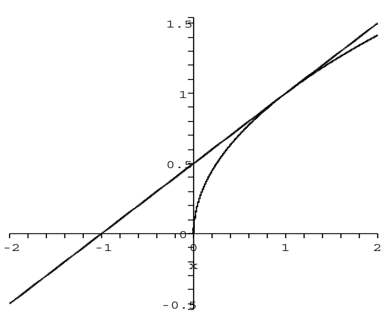
Chapter 3

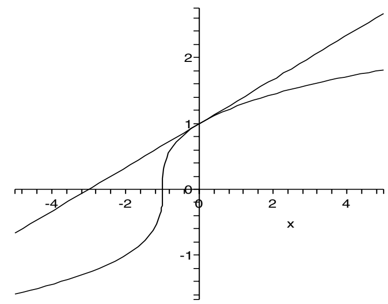
Applications of Differentiation

3.1 Linear Approximations and Newton's Method

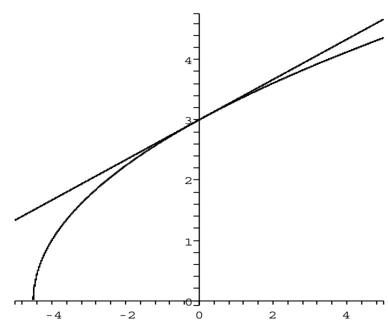
1. $f(x_0) = f(1) = \sqrt{1} = 1$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f'(x_0) = f'(1) = \frac{1}{2}$
So
$L(x) = f(x_0) + f'(x_0)(x - x_0)$
$= 1 + \frac{1}{2}(x - 1)$
$= \frac{1}{2} + \frac{1}{2}x$



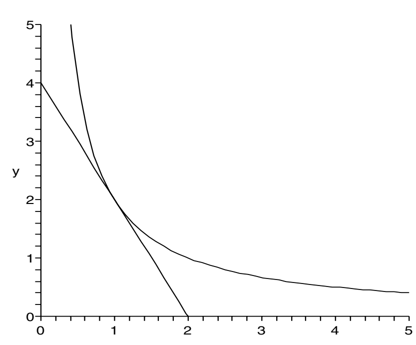
2. $f(0) = 1$, and $f'(x) = \frac{1}{3}(x+1)^{-2/3}$, so $f'(0) = \frac{1}{3}$. The linear approximation is
$L(x) = 1 + \frac{1}{3}(x - 0)$.



3. $f(x) = \sqrt{2x+9}$, $x_0 = 0$
$f(x_0) = f(0) = \sqrt{2 \cdot 0 + 9} = 3$
$f'(x) = \frac{1}{2}(2x+9)^{-1/2} \cdot 2 = (2x+9)^{-1/2}$
$f'(x_0) = f'(0) = (2 \cdot 0 + 9)^{-1/2} = \frac{1}{3}$
So
$L(x) = f(x_0) + f'(x_0)(x - x_0)$
$= 3 + \frac{1}{3}(x - 0)$
$= 3 + \frac{1}{3}x$

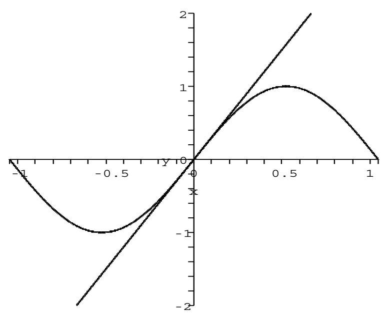


4. $f(1) = 2$, and $f'(x) = \frac{-2}{x^2}$, so $f'(1) = -2$. The linear approximation is
$L(x) = 2 + -2(x - 1)$.

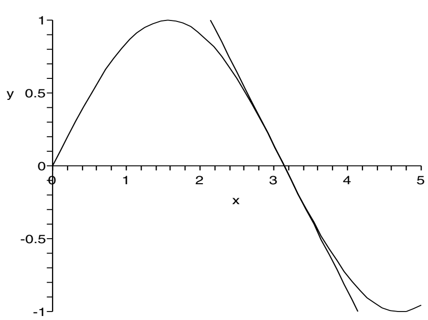


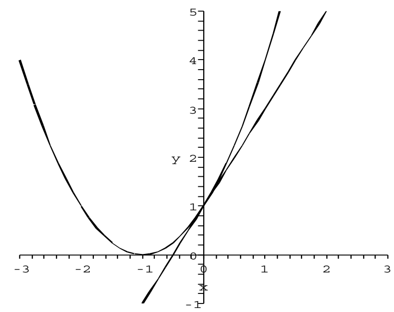
5. $f(x) = \sin 3x$, $x_0 = 0$
$f(x_0) = f(0) = \sin(3 \cdot 0) = \sin 0 = 0$
$f'(x) = 3\cos 3x$

$$
\begin{aligned}
f'(x_0) &= f'(0) = 3\cos 3\cdot 0 = 3\\
L(x) &= f(x_0) + f'(x_0)(x - x_0)\\
&= 0 + 3(x - 0)\\
&= 3x
\end{aligned}
$$

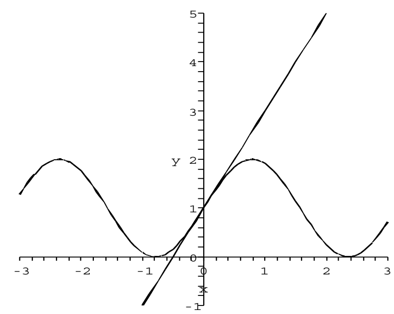


6. $f(\pi) = 0$, and $f'(x) = \cos x$, so $f'(\pi) = -1$. The linear approximation is
$L(x) = 0 + -1(x - \pi)$.

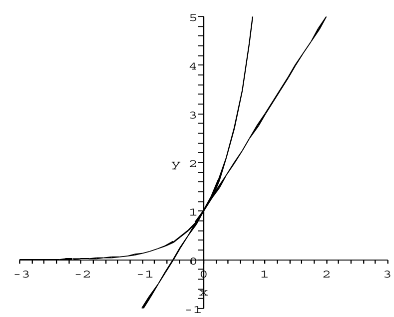




Graph of $f(x) = 1 + \sin(2x)$:



Graph of $f(x) = e^{2x}$:

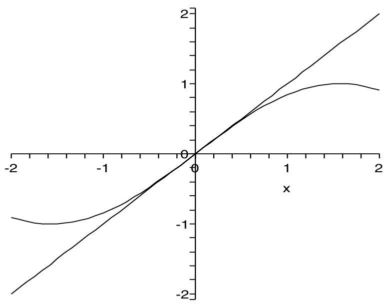


7. (a) $f(0) = g(0) = h(0) = 1$, so all pass through the point $(0, 1)$.
$f'(0) = 2(0 + 1) = 2$,
$g'(0) = 2\cos(2\cdot 0) = 2$, and
$h'(0) = 2e^{2\cdot 0} = 2$,
so all have slope 2 at $x = 0$.
The linear approximation at $x = 0$ for all three functions is
$L(x) = 1 + 2x$.

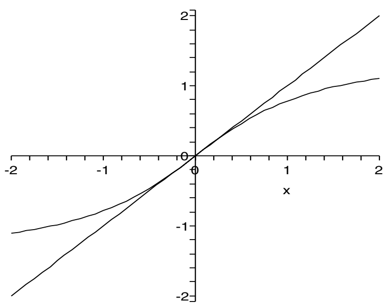
(b) Graph of $f(x) = (x + 1)^2$:

8. (a) $f(0) = g(0) = h(0) = 0$, so all pass through the point $(0, 0)$.
$f'(0) = \cos 0 = 1$,
$g'(0) = \frac{1}{1+0^2} = 1$, and
$h'(0) = \cosh 0 = 1$,
so all have slope 1 at $x = 0$.
The linear approximation at $x = 0$ for all three functions is
$L(x) = x$.

