the denominator is larger, the horizontal asymptote is $y = 0$.

69. Suppose the degree of q is n. If we divide both $p(x)$ and $q(x)$ by x^n, then the new denominator will approach a constant while the new numerator tends to ∞, so there is no horizontal asymptote.

70. If the horizontal asymptote is $y = 2$, the degrees of the numerator and denominator must be the same.

71. When we do long division, we get a remainder of $x + 2$, so the degree of p is one greater than the degree of q.

72. The function $q(x) = \dfrac{x^2}{2} - \dfrac{9}{2}$ satisfies the given conditions.

73. The function $q(x) = -2(x-2)(x-3)$ satisfies the given conditions.

74. The function $g(x) = \frac{2}{\pi} \cdot \tan^{-1} x \cdot (x-4)$ satisfies the given conditions.

75. True.

76. False if $b = 0$; otherwise true.

77. False.

78. True

79. True.

80. False. For example, $f(x) = 2x$ and $g(x) = x$.

**81.** Vertical asymptote at $x = 2$. Horizontal asymptotes at $y = 4$ and $y = 0$.

82. $x^2 - 4x = x(x - 4)$, so there is a vertical asymptote at $x = 0$.
$x^2 - 7x + 10 = (x - 2)(x - 5)$ so there are vertical asymptotes at $x = 2$ and $x = 5$.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 - 1}{x^2 - 7x + 10} = 1$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x + 3}{x^2 - 4x} = 0$$

So there are horizontal asymptotes at $y = 0$ and $y = 1$.

83. For any positive constant a, $e^{-at} \to 0$ as $t \to \infty$. Since $\sin t$ oscilates between -1 and 1, $e^{-at}\sin t \to 0$ as $t \to \infty$. In the following graph, we see that suspension system A damps out at about 5 seconds, while system B takes about 18 seconds to damp out.
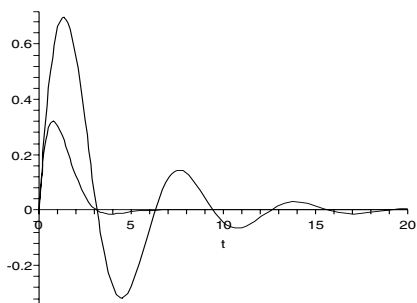
84. (a) $\displaystyle \lim_{x \to -\infty} p_n(x)$
$$= \lim_{x \to -\infty} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \right)$$
$$= \lim_{x \to -\infty} \left[x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x} \right) \right]$$
$$= \lim_{x \to -\infty} a_n x^n$$
When the degree n is odd, if a_n is positive, the limit as $x \to -\infty$ is $-\infty$, and if a_n is negative, the limit as $x \to -\infty$ is $+\infty$.

(b) As in part (a), we have
$$\lim_{x \to -\infty} p_n(x) = \lim_{x \to -\infty} a_n x^n$$
When the degree n is even, if a_n is positive, the limit as $x \to -\infty$ is $+\infty$, and if a_n is negative, the limit as $x \to -\infty$ is $-\infty$.

85. $g(x) = \sin x$, $h(x) = x$ at $a = 0$

86. The statement is not true if h and g are allowed to be any functions. For example if g has an asymptote at a, then $h(a)$ need not be zero. If h and g are polynomials, then the statement is true. The only way for $f(x) = \dfrac{g(x)}{h(x)}$ to have a vertical asymptote at $x = a$ is for $h(a) = 0$.

87. $\displaystyle \lim_{x \to 0^+} x^{1/(\ln x)} = e \approx 2.71828$
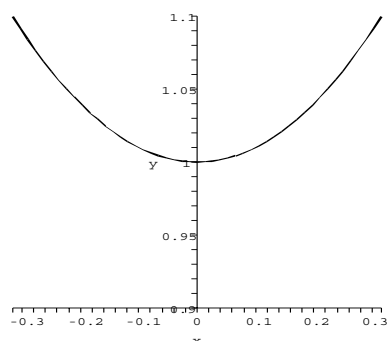
88. $\displaystyle \lim_{x \to 1^+} (\ln x)^{x^2 - 1} = 1$.

89. $\displaystyle \lim_{x \to \infty} x^{1/x} = 1$

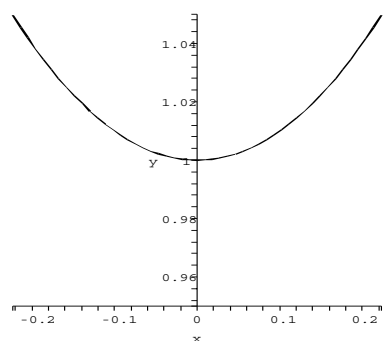**90.** $\displaystyle \lim_{x \to -\infty} \frac{\ln x}{x^2}$ is undefined, since the natural log is not defined for negative values.
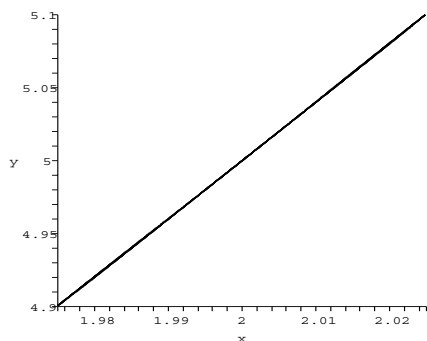
1.6 Formal Definition of the Limit

1. (a) From the graph, we determine that we can take $\delta = 0.316$, as shown below.
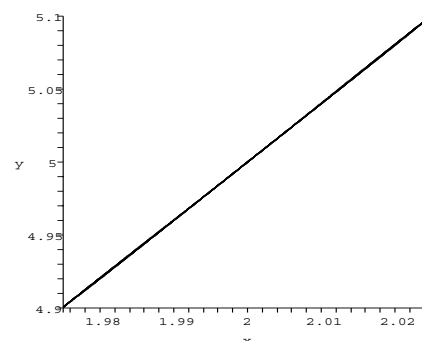
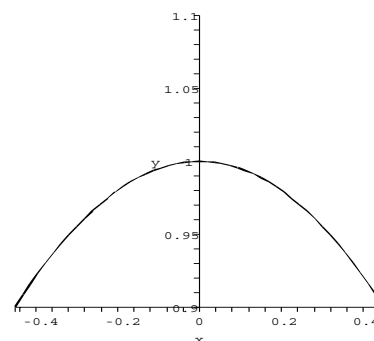(b) From the graph, we determine that we can take $\delta = 0.223$, as shown below.

3. (a) From the graph, we determine that we can take $\delta = 0.45$, as shown below.

(b) From the graph, we determine that we can take $\delta = 0.315$, as shown below.

2. (a) From the graph, we determine that we can take $\delta = 0.025$, as shown below.
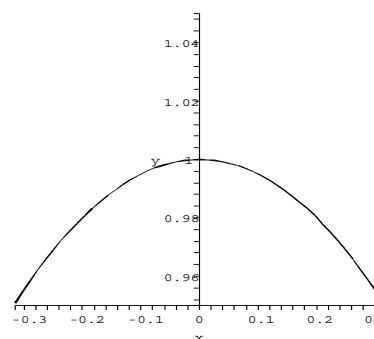

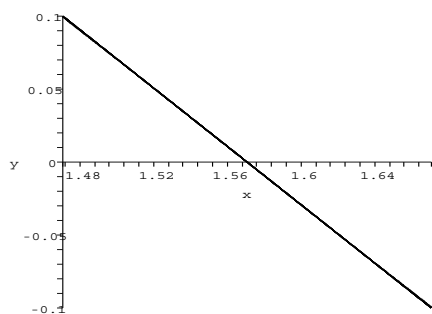(b) From the graph, we determine that we can take $\delta = 0.0125$, as shown below.

4. (a) From the graph, we determine that we can take $\delta = 0.1$, as shown below.

(b) From the graph, we determine that we can take $\delta = 0.05$, as shown below.

6. (a) From the graph, we determine that we can take $\delta = 0.19$, as shown below.

(b) From the graph, we determine that we can take $\delta = 0.095$, as shown below.

5. (a) From the graph, we determine that we can take $\delta = 0.38$, as shown below.
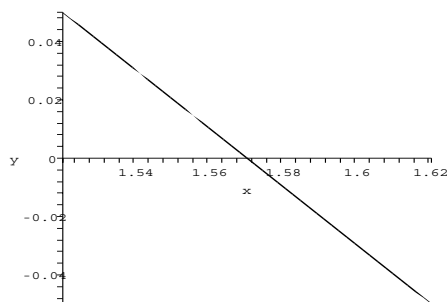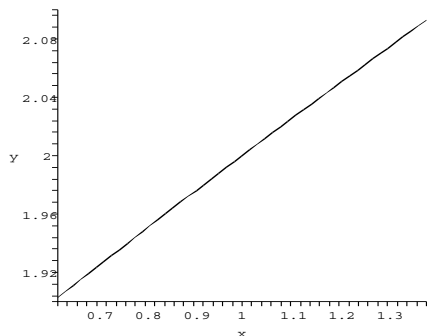

(b) From the graph, we determine that we can take $\delta = 0.2$, as shown below.

7. (a) From the graph, we determine that we can take $\delta = 0.02$, as shown below.