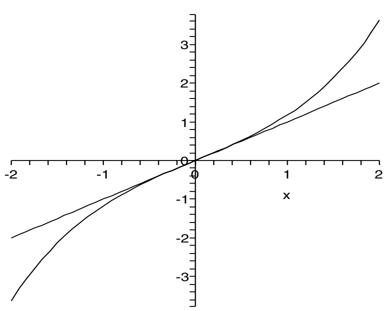
(b) Graph of $f(x) = \sin x$:



Graph of $g(x) = \tan^{-1} x$:



Graph of $h(x) = \sinh x$:



$\sin x$ is the closest fit, but $\sinh x$ is close.

9. (a) $f(x) = \sqrt[4]{16+x}, x_0 = 0$
$f(0) = \sqrt[4]{16+0} = 2$
$f'(x) = \frac{1}{4}(16+x)^{-3/4}$
$f'(0) = \frac{1}{4}(16+0)^{-3/4} = \frac{1}{32}$
$L(x) = f(0) + f'(0)(x-0)$
$= 2 + \frac{1}{32}x$
$L(0.04) = 2+\frac{1}{32}(0.04) = 2.00125$

(b) $L(0.08) = 2 + \frac{1}{32}(0.08) = 2.0025$

(c) $L(0.16) = 2 + \frac{1}{32}(0.16) = 2.005$

10. (a) $f(x) = \sin x,\ x_0 = 0$
$f(0) = 0$
$f'(x) = \cos x$
$f'(0) = \cos 0 = 1$
$L(x) = f(0) + f'(0)(x-0)$
$= 0 + 1 \cdot x$
$L(0.1) = 0.1$

(b) $f(x) = \sin x,\ x_0 = \frac{\pi}{3}$
$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$
$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$
$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$
$L(1) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1 - \frac{\pi}{3}\right) \approx 0.842$

(c) $f(x) = \sin x,\ x_0 = \frac{2\pi}{3}$
$f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'\left(\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$
$L(x) = f\left(\frac{2\pi}{3}\right) + f'\left(\frac{2\pi}{3}\right)\left(x - \frac{2\pi}{3}\right)$
$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{2\pi}{3}\right)$
$L\left(\frac{9}{4}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{9}{4} - \frac{2\pi}{3}\right) \approx 0.788$

11. (a) $\sqrt[4]{16.04} = 2.0012488$
$L(0.04) = 2.00125$
$|2.0012488 - 2.00125|$
$= .00000117$

(b) $\sqrt[4]{16.08} = 2.0024953$
$L(.08) = 2.0025$
$|2.0024953 - 2.0025|$
$= .00000467$

(c) $\sqrt[4]{16.16} = 2.0049814$
$L(.16) = 2.005$
$|2.0049814 - 2.005| = .0000186$

12. $e(0.04) \approx 0.00000117$, $\frac{e(0.04)}{0.04^2} \approx 0.000731$
$e(0.08) \approx 0.00000467$, $\frac{e(0.08)}{0.08^2} \approx 0.000730$
$e(0.16) \approx 0.00001864$, $\frac{e(0.16)}{0.16^2} \approx 0.000728$.
It seems that $e(\Delta x) \approx 0.00073(\Delta x)^2$.

13. (a) $L(x) = f(20) + \frac{18-14}{20-30}(x-20)$
$L(24) \approx 18 - \frac{4}{10}(24-20)$
$= 18 - 0.4(4)$
$= 16.4$ games

(b) $L(x) = f(40) + \frac{14-12}{30-40}(x-40)$
$f(36) \approx 12 - \frac{2}{10}(36-40)$
$= 12 - 0.2(-4)$
$= 12.8$ games

14. (a) $L(x) = f(80) + \frac{120-84}{80-60}(x-80)$
$L(72) = 120 + \frac{36}{20}(72-80)$
$= 120 + 1.8(-8)$
$= 105.6$ cans

(b) $L(x) = f(100) + \frac{168-120}{100-80}(x-100)$
$L(94) = 168 - \frac{48}{20}(94-100)$
$= 168 - 2.4(-6)$
$= 182.4$ cans

15. (a) $L(x) = f(200) + \frac{142-128}{220-200}(x-200)$
$L(208) = 128 + \frac{14}{20}(208-200)$
$= 128 + 0.7(8) = 133.6$

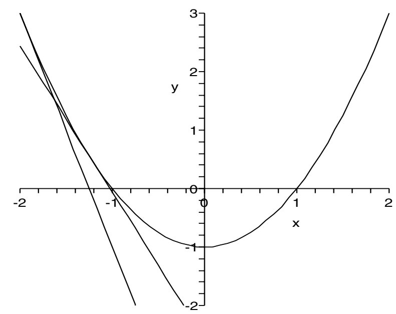
(b) $L(x) = f(240) + \frac{142-136}{220-240}(x-240)$
$L(232) = 136 - \frac{6}{20}(232-240)$
$= 136 - 0.3(-8) = 138.4$

16. (a) $L(x) = f(10) + \frac{14-8}{10-5}(x-10)$
$L(8) = 14 + \frac{6}{5}(-2) = 11.6$

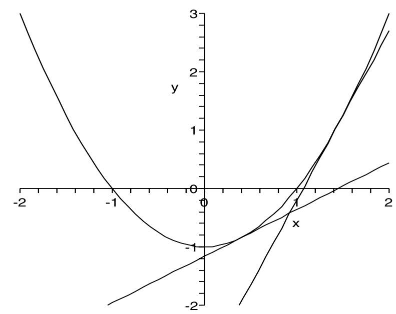
(b) $L(x) = f(10) + \frac{14-8}{10-5}(x-10)$
$L(12) = 14 + \frac{6}{5}(2) = 16.4$

17. The first tangent line intersects the x-axis at a point a little to the right of 1. So x_1 is about 1.25 (very roughly). The second tangent line intersects the x-axis at a point between 1 and x_1, so x_2 is about 1.1 (very roughly). Newton's Method will converge to the zero at $x = 1$.

18. Starting with $x_0 = -2$, Newton's method converges to $x = -1$.



Starting with $x_0 = 0.4$, Newton's method converges to $x = 1$.



19. It wouldn't work because $f'(0) = 0$.

20. $x_0 = 0.2$ works better as an initial guess. After jumping to $x_1 = 2.55$, the sequence rapidly decreases toward $x = 1$. Starting with $x_0 = 10$, it takes several steps to get to 2.5, on the way to $x = 1$.

21. $f(x) = x^3 + 3x^2 - 1 = 0$, $x_0 = 1$
$f'(x) = 3x^2 + 6x$

(a)
$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{1^3 + 3\cdot 1^2 - 1}{3\cdot 1^2 + 6\cdot 1} \\ &= 1 - \frac{3}{9} = \frac{2}{3} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{2}{3} - \frac{\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 - 1}{3\left(\frac{2}{3}\right)^2 + 6\left(\frac{2}{3}\right)} \\ &= \frac{79}{144} \approx 0.5486 \end{aligned}$$

(b) 0.53209

22. $f(x) = x^3 + 4x^2 - x - 1$, $x_0 = -1$
$f'(x) = 3x^2 + 8x - 1$

(a) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{3}{-6} = -\frac{1}{2}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{1}{2} - \frac{0.375}{-4.25} = -0.4117647$$

(b) The root is $x \approx -0.4064206546$.

23. $f(x) = x^4 - 3x^2 + 1 = 0$, $x_0 = 1$
$f'(x) = 4x^3 - 6x$

(a) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \left(\frac{1^4 - 3 \cdot 1^2 + 1}{4 \cdot 1^3 - 6 \cdot 1}\right) = \frac{1}{2}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \left(\frac{\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^2 + 1}{4\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)}\right) = \frac{5}{8}$$

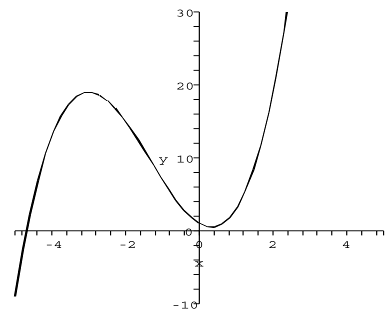
(b) 0.61803

24. $f(x) = x^4 - 3x^2 + 1$, $x_0 = -1$
$f'(x) = 4x^3 - 6x$

(a) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{-1}{2} = -\frac{1}{2}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{1}{2} - \frac{0.3125}{2.5} = -0.625$$

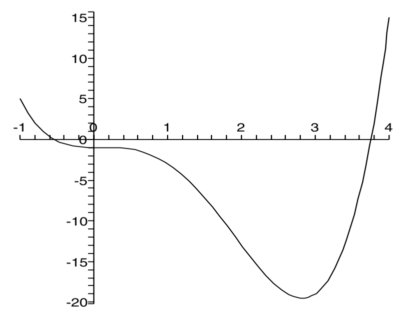
(b) The root is $x \approx -0.6180339887$.

25. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with $f(x) = x^3 + 4x^2 - 3x + 1$, and $f'(x) = 3x^2 + 8x - 3$.



Start with $x_0 = -5$ to find the root near -5:
$x_1 = -4.718750$, $x_2 = -4.686202$,
$x_3 = -4.6857796$, $x_4 = -4.6857795$

26. From the graph, we see two roots:



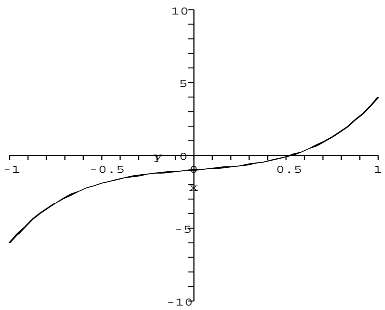
Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with $f(x) = x^4 - 4x^3 + x^2 - 1$, and $f'(x) = 4x^3 - 12x^2 + 2x$.

Start with $x_0 = 4$ to find the root below 4:
$x_1 = 3.791666667$, $x_2 = 3.753630030$,
$x_3 = 3.752433459$, $x_4 = 3.752432297$

Start with $x = -1$ to find the root just above -1:
$x_1 = -0.7222222222$,
$x_2 = -0.5810217936$,
$x_3 = -0.5416512863$,
$x_4 = -0.5387668233$,
$x_5 = -0.5387519962$

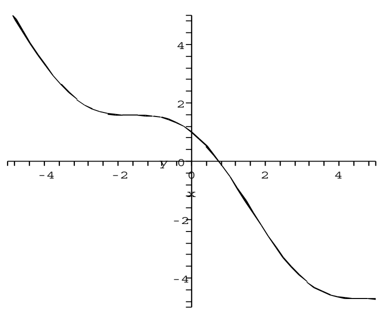
27. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with

$f(x) = x^5 + 3x^3 + x - 1$, and
$f'(x) = 5x^4 + 9x^2 + 1$.



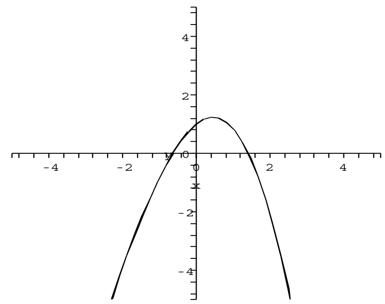
Start with $x_0 = 0.5$ to find the root near 0.5:
$x_1 = 0.526316$, $x_2 = 0.525262$,
$x_3 = 0.525261$, $x_4 = 0.525261$

28. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = \cos x - x$, and
$f'(x) = -\sin x - 1$.



Start with $x_0 = 1$ to find the root near 1:
$x_1 = 0.750364$, $x_2 = 0.739113$,
$x_3 = 0.739085$, $x_4 = 0.739085$

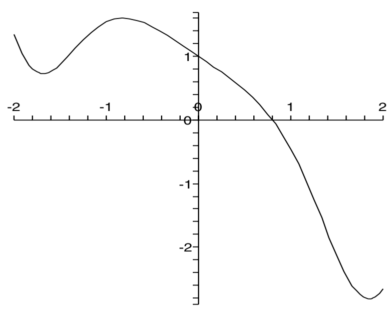
29. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = \sin x - x^2 + 1$, and
$f'(x) = \cos x - 2x$



Start with $x_0 = -0.5$ to find the root near -0.5:
$x_1 = -0.644108$, $x_2 = -0.636751$
$x_3 = -0.636733$, $x_4 = -0.636733$

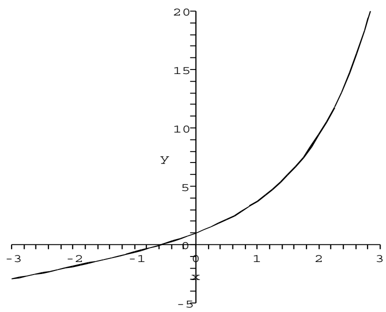
Start with $x_0 = 1.5$ to find the root near 1.5:
$x_1 = 1.413799$, $x_2 = 1.409634$
$x_3 = 1.409624$, $x_4 = 1.409624$

30. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = \cos x^2 - x$, and
$f'(x) = 2x \sin x^2 - 1$.



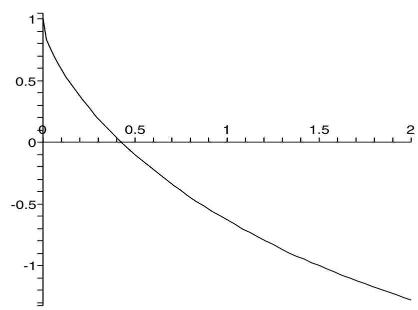
Start with $x_0 = 1$ to find the root between 0 and 1:
$x_1 = 0.8286590991$, $x_2 = 0.8016918647$,
$x_3 = 0.8010710854$, $x_4 = 0.8010707652$

31. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = e^x + x$, and
$f'(x) = e^x + 1$



Start with $x_0 = -0.5$ to find the root between 0 and -1:
$x_1 = -0.566311$, $x_2 = -0.567143$
$x_3 = -0.567143$, $x_4 = -0.567143$

32. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = e^{-x} - \sqrt{x}$, and
$f'(x) = -e^{-x} - \frac{1}{2\sqrt{x}}$.



Start with $x_0 = 0.5$ to find the root close to 0.5:
$x_1 = 0.4234369253$, $x_2 = 0.4262982542$,
$x_3 = 0.4263027510$

33.
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \left(\frac{x_n^2 - c}{2x_n}\right) \\ &= x_n - \frac{x_n^2}{2x_n} + \frac{c}{2x_n} \\ &= \frac{x_n}{2} + \frac{c}{2x_n} \\ &= \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) \end{aligned}$$
If $x_0 < \sqrt{a}$, then $a/x_0 > \sqrt{a}$, so $x_0 < \sqrt{a} < a/x_0$.

34. The root of $x^n - c$ is $\sqrt[n]{c}$, so Newton's method approximates this number. Newton's method gives
$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - c}{nx_i^{n-1}} \\ &= \frac{1}{n}(nx_i - x_i + cx_i^{1-n}), \end{aligned}$$
as desired.

35. $f(x) = x^2 - 11$; $x_0 = 3$; $\sqrt{11} \approx 3.316625$

36. Newton's method for $\sqrt{x}$ near $x = 23$ is $x_{n+1} = \frac{1}{2}(x_n + 23/x_n)$. Start with $x_0 = 5$ to get: $x_1 = 4.8$, $x_2 = 4.7958333$, and $x_3 = 4.7958315$.

37. $f(x) = x^3 - 11$; $x_0 = 2$; $\sqrt[3]{11} \approx 2.22398$

38. Newton's method for $\sqrt[3]{x}$ near $x = 23$ is $x_{n+1} = \frac{1}{3}(2x_n + 23/x_n^2)$. Start with $x_0 = 3$ to get:
$x_1 = 2.851851851$, $x_2 = 2.843889316$, and
$x_3 = 2.884386698$.

39. $f(x) = x^{4.4} - 24$; $x_0 = 2$; $\sqrt[4.4]{24} \approx 2.059133$

40. Newton's method for $\sqrt[4.6]{x}$ near $x = 24$ is $x_{n+1} = \frac{1}{4.6}(3.6x_n + 24/x_n^{3.6})$. Start with $x_0 = 2$ to get:
$x_1 = 1.995417100$, $x_2 = 1.995473305$, and
$x_3 = 1.995473304$.

41. $f(x) = 4x^3 - 7x^2 + 1 = 0$, $x_0 = 0$
$f'(x) = 12x^2 - 14x$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{0}$
The method fails because $f'(x_0) = 0$
Roots are 0.3454, 0.4362, 1.659.

42. Newton's method fails because $f' = 0$. As long as the sequence avoids $x_n = 0$ and $x_n = \frac{7}{6}$ (the zeros of

f'), Newton's method will succeed. Which root is found depends on the starting place.

43. $f(x) = x^2 + 1$, $x_0 = 0$
$f'(x) = 2x$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{0}$
The method fails because $f'(x_0) = 0$. There are no roots.

44. Newton's method fails because the function does not have a root!

45. $f(x) = \dfrac{4x^2 - 8x + 1}{4x^2 - 3x - 7} = 0$, $x_0 = -1$

Note: $f(x_0) = f(-1)$ is undefined, so Newton's Method fails because x_0 is not in the domain of f. Notice that $f(x) = 0$ only when $4x^2 - 8x + 1 = 0$. So using Newton's Method on $g(x) = 4x^2 - 8x + 1$ with $x_0 = -1$ leads to $x \approx .1339$. The other root is $x \approx 1.8660$.

46. The slope tends to infinity at the zero. For every starting point, the sequence does not converge.

47. (a) With $x_0 = 1.2$,
$x_1 = 0.800000000$,
$x_2 = 0.950000000$,
$x_3 = 0.995652174$,
$x_4 = 0.999962680$,
$x_5 = 0.999999997$,
$x_6 = 1.000000000$,
$x_7 = 1.000000000$

(b) With $x_0 = 2.2$,
$x_0 = 2.200000, x_1 = 2.107692$,
$x_2 = 2.056342, x_3 = 2.028903$,
$x_4 = 2.014652, x_5 = 2.007378$,
$x_6 = 2.003703, x_7 = 2.001855$,
$x_8 = 2.000928, x_9 = 2.000464$,
$x_{10} = 2.000232, x_{11} = 2.000116$,
$x_{12} = 2.000058, x_{13} = 2.000029$,
$x_{14} = 2.000015, x_{15} = 2.000007$,
$x_{16} = 2.000004, x_{17} = 2.000002$,
$x_{18} = 2.000001, x_{19} = 2.000000$,
$x_{20} = 2.000000$

The convergence is much faster with $x_0 = 1.2$.

48. Starting with $x_0 = 0.2$ we get a sequence that converges to 0 very slowly. (The 20th iteration is still not accurate past 7 decimal places.) Starting with $x_0 = 3$ we get a sequence that quickly converges to π. (The third iteration is already accurate to 10 decimal places!)

49. (a) With $x_0 = -1.1$
$x_1 = -1.0507937$,
$x_2 = -1.0256065$,
$x_3 = -1.0128572$,
$x_4 = -1.0064423$,
$x_5 = -1.0032246$,
$x_6 = -1.0016132$,
$x_7 = -1.0008068$,
$x_8 = -1.0004035$,
$x_9 = -1.0002017$,
$x_{10} = -1.0001009$,
$x_{11} = -1.0000504$,
$x_{12} = -1.0000252$,
$x_{13} = -1.0000126$,
$x_{14} = -1.0000063$,
$x_{15} = -1.0000032$,
$x_{16} = -1.0000016$,
$x_{17} = -1.0000008$,
$x_{18} = -1.0000004$,
$x_{19} = -1.0000002$,
$x_{20} = -1.0000001$,
$x_{21} = -1.0000000$,
$x_{22} = -1.0000000$

(b) With $x_0 = 2.1$
$x_0 = 2.100000000$,
$x_1 = 2.006060606$,
$x_2 = 2.000024340$,
$x_3 = 2.000000000$,
$x_4 = 2.000000000$

The rate of convergence in (a) is slower than the rate of convergence in (b).

50. From exercise 47, $f(x) = (x-1)(x-2)^2$. Newton's method converges slowly near the double root. From exercise 49, $f(x) = (x-2)(x+1)^2$. Newton's method again converges slowly near the double root. In exercise 48, Newton's method converges slowly near 0, which is a zero of both x and $\sin x$ but converges quickly near π, which is a zero only of $\sin x$.

51. $f(x) = \tan x$, $f(0) = \tan 0 = 0$
$f'(x) = \sec^2 x$, $f'(0) = \sec^2 0 = 1$
$L(x) = f(0) + f'(0)(x-0)$
$\quad = 0 + 1(x-0) = x$
$L(0.01) = 0.01$
$f(0.01) = \tan 0.01 \approx 0.0100003$
$L(0.1) = 0.1$
$f(0.1) = \tan(0.1) \approx 0.1003$
$L(1) = 1$
$f(1) = \tan 1 \approx 1.557$

52. The linear approximation for $\sqrt{1+x}$ at $x = 0$ is $L(x) = 1 + \frac{1}{2}x$. The error when $x = 0.01$ is 0.0000124, when $x = 0.1$ is 0.00119, and when $x = 1$ is 0.0858.

53. $f(x) = \sqrt{4+x}$
$f(0) = \sqrt{4+0} = 2$
$f'(x) = \frac{1}{2}(4+x)^{-1/2}$
$f'(0) = \frac{1}{2}(4+0)^{-1/2} = \frac{1}{4}$
$L(x) = f(0) + f'(0)(x-0) = 2 + \frac{1}{4}x$
$L(0.01) = 2 + \frac{1}{4}(0.01) = 2.0025$
$f(0.01) = \sqrt{4+0.01} \approx 2.002498$
$L(0.1) = 2 + \frac{1}{4}(0.1) = 2.025$
$f(0.1) = \sqrt{4+0.1} \approx 2.0248$
$L(1) = 2 + \frac{1}{4}(1) = 2.25$
$f(1) = \sqrt{4+1} \approx 2.2361$

54. The linear approximation for e^x at $x = 0$ is $L(x) = 1+x$. The error when $x = 0.01$ is 0.0000502, when $x = 0.1$ is 0.00517, and when $x = 1$ is 0.718.

55. If you graph $|\tan x - x|$, you see that the difference is less than .01 on the interval $-.306 < x < .306$ (In fact, a slightly larger interval would work as well.)

56. This can be solved by trial and error, or by using the CAS to plot $e^x - 1 - x - 0.01$, and solve for the intercepts. The interval is approximately
$-0.1448347511 \le x \le 0.1381651224$.

57. For small x we approximate e^x by $x+1$ (see exercise 54).

$$\frac{Le^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}} \approx \frac{L\left[\left(1+\frac{2\pi d}{L}\right) - \left(1-\frac{2\pi d}{L}\right)\right]}{\left(1+\frac{2\pi d}{L}\right) + \left(1-\frac{2\pi d}{L}\right)} \approx \frac{L\left(\frac{4\pi d}{L}\right)}{2} = 2\pi d$$

$$f(d) \approx \frac{4.9}{\pi} \cdot 2\pi d = 9.8d$$

58. If $f(x) = \dfrac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}$, then using the linear approximation we see that

$$f(x) \approx \frac{8\pi hcx^{-5}}{(1+\frac{hc}{kTx}) - 1} = 8\pi kTx^{-4}$$

as desired.

59. The smallest positive solution of the first equation is 0.132782, and for the second equation the smallest positive solution is 1, so the species modeled by the second equation is certain to go extinct. This is consistent with the models, since the expected number of offspring for the population modeled by the first equation is 2.2, while for the second equation it is only 1.3.

60. The linear approximation is given by

$$L(0) + L'(0)(v - 0)$$

so we first find $L'(v)$:

$$L'(v) = \frac{-L_0 v}{c^2\sqrt{1 - v^2/c^2}}.$$

Thus $L'(0) = 0$ and so the linear approximation is just $L(0) = L_0$. The linear approximation is constant and equal to the length of the object at rest, so this approximation suggests that there is never any velocity at which an object contracts to 90% of its original length.