(b) From the graph, we determine that we can take $\delta = 0.01$, as shown below.

8. (a) From the graph, we determine that we can take $\delta = 0.12$, as shown below.

(b) From the graph, we determine that we can take $\delta = 0.06$, as shown below.
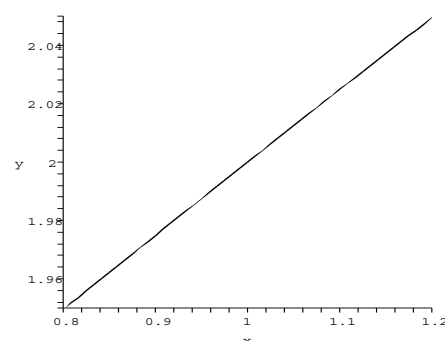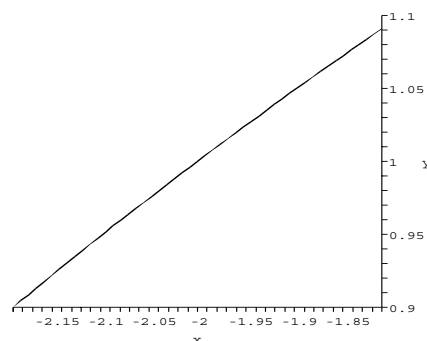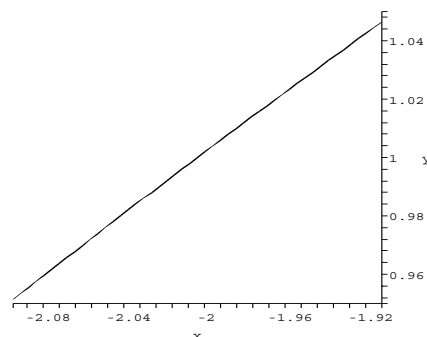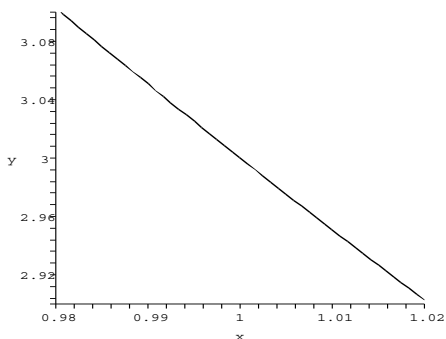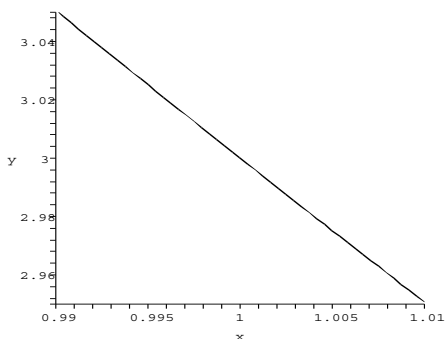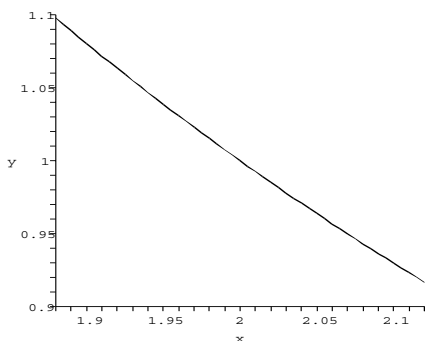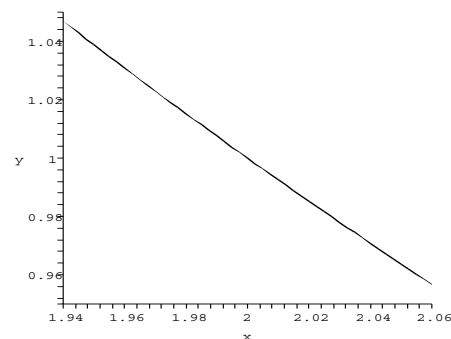
9. We want $|3x - 0| < \varepsilon$
$\Leftrightarrow 3|x| < \varepsilon$
$\Leftrightarrow |x| = |x - 0| < \varepsilon/3$
Take $\delta = \varepsilon/3$.

10. We want $|3x - 3| < \varepsilon$
$\Leftrightarrow 3|x - 1| < \varepsilon$
$\Leftrightarrow |x - 1| < \varepsilon/3$
Take $\delta = \varepsilon/3$.

11. We want $|3x + 2 - 8| < \varepsilon$
$\Leftrightarrow |3x - 6| < \varepsilon$
$\Leftrightarrow 3|x - 2| < \varepsilon$
$\Leftrightarrow |x - 2| < \varepsilon/3$
Take $\delta = \varepsilon/3$.

12. We want $|3x + 2 - 5| < \varepsilon$
$\Leftrightarrow |3x - 3| < \varepsilon$
$\Leftrightarrow 3|x - 1| < \varepsilon$
$\Leftrightarrow |x - 1| < \varepsilon/3$
Take $\delta = \varepsilon/3$.

13. We want $|3 - 4x - (-1)| < \varepsilon$
$\Leftrightarrow |-4x + 4| < \varepsilon$
$\Leftrightarrow 4|-x + 1| < \varepsilon$
$\Leftrightarrow 4|x - 1| < \varepsilon$
$\Leftrightarrow |x - 1| < \varepsilon/4$
Take $\delta = \varepsilon/4$.

14. We want $|3 - 4x - 7| < \varepsilon$
$\Leftrightarrow |-4x - 4| < \varepsilon$
$\Leftrightarrow 4|-x - 1| < \varepsilon$
$\Leftrightarrow 4|x + 1| < \varepsilon$
$\Leftrightarrow |x + 1| < \varepsilon/4$
Take $\delta = \varepsilon/4$.

15. We want $\left| \dfrac{x^2 + x - 2}{x - 1} - 3 \right| < \varepsilon$.

We have

$$\left| \frac{x^2 + x - 2}{x - 1} - 3 \right|$$
$$= \left| \frac{(x + 2)(x - 1)}{x - 1} - 3 \right|$$
$$= |x + 2 - 3|$$
$$= |x - 1|$$

Take $\delta = \varepsilon$.

16. We want $\left| \dfrac{x^2 - 1}{x + 1} - (-2) \right| < \varepsilon$.

We have

$$\left| \frac{x^2 - 1}{x + 1} + 2 \right| = \left| \frac{(x + 1)(x - 1)}{x + 1} + 2 \right|$$
$$= |x - 1 + 2|$$
$$= |x + 1|$$

Take $\delta = \varepsilon$.

17. We want $\left| x^2 - 1 - 0 \right| < \varepsilon$.
We have $|x^2 - 1| = |x - 1||x + 1|$. We require that $\delta < 1$, i.e., $|x - 1| < 1$ so $0 < x < 2$ and $|x + 1| < 3$. Then $|x^2 - 1| = |x - 1||x + 1| < 3|x - 1|$. Requiring this to be less than ε gives $|x - 1| < \varepsilon/3$, so $\delta = \min\{1, \varepsilon/3\}$.

18. We want $\left| x^2 - x + 1 - 1 \right| < \varepsilon$.
We have $|x^2 - x| = |x||x - 1|$. We require that $\delta < 1$, i.e., $|x - 1| < 1$ so $0 < x < 2$ and $|x| < 2$. Then $|x^2 - x| = |x||x - 1| < 2|x - 1|$. Requiring this to be less than ε gives $|x - 1| < \varepsilon/2$, so $\delta = \min\{1, \varepsilon/2\}$.

19. We want $\left| x^2 - 1 - 3 \right| < \varepsilon$.
We have $|x^2 - 4| = |x - 2||x + 2|$. We require that $\delta < 1$, i.e., $|x - 1| < 1$ so $1 < x < 3$ and $|x + 2| < 5$. Then $|x^2 - 4| = |x - 2||x + 2| < 5|x - 2|$. Requiring this to be less than ε gives $|x - 2| < \varepsilon/5$, so $\delta = \min\{1, \varepsilon/5\}$.

20. We want $\left| x^3 + 1 - 1 \right| < \varepsilon$, i.e., $|x^3| < \varepsilon$. Take $\delta = \sqrt[3]{\varepsilon}$.

21. Let $f(x) = mx + b$. Since $f(x)$ is continous, we know that $\lim\limits_{x \to a} f(x) = ma + b$. So we want to find a δ which forces $|mx + b - (ma + b)| < \varepsilon$. But

$$|mx + b - (ma + b)| = |mx - ma|$$
$$= |m||x - a|.$$

So as long as $|x - a| < \delta = \varepsilon/|m|$, we will have $|f(x) - (ma + b)| < \varepsilon$. This δ clearly does not depend on a. This is due to the fact that $f(x)$ is a linear function, so the slope is constant, which means that the ratio of the change in y to the change in x is constant.

22. Since the δ obtained in exercise 17 is different from that of exercise 19, we see immediately that the value of δ for $\lim\limits_{x \to a}(x^2 + b)$ does depend on a. In this case the ratio of the change in y to the change in x depends very much on the value of a. Near the origin, the graph is not very steep at all, while away from the origin the graph can become very steep indeed.