61. The only positive solution is 0.6407.

62. There are three positive solutions. Using a Newton's method or a CAS to solve for them gives: $x = 0.6492189$, $x = 3$, and $x = 3.8507811$.

63. $W(x) = \dfrac{PR^2}{(R+x)^2}, x_0 = 0$
$W'(x) = \dfrac{-2PR^2}{(R+x)^3}$
$L(x) = W(x_0) + W'(x_0)(x - x_0)$
$= \dfrac{PR^2}{(R+0)^2} + \left(\dfrac{-2PR^2}{(R+0)^3}\right)(x-0)$
$= P - \dfrac{2Px}{R}$
$L(x) = 120 - .01(120) = P - \dfrac{2Px}{R}$
$= 120 - \dfrac{2 \cdot 120x}{R}$
$.01 = \dfrac{2x}{R}$
$x = .005R = .005(20{,}900{,}000)$
$= 104{,}500$ ft

64. If $m = m_0(1 - v^2/c^2)^{1/2}$, then $m' = (m_0/2)(1 - v^2/c^2)^{-1/2}(-2v/c^2)$, and $m' = 0$ when $v = 0$. The linear approximation is the constant function $m = m_0$ for small values v.

65. To find the smallest positive solution of $\tan(\sqrt{x}) = \sqrt{x}$, plot $f(x) = \tan(\sqrt{x}) - \sqrt{x}$ to see that it crosses the x-axis at approximately $x = 20$. Newton's method (3 iterations) leads to $L \approx 20.19$.
$y = \sqrt{L} - \sqrt{L}x - \sqrt{L}\cos\sqrt{L}x + \sin\sqrt{L}x$
$= 4.493 - 4.493x - 4.493\cos 4.493x + \sin 4.493x$

66. The solution is $f = 0$. Newton's method is tricky to use because of the potential for rounding error when using $c = 10^{-13}$. In general, $S(f)$ is many orders of magnitude smaller than $S'(f)$ for f near zero, so $S(f)/S'(f)$ rounds to 0, and once this happens, Newton's method stops.

67. The linear approximation for the inverse tangent function at $x = 0$ is
$f(x) \approx f(0) + f'(0)(x - 0)$
$\tan^{-1}(x) \approx \tan^{-1}(0) + \frac{1}{1+0^2}(x - 0)$
$\tan^{-1}(x) \approx x$
Using this approximation,
$\phi = \tan^{-1}\left(\dfrac{3[1 - d/D] - w/2}{D - d}\right)$
$\phi \approx \dfrac{3[1 - d/D] - w/2}{D - d}$
If $d = 0$, then $\phi \approx \frac{3-w/2}{D}$. Thus, if w or D increase, then ϕ decreases.

68. $d'(\theta) = D(w/6\sin(\theta)$
$d(0) = D(1 - w/6)$ so
$d(\theta) \approx d(0) + d'(0)(\theta - 0)$
$= D(1 - w/6) + 0(\theta) = D(1 - w/6)$,
as desired.

69. (a) As we should expect, when we start with $x_0 = 0.1$, Newton's method converges to 0.

(b) When we start with $x_0 = 1.1$, Newton's method converges to 1.

(c) When we start with $x_0 = 2.1$, Newton's method converges to 2.

70. (a) 0

(b) 2

(c) 1

3.2 Indeterminate Forms and L'Hôpital's Rule

1. $\lim_{x\to -2} \frac{x+2}{x^2-4}$
$= \lim_{x\to -2} \frac{x+2}{(x+2)(x-2)}$
$= \lim_{x\to -2} \frac{1}{x-2} = -\frac{1}{4}$

2. $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$
$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-1)}$
$= \lim_{x\to 2} \frac{x+2}{x-1} = 4$

3. $\lim_{x\to\infty} \frac{3x^2+2}{x^2-4}$
$= \lim_{x\to\infty} \frac{3+\frac{2}{x^2}}{1-\frac{4}{x^2}}$
$= \frac{3}{1} = 3$

4. $\lim_{x\to -\infty} \frac{x+1}{x^2+4x+3}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to -\infty} \frac{1}{2x+4} = 0.$$

5. $\lim_{x\to 0} \frac{e^{2x}-1}{x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{2e^{2x}}{1} = \frac{2}{1} = 2.$$

6. $\lim_{x\to 0} \frac{\sin x}{e^{3x}-1}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\cos x}{3e^{3x}} = \frac{1}{3}.$$

7. $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{1/(1+x^2)}{\cos x} = \lim_{x\to 0} \frac{1}{1} = 1.$$

8. $\lim_{x\to 0} \frac{\sin x}{\sin^{-1} x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\cos x}{\frac{1}{\sqrt{1-x^2}}} = 1.$$

9. $\lim_{x\to \pi} \frac{\sin 2x}{\sin x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to \pi} \frac{2\cos 2x}{\cos x} = \frac{2(1)}{-1} = -2.$$

10. $\lim_{x\to -1} \frac{\cos^{-1} x}{x^2-1}$ is undefined (numerator goes to π, denominator goes to 0).

11. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule thrice to get
$= \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x}$
$= \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}.$

12. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}.$$

Apply L'Hôpital's Rule twice more to get

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{1}{3}.$$

13. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$
$= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \frac{\sqrt{x}+1}{\sqrt{x}+1}$
$= \lim_{x\to 1} \frac{x-1}{(x-1)\left(\sqrt{x}+1\right)}$
$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$

14. $\lim_{x\to 1} \frac{\ln x}{x-1}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

15. $\lim_{x\to\infty} \frac{x^3}{e^x}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule thrice to get

$$\lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

16. $\lim_{x\to\infty} \frac{e^x}{x^4}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule four times to get

$$\lim_{x\to\infty} \frac{e^x}{4x^3} = \lim_{x\to\infty} \frac{e^x}{12x^2} = \lim_{x\to\infty} \frac{e^x}{24x} = \lim_{x\to\infty} \frac{e^x}{24} = \infty.$$

17. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin^2 x}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule twice to get

$$\lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin^2 x + 2x\sin x\cos x}$$
$$= \lim_{x\to 0} \frac{-x\sin x}{\sin x\,(\sin x + 2x\cos x)}$$
$$= \lim_{x\to 0} \frac{-x}{\sin x + 2x\cos x}$$
$$= \lim_{x\to 0} \frac{-1}{\cos x + 2\cos x - 2x\sin x}$$
$$= -\frac{1}{3}.$$

18. Rewriting as one fraction we have

$$\lim_{x\to 0} \left(\cot^2 x - \frac{1}{x^2}\right) = \lim_{x\to 0} \left(\frac{x^2\cot^2 x - \sin^2 x}{x^2\sin^2 x}\right).$$

This is of the form $\frac{0}{0}$. We very carefully apply L'Hôpital's Rule four times to find that this is equal to

$$\lim_{x\to 0} \frac{f(x)}{g(x)}$$

where

$$f(x) = -16\cos^2 x + 16\sin^2 x + 64x\cos x\sin x - 8x^2\sin^2 x + 8x^2\cos^2 x$$

and

$$g(x) = 24\cos^2 x - 24\sin^2 x - 64x\cos x\sin x + 8x^2\sin^2 x - 8x^2\cos^2 x$$

so that the limit is equal to

$$\frac{-16}{24} = -\frac{2}{3}.$$

19. $\lim_{x\to 1} \frac{\sin \pi x}{x-1}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 1} \frac{\pi\cos\pi x}{1} = \frac{\pi(-1)}{1} = -\pi.$$

20. $\lim_{x\to 1} \frac{e^{x-1}-1}{x^2-1}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 1} \frac{e^{x-1}}{2x} = \frac{1}{2}.$$

21. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

22. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

23. $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \dfrac{x}{e^x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{1}{e^x} = 0.$$

24. $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\sin\frac{1}{x}}{1/x}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \cos\frac{1}{x} = 1.$$

25. As x approaches 1 from below, $\ln x$ is a small negative number. Hence $\ln(\ln x)$ is undefined, so the limit is undefined.