23. For a function $f(x)$ defined on some open interval (c, a) we say

$$\lim_{x \to a^-} f(x) = L$$

if, given any number $\varepsilon > 0$, there is another number $\delta > 0$ such that whenever $x \in (c, a)$ and $a - \delta < x < a$, we have $|f(x) - L| < \varepsilon$.

For a function $f(x)$ defined on some open interval (a, c) we say

$$\lim_{x \to a^+} f(x) = L$$

if, given any number $\varepsilon > 0$, there is another number $\delta > 0$ such that whenever $x \in (a, c)$ and $a < x < a + \delta$, we have $|f(x) - L| < \varepsilon$.

24. Note that $\left|\dfrac{1}{x} - 1\right| = \left|\dfrac{1-x}{x}\right|$. As $x \to 1^-$, we see that $1 - x > 0$ and $x > 0$ (we need not consider negative values of x). Thus we need to solve the inequality $\dfrac{1-x}{x} < 0.1$:

$$\frac{1-x}{x} < 0.1$$
$$1 - x < 0.1x$$
$$1 < 1.1x$$
$$\frac{1}{1.1} < x$$
$$0.909090\ldots < x$$

Thus we take
$\delta_1 = 1 - 0.909090\ldots = 0.090909\ldots$.
Similarly, as $x \to 1^+$, we have $x - 1 > 0$ and $x > 0$. Therefore we need

$$\frac{x-1}{x} < 0.1$$
$$x - 1 < 0.1x$$
$$0.9x < 1$$
$$x < \frac{1}{0.9}$$
$$x < 1.111111\ldots$$

Thus we take
$\delta_2 = 1.111111\ldots - 1 = 0.111111\ldots$.

In the definition of the limit we need to take the smaller δ (δ_1) to ensure that $|f(x) - L| < \varepsilon$ on both sides of $a = 1$.

To prove that $\lim\limits_{x \to 1} 1/x = 1$, we take $\delta < 1/2$, so that $1/2 < x < 3/2$. Then

$$\left|\frac{1-x}{x}\right| < \left|\frac{1-x}{\frac{1}{2}}\right|$$
$$= 2|1 - x|$$
$$= 2|x - 1|$$

To get this to be less than ε, we take $\delta = \min\{1/2, \varepsilon/2\}$.

25. As $x \to 1^+$, $x - 1 > 0$ so we compute

$$\frac{2}{x-1} > 100$$
$$2 > 100(x - 1)$$
$$\frac{2}{100} > x - 1$$

So take $\delta = 2/100$.

26. As $x \to 1^-$, $x - 1 < 0$ so we compute

$$\frac{2}{x-1} < -100$$
$$2 > -100(x - 1)$$
$$-\frac{2}{100} < x - 1$$
$$\frac{2}{100} > -x + 1 = |x - 1|$$

So take $\delta = 2/100$.

27. We look at the graph of $\cot x$ as $x \to 0^+$ and we find that we should take $\delta = 0.00794$.

28. We look at the graph of $\cot x$ as $x \to \pi^-$ and we find that we should take $\delta = 0.0098$.

29. As $x \to 2^-$, $4 - x^2 > 0$ so we compute

$$\frac{2}{\sqrt{4 - x^2}} > 100$$
$$2 > 100\sqrt{4 - x^2}$$
$$\frac{2}{100} > \sqrt{4 - x^2}$$
$$\frac{4}{10000} > 4 - x^2 = (2 - x)(2 + x)$$

Take $\delta < 1$ so that $1 < x < 3$ so we have $2 + x < 5$. Then $(2-x)(2+x) < (2-x)5$. Now, if we require $|x-2| < \frac{4}{50000}$ then $\frac{2}{\sqrt{4-x^2}} > 100$. So let $\delta = \frac{4}{50000}$.

30. We want $\ln x < -100$. This is true as long as $0 < x < e^{-100}$.

31. We want M such that if $x > M$,

$$\left| \frac{x^2 - 2}{x^2 + x + 1} - 1 \right| < 0.1$$

We have

$$\left| \frac{x^2 - 2}{x^2 + x + 1} - 1 \right|$$

$$= \left| \frac{x^2 - 2 - (x^2 + x + 1)}{x^2 + x + 1} \right|$$

$$= \left| \frac{-x - 3}{x^2 + x + 1} \right|$$

$$= \left| \frac{x + 3}{x^2 + x + 1} \right|$$

Now, as long as $x > 3$, we have

$$\left| \frac{x + 3}{x^2 + x + 1} \right| < \left| \frac{2x}{x^2 + x} \right|$$

$$= \left| \frac{2}{x + 1} \right|$$

We want $\left| \dfrac{2}{x + 1} \right| < 0.1$. Since $x \to \infty$, we can take $x > 0$, so we solve $\dfrac{2}{x + 1} < 0.1$ to get $x > 19$, i.e., $M = 19$.

32. Since $x \to \infty$, we can take $x > 4$ and then

$$\left| \frac{x - 2}{x^2 + x + 1} \right| < \left| \frac{x}{x^2 + x} \right|$$

$$= \left| \frac{1}{x + 1} \right|$$

Now we solve $\dfrac{1}{x + 1} < 0.1$ to get $x > 9$, i.e., $M = 9$.

33. We have

$$\left| \frac{x^2 + 3}{4x^2 - 4} - \frac{1}{4} \right| = \left| \frac{x^2 + 3 - (x^2 - 1)}{4x^2 - 4} \right|$$

$$= \left| \frac{4}{4x^2 - 4} \right|$$

$$= \left| \frac{1}{x^2 - 1} \right|$$

Since $x \to -\infty$, we may take $x < -1$ so that $x^2 - 1 > 0$. We now need $\dfrac{1}{x^2 - 1} < 0.1$. Solving for x gives $|x| > \sqrt{11} \approx 3.3166$. So we can take $N = -4$.

34. We have

$$\left| \frac{3x^2 - 2}{x^2 + 1} - 3 \right| = \left| \frac{3x^2 - 2 - (3x^2 + 3)}{x^2 + 1} \right|$$

$$= \left| \frac{-5}{x^2 + 1} \right|$$

$$= \left| \frac{5}{x^2 + 1} \right|$$

We now need $\dfrac{5}{x^2 + 1} < 0.1$. Solving for x gives $|x| > 7$, i.e., $N = -7$.

35. We want $|e^{-2x}| < 0.1$. Since $e^{-2x} > 0$ for any x, this is the same as $e^{-2x} < 0.1$ so $-2x < \ln(0.1)$ and then $x > \dfrac{\ln(0.1)}{-2} \approx 1.15$. We may take $M = 2$.

36. We look at the graph of $\dfrac{e^x + x}{e^x - x^2}$ as x gets larger and we find that we should take $M = 7$.

37. Let $\varepsilon > 0$ be given and let $M = \sqrt[3]{2/\varepsilon}$. Then if $x > M$,

$$\left| \frac{2}{x^3} \right| < \left| \frac{2}{\left(\sqrt[3]{2/\varepsilon} \right)^3} \right| = \varepsilon$$

38. Let $\varepsilon > 0$ be given and let $N = -\sqrt[3]{3/\varepsilon}$. Then if $x < N$,

$$\left| \frac{3}{x^3} \right| < \left| \frac{3}{\left(\sqrt[3]{3/\varepsilon} \right)^3} \right| = \varepsilon$$

39. Let $\varepsilon > 0$ be given and let $M = \varepsilon^{-1/k}$. Then if $x > M$,

$$\left|\frac{1}{x^k}\right| < \left|\frac{1}{\left(\varepsilon^{-1/k}\right)^k}\right| = \varepsilon$$

40. Let $\varepsilon > 0$ be given and let $N = -\varepsilon^{-1/2k}$. Then if $x < N$,

$$\left|\frac{1}{x^{2k}}\right| < \left|\frac{1}{\left(-\varepsilon^{-1/2k}\right)^{2k}}\right| = \varepsilon$$

41. Let $\varepsilon > 0$ be given and assume $\varepsilon \leq 1/2$. Let $N = -(\frac{1}{\varepsilon} - 2)^{1/2}$. Then if $x < N$,

$$\left|\frac{1}{x^2 + 2} - 3 - (-3)\right| = \left|\frac{1}{x^2 + 2}\right|$$

$$< \left|\frac{1}{\left(-(\frac{1}{\varepsilon} - 2)^{1/2}\right)^2 + 2}\right| = \varepsilon$$

42. Let $\varepsilon > 0$ be given and let $M = \varepsilon^{-1/2} + 7$. Then if $x > M$,

$$\left|\frac{1}{(x - 7)^2}\right| < \left|\frac{1}{\left(\varepsilon^{-1/2} + 7 - 7\right)^2}\right| = \varepsilon$$

43. Let $N < 0$ be given and let $\delta = \sqrt[4]{-2/N}$. Then for any x such that $|x + 3| < \delta$,

$$\left|\frac{-2}{(x + 3)^4}\right| > \left|\frac{-2}{(\sqrt[4]{-2/N})^4}\right| = |N|$$

44. Let $M > 0$ be given and let $\delta = \sqrt{3/M}$. Then for any x such that $|x - 7| < \delta$,

$$\left|\frac{3}{(x - 7)^2}\right| > \left|\frac{3}{(\sqrt{3/M})^2}\right| = |M|$$

45. Let $M > 0$ be given and let $\delta = \sqrt{4/M}$. Then for any x such that $|x - 5| < \delta$,

$$\left|\frac{4}{(x - 5)^2}\right| > \left|\frac{4}{(\sqrt{4/M})^2}\right| = |M|$$

46. Let $N < 0$ be given and let $\delta = \sqrt[6]{-6/N}$. Then for any x such that $|x + 4| < \delta$,

$$\left|\frac{-6}{(x + 4)^6}\right| > \left|\frac{-6}{(\sqrt[6]{-6/N})^6}\right| = |N|$$

47. We observe that $\lim\limits_{x \to 1^-} f(x) = 2$ and $\lim\limits_{x \to 1^+} f(x) = 4$. For any $x \in (1, 2)$,

$$|f(x) - 2| = |x^2 + 3 - 2| = |x^2 + 1| > 2.$$

So if $\varepsilon \leq 2$, there is no $\delta > 0$ to satisfy the definition of limit.

48. We observe that $\lim\limits_{x \to 0^-} f(x) = -1$ and $\lim\limits_{x \to 0^+} f(x) = -2$. For any $x \in (-1, 0)$,

$$|f(x) - (-2)| = |x^2 - 1 + 2| = |x^2 + 1| > 1.$$

So if $\varepsilon \leq 1$, there is no $\delta > 0$ to satisfy the definition of limit.

49. We observe that $\lim\limits_{x \to 1^-} f(x) = 2$ and $\lim\limits_{x \to 1^+} f(x) = 4$. For any $x \in (1, \sqrt{2})$,

$$|f(x) - 2| = |5 - x^2 - 2|$$
$$= |3 - x^2| > |3 - (\sqrt{2})^2| = 1.$$

So if $\varepsilon \leq 1$, there is no $\delta > 0$ to satisfy the definition of limit.

50. We observe that $\lim\limits_{x \to 2^-} f(x) = 1$ and $\lim\limits_{x \to 2^+} f(x) = 4$. For any $x \in (2, 3)$,

$$|f(x) - 1| = |x^2 - 1| > 3.$$

So if $\varepsilon \leq 3$, there is no $\delta > 0$ to satisfy the definition of limit.

51. We want to find, for any given $\varepsilon > 0$, a $\delta > 0$ such that whenever $0 < |r - 2| < \delta$, we have $|2r^2 - 8| < \varepsilon$. We see that

$$|2r^2 - 8| = 2|r^2 - 4| = 2|r - 2||r + 2|.$$

Since we want a radius close to 2, we may take $|r - 2| < 1$ which implies $|r + 2| < 5$ and so

$$|2r^2 - 8| < 10|r - 2|$$

whenever $|r - 2| < 1$. If we then take $\delta = \min\{1, \varepsilon/10\}$, we see that whenever $0 < |r - 2| < \delta$, we have

$$|2r^2 - 8| < 10 \cdot \delta \le 10 \cdot \frac{\varepsilon}{10} = \varepsilon.$$

52. We want to find, for any given $\varepsilon > 0$, a $\delta > 0$ such that whenever $0 < |r - \frac{1}{2}| < \delta$, we have $|\frac{4}{3}\pi r^3 - \frac{\pi}{6}| < \varepsilon$. We see that

$$\left|\frac{4}{3}\pi r^3 - \frac{\pi}{6}\right| = \frac{4\pi}{3}\left|r - \frac{1}{2}\right|\left|r^2 + \frac{r}{2} + \frac{1}{4}\right|.$$

Since we want a radius close to $1/2$, we may take $|r - 1/2| < 1/2$ so $0 < r < 1$. Since the function $r^2 + r/2 + 1/4$ is increasing on the interval $(0, 1)$, we see that

$$\left|r^2 + \frac{r}{2} + \frac{1}{4}\right| < 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

whenever $|r - 1/2| < 1/2$. If we then take $\delta = \min\left\{\frac{1}{2}, \frac{3\varepsilon}{7\pi}\right\}$, we have

$$\left|\frac{4}{3}\pi r^3 - \frac{\pi}{6}\right| < \frac{7\pi}{3}\left|r - \frac{1}{2}\right|$$
$$< \frac{7\pi}{3} \cdot \frac{3\varepsilon}{7\pi} = \varepsilon.$$

53. Let $L = \lim_{x \to a} f(x)$. Given any $\varepsilon > 0$, we know there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have

$$|f(x) - L| < \frac{\varepsilon}{|c|}.$$

Here, we can take $\varepsilon/|c|$ instead of ε because there is such a δ for *any* ε, including $\varepsilon/|c|$. But now we have

$$|c \cdot f(x) - c \cdot L| = |c| \cdot |f(x) - L|$$
$$< |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Therefore, $\lim_{x \to a} c \cdot f(x) = c \cdot L$, as desired.

54. Let $L_1 = \lim_{x \to a} f(x)$. Then, given any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that whenever $0 < |x - a| < \delta_1$, we have

$$|f(x) - L_1| < \frac{\varepsilon}{2}.$$

Similarly, let $L_2 = \lim_{x \to a} g(x)$. Then, given any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that whenever $0 < |x - a| < \delta_2$, we have

$$|g(x) - L_2| < \frac{\varepsilon}{2}.$$

Note that
$$|(f(x) + g(x)) - (L_1 + L_2)|$$
$$= |(f(x) - L_1) + (g(x) - L_2)|$$
$$\le |f(x) - L_1| + |g(x) - L_2|$$

by the triangle inequality. So whenever $\delta = \min\{\delta_1, \delta_2\}$, we have

$$|(f(x) + g(x)) - (L_1 + L_2)|$$
$$\le |f(x) - L_1| + |g(x) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as desired. The proof for $f(x) - g(x)$ is similar, noting that

$$|(f(x) - g(x)) - (L_1 - L_2)|$$
$$= |(f(x) - L_1) + (-1)(g(x) - L_2)|$$
$$\le |f(x) - L_1| + |g(x) - L_2|.$$

55. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever $0 < |x - a| < \delta_1$, we have

$$|f(x) - L| < \varepsilon.$$

In particular, we know that

$$L - \varepsilon < f(x).$$

Similarly, since $\lim\limits_{x \to a} h(x) = L$, there exists $\delta_2 > 0$ such that whenever $0 < |x - a| < \delta_2$, we have

$$|h(x) - L| < \varepsilon.$$

In particular, we know that

$$h(x) < L + \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |x - a| < \delta$, we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon.$$

Therefore

$$|g(x) - L| < \varepsilon$$

and so $\lim\limits_{x \to a} g(x) = L$ as desired.

56. Let $\varepsilon > 0$ be given. If $x < a$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon$. Likewise, if $x > a$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|f(x) - L| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for any x such that $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

57. Let $\varepsilon > 0$ be given. If $\lim\limits_{x \to a} f(x) = L$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Since $|f(x) - L - 0| = |f(x) - L|$, this is precisely what we need to see that $\lim\limits_{x \to a}[f(x) - L] = 0$.

58. Let $\varepsilon > 0$ be given. If $\lim\limits_{x \to a}[f(x) - L] = 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L - 0| < \varepsilon$. Since $|f(x) - L - 0| = |f(x) - L|$, this is precisely what we need to see that $\lim\limits_{x \to a} f(x) = L$.

59. If $2 < x < \sqrt{4.1}$ then $4 < x^2 < 4.1$ so (for $x \in (2, \sqrt{4.1})$), $x^2 - 4 < 4.1 - 4 = 0.1$.

If $\sqrt{3.9} < x < 2$ then $3.9 < x^2 < 4$ so (for $x \in (\sqrt{3.9}, 2)$), $x^2 - 4 > 3.9 - 4 = -0.1$.

For the limit definition, we need to take $\delta = \min\{\delta_1, \delta_2\} = \delta_1$ to ensure that x^2 is within 0.1 of 4 on *both* sides of $x = 2$.

60. If $2 < x < \sqrt{4 + \varepsilon}$ then $4 < x^2 < 4 + \varepsilon$ so (for $x \in (2, \sqrt{4 + \varepsilon})$), $x^2 - 4 < 4 + \varepsilon - 4 = \varepsilon$.

If $\sqrt{4 - \varepsilon} < x < 2$ then $4 - \varepsilon < x^2 < 4$ so (for $x \in (\sqrt{4 - \varepsilon}, 2)$), $x^2 - 4 > 4 - \varepsilon - 4 = -\varepsilon$.

Let $\delta = \min\{\sqrt{4 + \varepsilon}, \sqrt{4 - \varepsilon}\}$.