26. $\lim\limits_{x\to 0} \dfrac{\sin(\sin x)}{\sin(x)}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\cos(\sin x)\cos(x)}{\cos(x)} = 1.$$

27. $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

28. $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0^+} \frac{\cos x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0.$$

29. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\cot x}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} \\
&= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} \\
&= \lim_{x\to 0^+} \left(-\sin x \frac{\sin x}{x}\right) = (0)(1) = 0.
\end{aligned}$$

30. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x}}{\ln x} = 0$ (numerator goes to 0 and denominator goes to $-\infty$).

31.
$$\begin{aligned}
&\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \\
&= \lim_{x\to\infty} \left(\left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right) \\
&= \lim_{x\to\infty} \left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right) \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0.
\end{aligned}$$

32. $\lim\limits_{x\to\infty} \ln x - x = \lim\limits_{x\to\infty} \dfrac{\frac{\ln x}{x} - 1}{\frac{1}{x}} = -\infty$ since the numerator goes to -1 and the denominator goes to 0^+. (Recall Example 2.8 which shows $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$.)

33. Let $y = \left(1 + \frac{1}{x}\right)^x$.
Then $\ln y = x\ln\left(1 + \frac{1}{x}\right)$. Then

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}
\end{aligned}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{-1/x^2}$$
$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Hence $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e.$

34. Notice that the limit in question has the indeterminate form 1^{∞}. Also, note that as x gets large,

$$\left|\frac{x+1}{x-2}\right| = \frac{x+1}{x-2}.$$

Define

$$y = \left(\frac{x+1}{x-2}\right)^{\sqrt{x^2-4}}.$$

Then

$$\ln y = \sqrt{x^2-4}\,\ln\left(\frac{x+1}{x-2}\right)$$

and

$$\lim_{x\to\infty} \ln y$$
$$= \lim_{x\to\infty}\left(\sqrt{x^2-4}\,\ln\left(\frac{x+1}{x-2}\right)\right)$$
$$= \lim_{x\to\infty}\left(\frac{\ln\left(\frac{x+1}{x-2}\right)}{\frac{1}{\sqrt{x^2-4}}}\right)$$

This last limit has indeterminate form $\frac{0}{0}$, so we can apply L'Hôpital's Rule and simplify to find that the above is equal to

$$\lim_{x\to\infty} \frac{-3(x^2-4)^{3/2}}{-x^3+x^2+2x}$$

and this is equal to 3. So $\lim_{x\to\infty} \ln y = 3$. Thus

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^3 \approx 20.086.$$

35. $\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x+1}}\right)$
$= \lim_{x\to 0^+}\left(\frac{\sqrt{x+1}-(\sqrt{x})^2}{\sqrt{x}\sqrt{x+1}}\right)$
$= \lim_{x\to 0^+}\left(\frac{\sqrt{x+1}-x}{\sqrt{x}\sqrt{x+1}}\right)$
$= \infty.$

36. $\lim_{x\to 1} \frac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get
$\lim_{x\to 1} \frac{\frac{1}{2}(5-x)^{-1/2}(-1)}{\frac{1}{2}(10-x)^{-1/2}(-1)}$
$= \lim_{x\to 1} \frac{\sqrt{10-x}}{\sqrt{5-x}} = \frac{3}{2}.$

37. Let $y = (1/x)^x$. Then $\ln y = x\ln(1/x)$. Then

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln(1/x) = 0,$$

by Exercise 27. Thus

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = 1.$$

38. Let $y = \lim_{x\to 0^+} (\cos x)^{1/x}$.
Then
$\ln y = \lim_{x\to 0^+} \frac{1}{x}\ln\cos x$
$= \lim_{x\to 0^+} \frac{\ln(\cos x)}{x}$ is type $\frac{0}{0}$
so apply L'Hôpital's Rule to get

$$\lim_{x\to 0^+} \frac{-\tan x}{1} = 0.$$

Therefore the limit is $y = e^0 = 1$.

39. L'Hôpital's rule does not apply. As $x \to 0$, the numerator gets close to 1 and the denominator is small and positive. Hence the limit is ∞.

40. $\lim_{x\to 0} \frac{e^x-1}{x^2}$ is type $\frac{0}{0}$, but $\lim_{x\to 0} \frac{e^x}{2x}$ is not,

so L'Hôpital's Rule does not apply to this limit.

41. L'Hôpital's rule does not apply. As $x \to 0$, the numerator is small and positive while the denominator goes to $-\infty$. Hence the limit is 0. Also $\lim_{x\to 0} \frac{2x}{2/x}$, which equals $\lim_{x\to 0} x^2$, is not of the form $\frac{0}{0}$ so L'Hôpital's rule doesn't apply here either.

42. $\lim_{x\to 0} \frac{\sin x}{x^2}$ is type $\frac{0}{0}$, but $\lim_{x\to 0} \frac{\cos x}{2x}$ is not,

so L'Hôpital's rule does not apply. This limit is undefined because the numerator goes to 1 and the denominator goes to 0.

43. Starting with $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$, we cannot "cancel sin" to get $\lim_{x\to 0} \frac{3x}{2x}$. We can cancel the x's in the last limit to get the final answser of 3/2. The first step is likely to give a correct answer because the linear approximation of $\sin 3x$ is $3x$, and the linear approximation of $\sin 2x$ is $2x$. The linear approximations are better the closer x is to zero, so the limits are likely to be the same.

44. $\lim_{x\to 0} \frac{\sin nx}{\sin mx}$ is type $\frac{0}{0}$;
we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{n\cos nx}{m\cos mx} = \frac{n}{m}.$$

45. (a) $\lim_{x\to 0} \frac{\sin x^2}{x^2} = \lim_{x\to 0} \frac{2x\cos x^2}{2x}$
$= \lim_{x\to 0} \cos x^2 = 1,$
which is the same as $\lim_{x\to 0} \frac{\sin x}{x}$.

(b) $\lim_{x\to 0} \frac{1-\cos x^2}{x^4}$
$= \lim_{x\to 0} \frac{2x\sin x^2}{4x^3} = \lim_{x\to 0} \frac{\sin x^2}{2x^2}$
$= \frac{1}{2}\lim_{x\to 0} \frac{\sin x^2}{x^2}$
$= (1/2)(1) = 1/2$ (by part (a)),
while
$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}$
$= \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x}$
$= \frac{1}{2}(1) = \frac{1}{2}$
so both of these limits are the same.

46. Based on the patterns found in exercise 45, we should guess
$\lim_{x\to 0} \frac{\sin x^3}{x^3} = 1$ and
$\lim_{x\to 0} \frac{1-\cos x^3}{x^6} = \frac{1}{2}$.

47. $\lim_{x\to 0} \frac{\sin kx^2}{x^2}$
$= \lim_{x\to 0} \frac{2kx\cos kx^2}{2x}$
$= \lim_{x\to 0} k\cos kx^2 = k(1) = k$

48. Based on the result of exercise 47, we see that a limit of type $\frac{0}{0}$ can have any real value, be $\pm\infty$, or be undefined.

49. $\lim_{x\to\infty} e^x = \lim_{x\to\infty} x^n = \infty$
$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ since n applications of L'Hôpital's rule yields

$$\lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

Hence e^x dominates x^n.

50. $\lim_{x\to\infty} \ln x = \lim_{x\to\infty} x^p = \infty$.
$\lim_{x\to\infty} \frac{\ln x}{x^p}$ is of type $\frac{\infty}{\infty}$;
we use L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$$

(since $p > 0$).
Therefore, x^p dominates $\ln x$.