1.7 Limits and Loss-of-Significance Errors

1. The limit is $\frac{1}{4}$.
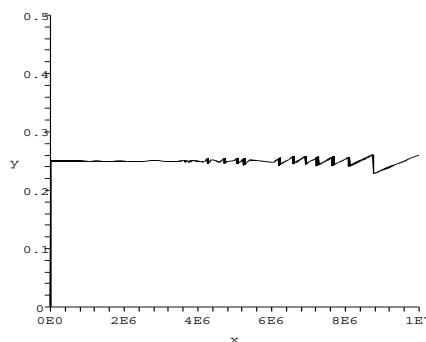
We can rewrite the function as

$$f(x) =$$

$$x(\sqrt{4x^2 + 1} - 2x) \cdot \frac{\sqrt{4x^2 + 1} + 2x}{\sqrt{4x^2 + 1} + 2x}$$
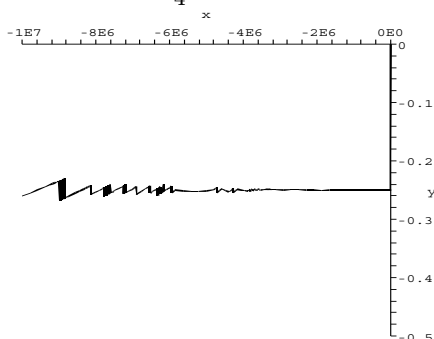
$$= \frac{x(4x^2 + 1 - 4x^2)}{\sqrt{4x^2 + 1} + 2x}$$

$$= \frac{x}{\sqrt{4x^2 + 1} + 2x}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = x(\sqrt{4x^2 + 1} - 2x)$, while the third column contains values calculated using the rewritten $f(x)$.

x	old $f(x)$	new $f(x)$
1	0.236068	0.236068
10	0.249844	0.249844
100	0.249998	0.249998
1000	0.250000	0.250000
10000	0.250000	0.250000
100000	0.249999	0.250000
1000000	0.250060	0.250000
10000000	0.260770	0.250000
100000000	0.000000	0.250000
1000000000	0.000000	0.250000

2. The limit is $-\frac{1}{4}$.
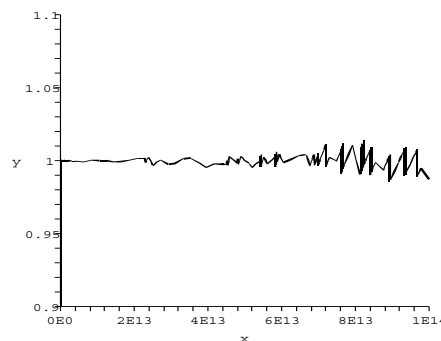

We can rewrite the function as

$$x(\sqrt{4x^2 + 1} + 2x)\frac{(\sqrt{4x^2 + 1} - 2x)}{(\sqrt{4x^2 + 1} - 2x)}$$

$$= \frac{x}{(\sqrt{4x^2 + 1} - 2x)}$$

to avoid loss-of-significance errors.

3. The limit is 1.

We can rewrite the function as

$$\sqrt{x}(\sqrt{x+4} - \sqrt{x+2}) \cdot \frac{\sqrt{x+4} + \sqrt{x+2}}{\sqrt{x+4} + \sqrt{x+2}}$$

$$= \frac{\sqrt{x}[(x+4) - (x+2)]}{\sqrt{x+4} + \sqrt{x+2}}$$

$$= \frac{2\sqrt{x}}{\sqrt{x+4} + \sqrt{x+2}}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = \sqrt{x}\left(\sqrt{x+4} - \sqrt{x+2}\right)$, while the third column contains values calculated using the rewritten $f(x)$.

x	old $f(x)$	new $f(x)$
1	0.504017	0.504017
10	0.877708	0.877708
100	0.985341	0.985341
1000	0.998503	0.998503
10000	0.999850	0.999850
100000	0.999985	0.999985
1000000	0.999998	0.999999
10000000	1.000000	1.000000
100000000	1.000000	1.000000
1000000000	1.000000	1.000000
10000000000	1.000000	1.000000
1E+11	0.999990	1.000000
1E+12	1.000008	1.000000
1E+13	0.999862	1.000000
1E+14	0.987202	1.000000
1E+15	0.942432	1.000000
1E+16	0.000000	1.000000
1E+17	0.000000	1.000000

4. The limit is 4.
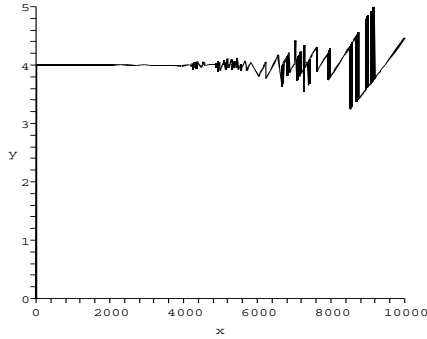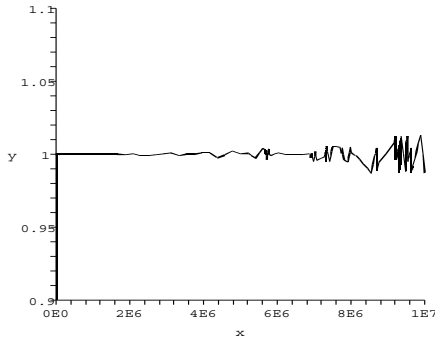

We can rewrite the function as

$$x^2(\sqrt{x^4+8}-x^2)\frac{(\sqrt{x^4+8}+x^2)}{(\sqrt{x^4+8}+x^2)}$$

$$=\frac{8x^2}{(\sqrt{x^4+8}+x^2)}$$

to avoid loss-of-significance errors.

5. The limit is 1.

We can multiply $f(x)$ by

$$\frac{\sqrt{x^2+4}+\sqrt{x^2+2}}{\sqrt{x^2+4}+\sqrt{x^2+2}}$$
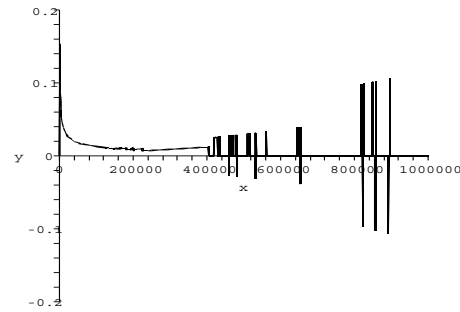
to rewrite the function as

$$\frac{x[x^2+4-(x^2+2)]}{\sqrt{x^2+4}+\sqrt{x^2+2}}$$

$$=\frac{2x}{\sqrt{x^2+4}+\sqrt{x^2+2}}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = \left(\sqrt{x^2+4}-\sqrt{x^2+2}\right)$, while the third column contains values calculated using the rewritten $f(x)$.

x	old $f(x)$	new $f(x)$
1	0.504017	0.504017
10	0.985341	0.985341
100	0.999850	0.999850
1000	0.999998	0.999999
10000	1.000000	1.000000
100000	1.000000	1.000000
1000000	1.000008	1.000000
10000000	0.987202	1.000000
100000000	0.000000	1.000000
1000000000	0.000000	1.000000

6. The limit is 0.

We can rewrite the function as

$$x(\sqrt{x^3+8}-x^{3/2})\frac{(\sqrt{x^3+8}+x^{3/2})}{(\sqrt{x^3+8}+x^{3/2})}$$

$$=\frac{8x}{(\sqrt{x^3+8}+x^{3/2})}$$

to avoid loss-of-significance errors.

7. The limit is 1/6.
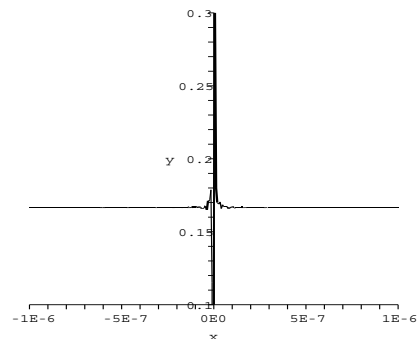
We can rewrite the function as

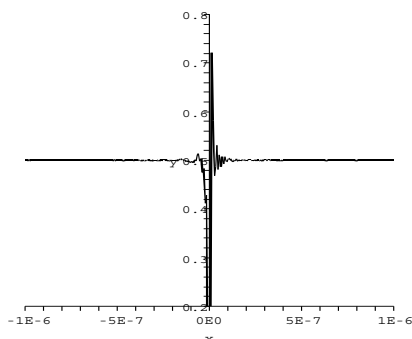$$\frac{1-\cos 2x}{12x^2}\cdot\frac{1+\cos 2x}{1+\cos 2x}$$

$$= \frac{\sin^2 2x}{12x^2(1 + \cos 2x)}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = \dfrac{1 - \cos 2x}{12x^2}$, while the third column contains values calculated using the rewritten $f(x)$. Note that $f(x) = f(-x)$ and so we get the same values when x is negative (which allows us to conjecture the two-sided limit as $x \to 0$).

x	old $f(x)$	new $f(x)$
1	0.118012	0.118012
0.1	0.166112	0.166112
0.01	0.166661	0.166661
0.001	0.166667	0.166667
0.0001	0.166667	0.166667
0.00001	0.166667	0.166667
0.000001	0.166663	0.166667
0.0000001	0.166533	0.166667
0.00000001	0.185037	0.166667
0.000000001	0	0.166667
1E-10	0	0.166667

8. The limit is $\frac{1}{2}$.

We can rewrite the function as
$$\frac{(1 - \cos x)}{x^2} \frac{(1 + \cos x)}{(1 + \cos x)}$$
$$= \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$
$$= \frac{\sin^2 x}{x^2(1 + \cos x)}$$

to avoid loss-of-significance errors.

9. The limit is $\frac{1}{2}$.
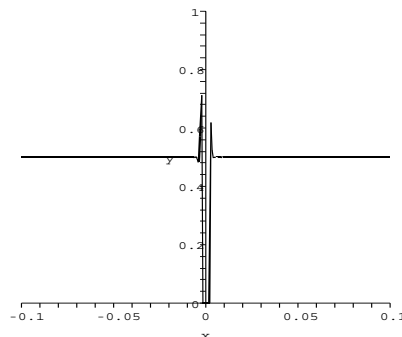
We can rewrite the function as

$$\frac{1 - \cos x^3}{x^6} \frac{1 + \cos x^3}{1 + \cos x^3}$$
$$= \frac{\sin^2(x^3)}{x^6(1 + \cos x^3)}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = \frac{1 - \cos x^3}{x^6}$, while the third column contains values calculated using the rewritten $f(x)$. Note that $f(x) = f(-x)$ and so we get the same values when x is negative (which allows us to conjecture the two-sided limit as $x \to 0$).

x	old $f(x)$	new $f(x)$
1	0.459698	0.459698
0.1	0.500000	0.500000
0.01	0.500044	0.500000
0.001	0.000000	0.500000
0.0001	0.000000	0.500000

10. The limit is $\frac{1}{2}$.
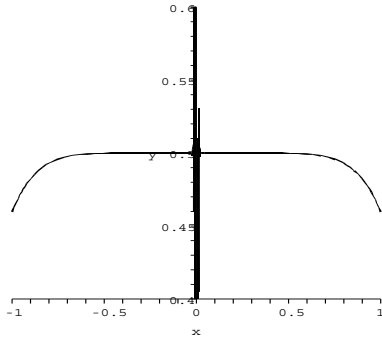
We can rewrite the function as

$$\frac{(1 - \cos x^4)}{x^8} \frac{(1 + \cos x^4)}{(1 + \cos x^4)}$$

$$= \frac{1 - \cos^2 x^4}{x^8(1 + \cos x^4)}$$

$$= \frac{\sin^2 x^4}{x^8(1 + \cos x^4)}$$

to avoid loss-of-significance errors.

11. The limit is $2/3$.

We can multiply $f(x)$ by

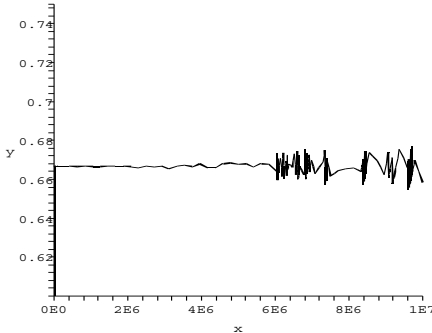$$1 = \frac{g(x)}{g(x)}$$

where

$$g(x) = (x^2 + 1)^{\frac{2}{3}} + (x^2 + 1)^{\frac{1}{3}}(x^2 - 1)^{\frac{1}{3}} + (x^2 - 1)^{\frac{2}{3}}$$

to rewrite the function as

$$\frac{2x^{4/3}}{g(x)}$$

to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = x^{4/3}(\sqrt[3]{x^2 + 1} - \sqrt[3]{x^2 - 1})$, while the third column contains values calculated using the rewritten $f(x)$.

x	old $f(x)$	new $f(x)$
1	1.259921	1.259921
10	0.666679	0.666679
100	0.666667	0.666667
1000	0.666667	0.666667
10000	0.666668	0.666667
100000	0.666532	0.666667
1000000	0.63	0.666667
10000000	2.154435	0.666667
100000000	0.000000	0.666667
1000000000	0.000000	0.666667

12. The limit is $7/3$.

We can multiply $f(x)$ by

$$\frac{(x + 4)^{\frac{2}{3}} + (x + 4)^{\frac{1}{3}}(x - 3)^{\frac{1}{3}} + (x - 3)^{\frac{2}{3}}}{(x + 4)^{\frac{2}{3}} + (x + 4)^{\frac{1}{3}}(x - 3)^{\frac{1}{3}} + (x - 3)^{\frac{2}{3}}}$$

to rewrite the function as

$$\frac{7x^{2/3}}{(x + 4)^{\frac{2}{3}} + (x + 4)^{\frac{1}{3}}(x - 3)^{\frac{1}{3}} + (x - 3)^{\frac{2}{3}}}$$
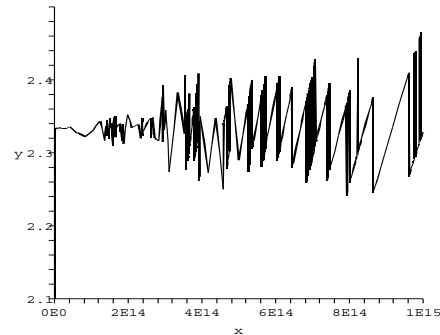
to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = x^{2/3}(\sqrt[3]{x + 4} - \sqrt[3]{x - 3})$, while the third column contains values calculated using the rewritten $f(x)$.

x	old $f(x)$	new $f(x)$
1	2.969897	1.259921
10	2.307850	2.307850
100	2.326111	2.326110
1000	2.332561	2.332561
10000	2.333256	2.333256
100000	2.333326	2.333326
1000000	2.333333	2.333333
10000000	2.333333	2.333333
100000000	2.333332	2.333333
1000000000	2.33337	2.333333
10000000000	2.333327	2.333333
1E+11	2.333253	2.333333
1E+12	2.3	2.333333
1E+13	2.320794	2.333333
1E+14	2.154435	2.333333
1E+15	0.000000	2.333333
1E+16	0.000000	2.333333

13. $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1}$

$= \lim\limits_{x \to 1} \dfrac{(x + 2)(x - 1)}{x - 1}$

$= \lim\limits_{x \to 1} (x + 2) = 3$

$\lim\limits_{x \to 1} \dfrac{x^2 + x - 2.01}{x - 1}$ does not exist, since when x is close to 1, the numerator is close to $-.01$ (a small but non-zero number) and the denominator is close to 0.

14. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

$= \lim\limits_{x \to 2} \dfrac{x - 2}{(x - 2)(x + 2)} = \dfrac{1}{4}$

and $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4.01} = 0$.

15. $f(1) = 0; g(1) = 0.00159265$
$f(10) = 0; g(10) = -0.0159259$
$f(100) = 0; g(100) = -0.158593$
$f(1000) = 0; g(1000) = -0.999761$

16. For $x = 10^5$, the computation procedes as follows: Compute $4 \times (10^5)^2 = 4 \times 10^{10}$. Adding one does

not change the 10 digit mantissa (this is the round-off error). Now computing the square root gives 2×10^5, and we subtract this same amount resulting in zero. This is multiplied by 10^5 yielding zero.

17. $(1.000003 - 1.000001) \times 10^7 = 20$
On a computer with a 6-digit mantissa, the calculation would be $(1.00000 - 1.00000) \times 10^7 = 0$.

18. The answer with a six-digit mantissa is 0. The exact answer is 50.

Ch. 1 Review Exercises

1. The slope appears to be 2.

Second point	m_{sec}
$(3, 3)$	3
$(2.1, 0.21)$	2.1
$(2.01, 0.0201)$	2.01
$(1, -1)$	1
$(1.9, -0.19)$	1.9
$(1.99, -0.0199)$	1.99

2. The slope appears to be 2.

Second point	m_{sec}
$(-0.2, -0.3894)$	1.9471
$(-0.1, -0.1987)$	1.9867
$(-0.01, -0.02)$	2
$(0.2, 0.3894)$	1.9471
$(0.1, 0.1987)$	1.9876
$(0.01, 0.02)$	2

3. (a) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{16}$, $\frac{\pi}{8}$, $\frac{3\pi}{16}$, and $\frac{\pi}{4}$.

Left	Right	Length
$(0, 0)$	$(0.2, 0.2)$	0.276
$(0.2, 0.2)$	$(0.39, 0.38)$	0.272
$(0.39, 0.38)$	$(0.59, 0.56)$	0.262
$(0.59, 0.56)$	$(0.785, 0.71)$	0.248
	Total	1.058

(b) For the x-values of our points here we use (approximations of) 0, $\frac{\pi}{32}$, $\frac{\pi}{16}$, $\frac{3\pi}{32}$, $\frac{\pi}{8}$, $\frac{5\pi}{32}$, $\frac{3\pi}{16}$, $\frac{7\pi}{32}$, and $\frac{\pi}{4}$.