51. $\lim_{x\to 0} \frac{e^{cx}-1}{x} = \lim_{x\to 0} \frac{ce^{cx}}{1} = c$

52. $\lim_{x\to 0} \frac{\tan cx - cx}{x^3}$ is type $\frac{0}{0}$; we use L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{c\sec^2 cx - c}{3x^2} = \lim_{x\to 0} \frac{2c^2\sec^2 cx \tan cx}{6x} = \lim_{x\to 0} \frac{4c^3\sec^2 cx\tan^2 cx + 2c^3\sec^4 cx}{6} = \frac{c^3}{3}.$$

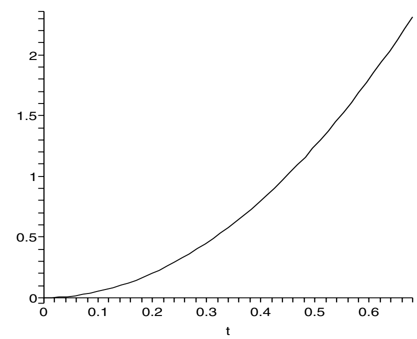
53. If $x \to 0$, then $x^2 \to 0$, so if $\lim_{x\to 0} \frac{f(x)}{g(x)} = L$, then $\lim_{x\to 0} \frac{f(x^2)}{g(x^2)} = L$ (but not conversely). If $a \neq 0$ or 1, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ involves the behavior of the quotient near a, while $\lim_{x\to a} \frac{f(x^2)}{g(x^2)}$ involves the behavior of the quotient near the different point a^2.

54. Functions $f(x) = |x|$ and $g(x) = x$ work. $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist as it approaches -1 from the left and it approaches 1 from the right, but $\lim_{x\to 0} \frac{f(x^2)}{g(x^2)} = 1$.

55. $$\lim_{\omega\to 0} \frac{2.5(4\omega t - \sin 4\omega t)}{4\omega^2} = \lim_{\omega\to 0} \frac{2.5(4t - 4t\cos 4\omega t)}{8\omega} = \lim_{\omega\to 0} \frac{2.5(16t^2\sin 4\omega t)}{8} = 0$$

56. $\lim_{\omega\to 0} \frac{2.5 - 2.5\sin(4\omega t + \frac{\pi}{2})}{4\omega^2}$ is type $\frac{0}{0}$; we apply L'Hôpital's Rule to get

$$\lim_{\omega\to 0} \frac{-10t\cos(4\omega t + \frac{\pi}{2})}{8\omega} = \lim_{\omega\to 0} \frac{40t^2\sin(4\omega t + \frac{\pi}{2})}{8} = 5t^2.$$



The pitch curves drastically left to right.

57. (a) $\frac{(x+1)(2+\sin x)}{x(2+\cos x)}$

(b) $\frac{x}{e^x}$

(c) $\frac{3x+1}{x-7}$

(d) $\frac{3-8x}{1+2x}$

58. (a) $\lim_{x\to\infty} x - \ln x = \infty$ (see exercise 32).

(b) $\lim_{x\to\infty} \sqrt{x^2+1} - x = 0$ (see exercise 31).

(c) $\lim_{x\to\infty} \sqrt{x^2+4x} - x$

$$= \lim_{x\to\infty} (\sqrt{x^2+4x} - x) = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x\frac{1}{x}}{(\sqrt{x^2+4x}+x)\frac{1}{x}} = \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} = 2,$$

where to get from the second to the third line, we have multiplied by

$$\frac{(\sqrt{x^2+4x}+x)}{(\sqrt{x^2+4x}+x)}.$$

59. The area of triangular region 1 is (1/2)(base)(height)

$= (1/2)(1-\cos\theta)(\sin\theta)$.
Let P be the center of the circle. The area of region 2 equals the area of sector APC minus the area of triangle APB. The area of the sector is $\theta/2$, while the area of triangle APB is
$(1/2)(\text{base})(\text{height})$
$= (1/2)(\cos\theta)(\sin\theta)$.
Hence the area of region 1 divided by the area of region 2 is

$$\frac{(1/2)(1-\cos\theta)(\sin\theta)}{\theta/2-(1/2)(\cos\theta)(\sin\theta)}$$
$$= \frac{(1-\cos\theta)(\sin\theta)}{\theta-\cos\theta\sin\theta}$$
$$= \frac{\sin\theta-\cos\theta\sin\theta}{\theta-\cos\theta\sin\theta}$$
$$= \frac{\sin\theta-(1/2)\sin 2\theta}{\theta-(1/2)\sin 2\theta}$$

Then

$$\lim_{\theta\to 0}\frac{\sin\theta-(1/2)\sin 2\theta}{\theta-(1/2)\sin 2\theta}$$
$$= \lim_{\theta\to 0}\frac{\cos\theta-\cos 2\theta}{1-\cos 2\theta}$$
$$= \lim_{\theta\to 0}\frac{-\sin\theta+2\sin 2\theta}{2\sin 2\theta}$$
$$= \lim_{\theta\to 0}\frac{-\cos\theta+4\cos 2\theta}{4\cos 2\theta}$$
$$= \frac{-1+4(1)}{4(1)} = \frac{3}{4}$$

60. $$\lim_{x\to 0^+}\frac{160x^{-0.4}+90}{8x^{-0.4}+10}$$
$$= \lim_{x\to 0^+}\frac{160+90x^{0.4}}{8+10x^{0.4}} = \frac{160}{8} = 20.$$

If there is no light, the pupils will expand to this size. This is the largest the pupils can get.

$$\lim_{x\to\infty}\frac{160x^{-0.4}+90}{8x^{-0.4}+10} = \frac{90}{10} = 9.$$

As the amount of light grows, the pupils shrink, and the size approaches 6mm in the limit. This is the smallest possible size of the pupils.

3.3 Maximum and Minimum Values

1. (a) No absolute extrema.

(b) $f(0) = -1$ is absolute max. There is no absolute minimum (vertical asymptotes at $x = \pm 1$).

(c) No absolute extrema. (They would be at the endpoints which are not included in the interval.)

2. (a) The minimum is $f(0) = 0$, and the function has no maximum.

(b) The minimum is $f(0) = 0$. There is no absolute maximum (vertical asymptote at $x = 1$).

(c) The function does not have a maximum or minimum. The minimum would be at $x = 0$ (not included in this interval) while the asymptote at $x = 1$ precludes an absolute maximum.

3. (a) $f\left(\frac{\pi}{2}+2n\pi\right) = 1$ for any integer n is abs max;
$f\left(\frac{3\pi}{2}+2n\pi\right) = -1$ for any integer n is abs min

(b) $f(0) = 0$ is abs min; $f(\pi/4) = \frac{\sqrt{2}}{2}$ is abs max

(c) $f(\pi/2) = 1$ is abs max; there is no abs min, which would occur at both endpoints (not included in the interval).

4. (a) The function has no absolute extrema.

(b) The absolute minimum is -1 which occurs at $x = 1$ and $x = -2$. The absolute maximum is 3 which occurs at $x = -1$ and $x = 2$.

(c) On $(0,2)$ the minimum is $f(1) = -1$, and the function has no maxima. The maxima would be at the right endpoint (not included in this interval).

5. $f(x) = x^2 + 5x - 1$
$f'(x) = 2x + 5$
$2x + 5 = 0$
$x = -5/2$ is a critical number. This is a parabola opening upward, so we have a minimum.

6. $f(x) = -x^2 + 4x + 2$
$f'(x) = -4x + 4 = 0$ when $x = 1$.
This is a parabola opening downward, so we have a maximum.

7. $f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3$
$3x^2 - 3 = 3(x^2 - 1)$
$= 3(x+1)(x-1) = 0$
$x = \pm 1$ are critical numbers.
This is a cubic with a positive leading coefficient so $x = -1$ is a local max, $x = 1$ is a local min.

8. $f(x) = -x^3 + 6x^2 + 2$
$f'(x) = -3x^2 + 12x = -3x(x+4) = 0$ when $x = 0$ and $x = -4$.
This is a cubic with a positive leading coefficient so $x = 0$ is a local max and $x = -4$ is a local min.

9. $f(x) = x^3 - 3x^2 + 6x$
$f'(x) = 3x^2 - 6x + 6$
$3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 0$
We can use the quadratic formula to find the roots, which are $x = 1 \pm \sqrt{-1}$. These are imaginary so there are no real critical numbers.

10. $f(x) = x^3 - 3x^2 + 3x$
$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 = 0$ when $x = 1$. Since $f(x)$ is a cubic with only one critical number, it is neither a local min nor max.