Left	Right	Length
(0, 0)	(0.1, 0.1)	0.139
(0.1, 0.1)	(0.2, 0.2)	0.138
(0.2, 0.2)	(0.29, 0.29)	0.137
(0.29, 0.29)	(0.39, 0.38)	0.135
(0.39, 0.38)	(0.49, 0.47)	0.132
(0.49, 0.47)	(0.59, 0.56)	0.129
(0.59, 0.56)	(0.69, 0.63)	0.126
(0.69, 0.63)	(0.785, 0.71)	0.122
	Total	1.058

4. (a)

Left	Right	Length
(0, 0)	$(\frac{\pi}{16}, 0.1951)$	0.2768
$(\frac{\pi}{16}, 0.1951)$	$(\frac{2\pi}{16}, 0.3827)$	0.2716
$(\frac{2\pi}{16}, 0.3827)$	$(\frac{3\pi}{16}, 0.5556)$	0.2616
$(\frac{3\pi}{16}, 0.5556)$	$(\frac{\pi}{4}, 0.7071)$	0.2480
	Total	1.058

(b)

Left	Right	Length
(0, 0)	$(\frac{\pi}{32}, 0.0980)$	0.1387
$(\frac{\pi}{32}, 0.0980)$	$(\frac{2\pi}{32}, 0.1951)$	0.1381
$(\frac{2\pi}{32}, 0.1951)$	$(\frac{3\pi}{32}, 0.2903)$	0.1368
$(\frac{3\pi}{32}, 0.2903)$	$(\frac{4\pi}{32}, 0.3827)$	0.1348
$(\frac{4\pi}{32}, 0.3827)$	$(\frac{5\pi}{32}, 0.4714)$	0.1323
$(\frac{5\pi}{32}, 0.4714)$	$(\frac{6\pi}{32}, 0.5556)$	0.1293
$(\frac{6\pi}{32}, 0.5556)$	$(\frac{7\pi}{32}, 0.6344)$	0.1259
$(\frac{7\pi}{32}, 0.6344)$	$(\frac{\pi}{4}, 0.7071)$	0.1222
	Total	1.0581

5. Let $f(x) = \dfrac{\tan^{-1} x^2}{x^2}$.

x	$f(x)$
0.1	0.999966669
0.01	0.999999997
0.001	1.000000000
0.0001	1.000000000
0.00001	1.000000000
0.000001	1.000000000

Note that $f(x) = f(-x)$, so the results for negative x will be the same as above. The limit appears to be 1.

6. $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{\ln x^2} = 1$.

7. Let $f(x) = \dfrac{x + 2}{|x + 2|}$.

x	$f(x)$
-1.9	1
-1.99	1
-1.999	1
-2.1	-1
-2.01	-1
-2.001	-1

$\displaystyle\lim_{x \to -2} \frac{x + 2}{|x + 2|}$ does not exist.

8. $\displaystyle\lim_{x \to 0}(1 + 2x)^{1/x} = e^2 \approx 7.389$.

9. Let $f(x) = \left(1 + \dfrac{2}{x}\right)^x$.

x	$f(x)$
10	6.1917
100	7.2446
1000	7.3743
10,000	7.3876

$\displaystyle\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^2 \approx 7.4$

10. $\displaystyle\lim_{x \to \infty} x^{2/x} = 1$.

11. (a) $\displaystyle\lim_{x \to -1^-} f(x) = 1$.

(b) $\displaystyle\lim_{x \to -1^+} f(x) = -2$.

(c) $\displaystyle\lim_{x \to -1} f(x)$ does not exist.

(d) $\displaystyle\lim_{x \to 0} f(x) = 0$.

12. (a) $\displaystyle\lim_{x \to 1^-} f(x) = 1$.

(b) $\displaystyle\lim_{x \to 1^+} f(x) = 3$.

(c) $\displaystyle\lim_{x \to 1} f(x)$ does not exist.

(d) $\displaystyle\lim_{x \to 2} f(x) = 2$.

13. $x = -1, x = 1$

14. One possible graph:
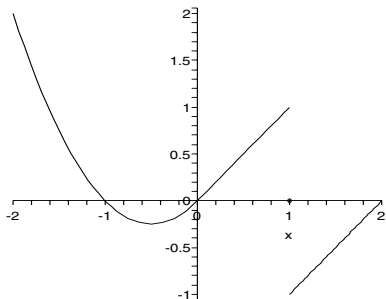
15. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$
$= \lim\limits_{x \to 2} \dfrac{(x - 2)(x + 1)}{(x + 2)(x - 2)}$
$= \lim\limits_{x \to 2} \dfrac{x + 1}{x + 2} = \dfrac{3}{4}.$

16. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^2 + x - 2}$
$= \lim\limits_{x \to 1} \dfrac{(x - 1)(x + 1)}{(x + 2)(x - 1)}$
$= \lim\limits_{x \to 1} \dfrac{x + 1}{x + 2} = \dfrac{2}{3}.$

17. $\lim\limits_{x \to 0^+} \dfrac{x^2 + x}{\sqrt{x^4 + 2x^2}}$
$= \lim\limits_{x \to 0^+} \dfrac{x(x + 1)}{x\sqrt{x^2 + 2}}$
$= \lim\limits_{x \to 0^+} \dfrac{x + 1}{\sqrt{x^2 + 2}}$
$= \dfrac{1}{\sqrt{2}}$
but
$\lim\limits_{x \to 0^-} \dfrac{x^2 + x}{\sqrt{x^4 + 2x^2}}$
$= \lim\limits_{x \to 0^-} \dfrac{x(x + 1)}{(-x)\sqrt{x^2 + 2}}$
$= \lim\limits_{x \to 0^-} -\dfrac{x + 1}{\sqrt{x^2 + 2}}$
$= -\dfrac{1}{\sqrt{2}}$

Since the left and right limits are not equal, $\lim\limits_{x \to 0} \dfrac{x^2 + x}{\sqrt{x^4 + 2x^2}}$ does not exist.

18. $\lim\limits_{x \to 0^+} e^{-\cot x} = \lim\limits_{x \to \infty} e^{-x} = 0$
but
$\lim\limits_{x \to 0^-} e^{-\cot x} = \lim\limits_{x \to -\infty} e^{-x} = \infty$
Since the left and right limits are not equal, $\lim\limits_{x \to 0} e^{-\cot x}$ does not exist.

19. $\lim\limits_{x \to 0} (2 + x)\sin(1/x)$
$= \lim\limits_{x \to 0} 2\sin(1/x);$
however, since $\lim\limits_{x \to 0} \sin(1/x)$ does not exist, it follows that $\lim\limits_{x \to 0} (2 + x)\sin(1/x)$ also does not exist.

20. $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x^2} = 1.$

21. $\lim\limits_{x \to 2} f(x) = 5.$

22. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (2x + 1) = 3$
$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2 + 1) = 2$
$\lim\limits_{x \to 1} f(x)$ does not exist.

23. Multiply the function by
$\dfrac{(1 + 2x)^{\frac{2}{3}} + (1 + 2x)^{\frac{1}{3}} + 1}{(1 + 2x)^{\frac{2}{3}} + (1 + 2x)^{\frac{1}{3}} + 1}$
to get
$\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1 + 2x} - 1}{x}$
$= \lim\limits_{x \to 0} \dfrac{2}{(1 + 2x)^{\frac{2}{3}} + (1 + 2x)^{\frac{1}{3}} + 1} = \dfrac{2}{3}$

24. $\lim\limits_{x \to 1} \dfrac{x - 1}{\sqrt{10 - x} - 3}$
$\lim\limits_{x \to 1} \dfrac{x - 1}{\sqrt{10 - x} - 3} \cdot \dfrac{\sqrt{10 - x} + 3}{\sqrt{10 - x} + 3}$
$\lim\limits_{x \to 1} \dfrac{(x - 1)(\sqrt{10 - x} + 3)}{10 - x - 9}$
$\lim\limits_{x \to 1} \dfrac{(x - 1)(\sqrt{10 - x} + 3)}{1 - x}$

$$\lim_{x \to 1} \frac{-(1-x)(\sqrt{10-x}+3)}{1-x}$$
$$\lim_{x \to 1} -(\sqrt{10-x}+3) = -6$$

25. $\displaystyle \lim_{x \to 0} \cot(x^2) = \infty$

26. $\displaystyle \lim_{x \to 1} \tan^{-1}\left(\frac{x}{x^2 - 2x + 1}\right)$
$$= \lim_{x \to 1} \tan^{-1}\left(\frac{x}{(x-1)^2}\right)$$
$$= \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

27. $\displaystyle \lim_{x \to \infty} \frac{x^2 - 4}{3x^2 + x + 1}$
$$= \lim_{x \to \infty} \frac{x^2\left(1 - \frac{4}{x^2}\right)}{x^2\left(3 + \frac{1}{x} + \frac{1}{x^2}\right)}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{4}{x^2}}{3 + \frac{1}{x} + \frac{1}{x^2}} = \frac{1}{3}$$

28. $\displaystyle \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + 4}} \frac{1/x}{1/x}$
$$= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 4/x^2}} = 2$$

29. Since $\displaystyle \lim_{x \to \pi/2} \tan^2 x = +\infty$, it follows that $\displaystyle \lim_{x \to \pi/2} e^{-\tan^2 x} = 0.$

30. $\displaystyle \lim_{x \to -\infty} e^{-x^2} = 0.$

31. $\displaystyle \lim_{x \to \infty} \ln 2x = \lim_{x \to \infty} (\ln 2 + \ln x)$
$$= \ln 2 + \lim_{x \to \infty} \ln x = \infty$$

32. $\displaystyle \lim_{x \to 0^+} \ln 3x = -\infty$

33. $\displaystyle \lim_{x \to -\infty} \frac{2x}{x^2 + 3x - 5}$
$$= \lim_{x \to -\infty} \frac{2x}{x^2\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}$$
$$= \lim_{x \to -\infty} \frac{2}{x\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)} = 0$$

34. $\displaystyle \lim_{x \to -2} \frac{2x}{x^2 + 3x + 2}$
$$= \lim_{x \to -2} \frac{2x}{(x+2)(x+1)}$$
does not exist. Approaches $-\infty$ from the left, and ∞ from the right.

35. Let $u = -\dfrac{1}{3x}$, so that $\dfrac{2}{x} = -6u$. Then,
$$\lim_{x \to 0^+} (1 - 3x)^{2/x}$$
$$= \lim_{u \to -\infty} \left(1 + \frac{1}{u}\right)^{-6u}$$
$$= \left[\lim_{u \to -\infty} \left(1 + \frac{1}{u}\right)^{u}\right]^{-6} = e^{-6}$$
and
$$\lim_{x \to 0^-} (1 - 3x)^{2/x}$$
$$= \lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^{-6u}$$
$$= \left[\lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^{u}\right]^{-6} = e^{-6}$$
Thus, $\displaystyle \lim_{x \to 0} (1 - 3x)^{2/x} = e^{-6}.$

36. $\displaystyle \lim_{x \to 0^+} \frac{2x - |x|}{|3x| - 2x}$
$$= \lim_{x \to 0^+} \frac{2x - x}{3x - 2x}$$
$$= \lim_{x \to 0^+} \frac{x}{x} = 1$$
but
$$\lim_{x \to 0^-} \frac{2x - |x|}{|3x| - 2x}$$
$$\lim_{x \to 0^-} \frac{2x - (-x)}{-3x - 2x}$$
$$\lim_{x \to 0^-} \frac{3x}{-5x} = -\frac{3}{5}$$
Thus the limit does not exit.

37. $0 \le \dfrac{x^2}{x^2 + 1} < 1$
$$\Rightarrow -2|x| \le \frac{2x^3}{x^2 + 1} < 2|x|$$
$$\lim_{x \to 0} -2|x| = 0; \lim_{x \to 0} 2|x| = 0$$
By the Squeeze Theorem,

$$\lim_{x \to 0} \frac{2x^3}{x^2 + 1} = 0.$$

38. The first two rows of the following table show that $f(x)$ has a root in $[1, 2]$. In the following rows, we use the midpoint of the previous interval as our new x. When $f(x)$ is positive, we use the left half, and when $f(x)$ is negative, we use the right half of the interval.

x	$f(x)$
1	-1
2	5
1.5	0.875
1.25	-0.2969
1.375	0.22246
1.3125	-0.0515
1.34375	0.0826

The zero is in the interval $(1.3125, 1.34375)$.

39. $f(x) = \dfrac{x - 1}{x^2 + 2x - 3} = \dfrac{x - 1}{(x + 3)(x - 1)}$ has a non-removable discontinuity at $x = -3$ and a removable discontinuity at $x = 1$.

40. $f(x) = \dfrac{x + 1}{(x - 2)(x + 2)}$ is discontinuous at $x = \pm 2$. Not removable.

41. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x - 3) = 5$

f has a non-removable discontinuity at $x = 2$.

42. $f(x) = x \cot x$ has discontinuities wherever $\sin x$ is zero, namely $x = k\pi$ for any integer k. The discontinuity at $x = 0$ is removable because $\lim_{x \to 0} x \cot x = 1$. The other discontinuities are not removable.

43. $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$ continuous on $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

44. $f(x)$ is continuous wherever $3x - 4 > 0$ i.e., on the interval $(\frac{4}{3}, \infty)$.

45. $f(x) = \sin(1 + e^x)$ is continuous on the interval $(-\infty, \infty)$.

46. $f(x)$ is continuous wherever $x^2 - 4 \geq 0$ i.e., on the intervals $(-\infty, -2]$ and $[2, \infty)$.

47. $f(x) = \dfrac{x + 1}{(x - 2)(x - 1)}$ has vertical asymptotes at $x = 1$ and $x = 2$.

$\lim_{x \to \pm\infty} \dfrac{x + 1}{x^2 - 3x + 2}$

$= \lim_{x \to \pm\infty} \dfrac{x \left(1 + \frac{1}{x}\right)}{x^2 \left(1 - \frac{3}{x} + \frac{2}{x^2}\right)}$

$= \lim_{x \to \pm\infty} \dfrac{1 + \frac{1}{x}}{x \left(1 - \frac{3}{x} + \frac{2}{x^2}\right)} = 0$

So $f(x)$ has a horizontal asymptote at $y = 0$.

48. Vertical asymptote at $x = 4$. Horizontal asymptote at $y = 0$. (Removable discontinuity at $x = -2$.)

49. $f(x) = \dfrac{x^2}{x^2 - 1} = \dfrac{x^2}{(x + 1)(x - 1)}$ has vertical asymptotes at $x = -1$ and $x = 1$.

$\lim_{x \to \pm\infty} \dfrac{x^2}{x^2 - 1}$

$= \lim_{x \to \pm\infty} \dfrac{x^2}{x^2 \left(1 - \frac{1}{x^2}\right)}$

$= \lim_{x \to \pm\infty} \dfrac{1}{1 - \frac{1}{x^2}}$

$= \dfrac{1}{1} = 1$

So $f(x)$ has a horizontal asymptote at $y = 1$.

50. Vertical asymptotes at $x = 2$ and $x = -1$. Long division reveals the slant asymptote $y = x + 1$.

51. $\lim\limits_{x \to 0^+} 2e^{1/x} = \infty$, so $x = 0$ is a vertical asymptote.

$$\lim_{x \to \infty} 2e^{1/x} = 2, \qquad \lim_{x \to -\infty} 2e^{1/x} = 2,$$

so $y = 2$ is a horizontal asymptote.

52. Horizontal asymptotes at $y = \pm\frac{3\pi}{2}$.

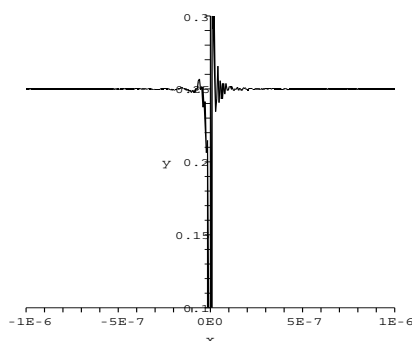**53.** $f(x)$ has a vertical asymptote when $e^x = 2$, that is, $x = \ln 2$.

$$\lim_{x \to \infty} \frac{3}{e^x - 2} = 0$$
$$\lim_{x \to -\infty} \frac{3}{e^x - 2} = -\frac{3}{2}$$

so $y = 0$ and $y = -3/2$ are horizontal asymptotes.

54. Vertical asymptote at $x = 2$. No horizontal or slant asymptotes.

55. The limit is $\frac{1}{4}$.

We can rewrite the function as
$$\frac{1 - \cos x}{2x^2} = \left(\frac{1 - \cos x}{2x^2}\right)\left(\frac{1 + \cos x}{1 + \cos x}\right)$$
$$= \frac{1 - \cos^2 x}{2x^2(1 + \cos x)} = \frac{\sin^2 x}{2x^2(1 + \cos x)}$$
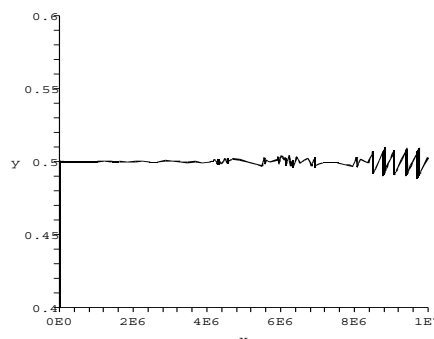to avoid loss-of-significance errors.

In the table below, the middle column contains values calculated using $f(x) = \frac{1 - \cos x}{2x^2}$, while the third column contains values calculated using

the rewritten $f(x)$. Note that $f(x) = f(-x)$ and so we get the same values when x is negative (which allows us to conjecture the two-sided limit as $x \to 0$).

x	old $f(x)$	new $f(x)$
1	0.229849	0.229849
0.1	0.249792	0.249792
0.01	0.249998	0.249998
0.001	0.250000	0.250000
0.0001	0.250000	0.250000
0.00001	0.250000	0.250000
0.000001	0.250022	0.250000
0.0000001	0.249800	0.250000
0.00000001	0.000000	0.250000
0.000000001	0.000000	0.250000

56. The limit is $\frac{1}{2}$.

We can rewrite the function as
$$x(\sqrt{x^2 + 1} - x)\frac{(\sqrt{x^2 + 1} + x)}{(\sqrt{x^2 + 1} + x)}$$
$$= \frac{x}{(\sqrt{x^2 + 1} + x)}$$
to avoid loss-of-significance errors.

57. The limit of θ' as x approaches 0 is 66 radians per second, far faster than the player can maintain focus. From about 9 feet on in to the plate the player can't keep her eye on the ball.