11. $f(x) = x^4 - 3x^3 + 2$
$f'(x) = 4x^3 - 9x^2$
$4x^3 - 9x^2 = x^2(4x - 9) = 0$
$x = 0, 9/4$ are critical numbers
$x = 9/4$ is a local min; $x = 0$ is neither a local max nor min.

12. $f(x) = x^4 + 6x^2 - 2$
$f'(x) = 4x^3 + 12x = 0$ when $x = 0$ (minimum).

13. $f(x) = x^{3/4} - 4x^{1/4}$
$f'(x) = \dfrac{3}{4x^{1/4}} - \dfrac{1}{x^{3/4}}$
If $x \neq 0$, $f'(x) = 0$ when $3x^{3/4} = 4x^{1/4}$ $x = 0, 16/9$ are critical numbers.
$x = 16/9$ is a local min, $x = 0$ is neither a local max nor min.

14. $f(x) = (x^{2/5} - 3x^{1/5})^2$
$f'(x) = 2(x^{2/5} - 3x^{1/5}) \left(\dfrac{2}{5x^{3/5}} - \dfrac{3}{5x^{4/5}} \right)$
$f'(x) = 0$ when $x = 3^5$ (minimum) and $x = (\frac{3}{2})^5$ (maximum). $f'(x)$ is undefined when $x = 0$ (minimum).

15. $f(x) = \sin x \cos x$ on $[0, 2\pi]$
$f'(x) = \cos x \cos x + \sin x(-\sin x)$
$= \cos^2 x - \sin^2 x$

$\cos^2 x - \sin^2 x = 0$
$\cos^2 x = \sin^2 x$
$\cos x = \pm \sin x$

$x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ are critical numbers.
$x = \pi/4, 5\pi/4$ are local max, $x = 3\pi/4, 7\pi/4$ are local min.

16. $f(x) = \sqrt{3}\sin x + \cos x$
$f'(x) = \sqrt{3}\cos x - \sin x = 0$ when $\tan(x) = \sqrt{3}$ or $x = \pi/3 + k\pi$ for any integer k (maxima for even k and minima for odd k).

17. $f(x) = \dfrac{x^2-2}{x+2}$
Note that $x = -2$ is not in the domain of f.
$$f'(x) = \frac{(2x)(x+2)-(x^2-2)(1)}{(x+2)^2} = \frac{2x^2+4x-x^2+2}{(x+2)^2} = \frac{x^2+4x+2}{(x+2)}$$
$f'(x) = 0$ when $x^2+4x+2 = 0$, so the critical numbers are $x = -2 \pm \sqrt{2}$.
$x = -2+\sqrt{2}$ is a local min; $x = -2+\sqrt{2}$ is a local max.

18. $f(x) = \dfrac{x^2-x+4}{x-1}$
$$f'(x) = \frac{(x-1)(2x-1)-(x^2-x+4)}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2} = 0$$
when $x = -1$ (maximum) and $x = 3$ (minimum). $f'(x)$ is undefined when $x = 1$ (not in domain of f).

19. $f(x) = \dfrac{x}{x^2+1}$
$$f'(x) = \frac{1(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$
$f'(x) = 0$ for $1-x^2 = 0, x = 1, -1$; $f'(x)$ is defined for all x, so $x = 1, -1$ are the critical numbers.
$x = -1$ is local min, $x = 1$ is local max.

20. $f(x) = \dfrac{3x}{x^2-1}$
$$f'(x) = \frac{(x^2-1)3-3x(2x)}{(x^2-1)^2} = \frac{-3(x^2+1)}{(x^2-1)^2} \neq 0 \text{ for any } x.$$
$f'(x)$ is undefined when $x = \pm 1$ (not in domain of f).

21. $f(x) = \dfrac{e^x+e^{-x}}{2}$
$f'(x) = \dfrac{e^x-e^{-x}}{2}$
$f'(x) = 0$ when $e^x = e^{-x}$, that is, $x = 0$.
$f'(x)$ is defined for all x, so $x = 0$ is a critical number.
$x = 0$ is a local min.

22. $f(x) = xe^{-2x}$
$f'(x) = e^{-2x} - 2xe^{-2x} = 0$ when $x = \frac{1}{2}$ (maximum).

23. $f(x) = x^{4/3} + 4x^{1/3} + 4x^{-2/3}$
f is not defined at $x = 0$.
$$f'(x) = \tfrac{4}{3}x^{1/3} + \tfrac{4}{3}x^{-2/3} - \tfrac{8}{3}x^{-5/3} = \tfrac{4}{3}x^{-5/3}(x^2+x-2) = \tfrac{4}{3}x^{-5/3}(x-1)(x+2)$$
$x = -2, 1$ are critical numbers.
$x = -2$ and $x = 1$ are local minima.

24. $f(x) = x^{7/3} - 28x^{1/3}$
$f'(x) = \frac{7}{3}x^{4/3} - \frac{28}{3}x^{-2/3} = 0$ when $x = -2$ (maximum) and $x = 2$ (minimum).
$f'(x)$ is undefined at $x = 0$ (neither)

25. $f(x) = 2x\sqrt{x+1} = 2x(x+1)^{1/2}$
Domain of f is all $x \geq -1$.
$$f'(x) = 2(x+1)^{1/2} + 2x\left(\frac{1}{2}(x+1)^{-1/2}\right) = \frac{2(x+1)+x}{\sqrt{x+1}} = \frac{3x+2}{\sqrt{x+1}}$$
$f'(x) = 0$ for $3x+2 = 0$, $x = -2/3$.
$f'(x)$ is undefined for $\sqrt{x+1} = 0$, $x = -1$ so $x = -2/3, -1$ are critical numbers.
$x = -2/3$ is a local min. $x = -1$ is an endpoint so is neither a local min nor a local max, though it is a maximum on the interval $[-1, 0)$.

26. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$f'(x) = \dfrac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$

$= \dfrac{1}{(x^2+1)^{3/2}} \neq 0$ for any x, and is not undefined for any x.

27. Because of the absolute value sign, there may be critical numbers where the function x^2-1 changes sign; that is, at $x = \pm 1$. For $x > 1$ and for $x < -1$, $f(x) = x^2-1$ and $f'(x) = 2x$, so there are no critical numbers on these intervals. For $-1 < x < 1$, $f(x) = 1 - x^2$ and $f'(x) = -2x$, so 0 is a critical number. A graph confirms this analysis and shows there is a local max at $x = 0$ and local min at $x = \pm 1$.

28. $f(x) = \sqrt[3]{x^3 - 3x^2 + 2x}$

$f'(x) =$

$\frac{1}{3}(x^3 - 3x^2 + 2x)^{-2/3}(3x^2 - 6x + 2)$

$= \dfrac{3x^2 - 6x + 2}{3x^{2/3}((x-2)(x-1))^{2/3}}$

$f'(x) = 0$ when $x = \dfrac{3 \pm \sqrt{3}}{3}$ and $f'(x)$ is undefined at $x = 0$, $x = 2$ and $x = 1$. $x = \dfrac{3 - \sqrt{3}}{3}$ is a local max and $x = \dfrac{3 + \sqrt{3}}{3}$ is a local min.

29. First, let's find the critical numbers for $x < 0$. In this case,

$f(x) = x^2 + 2x - 1$

$f'(x) = 2x + 2 = 2(x+1)$

so the only critical number in this interval is $x = -1$ and it is a local minimum.

Now for $x > 0$,

$f(x) = x^2 - 4x + 3$

$f'(x) = 2x - 4 = 2(x-2)$

so the only critical number is $x = 2$ and it is a local minimum.

Finally, $x = 0$ is also a critical number, since f is not continuous and hence not differentiable at $x = 0$. Indeed, $x = 0$ is a local maximum.

30. $f'(x) = \cos x$ for $-\pi < x < \pi$, and $f'(x) = -\sec^2 x$ for $|x| \geq \pi$.

$f'(x) = 0$ for $x = -\pi/2$ (minimum) and $x = \pi/2$ (maximum). $f'(x)$ is undefined for $x = (2k+1)\frac{\pi}{2}$ for integers $k \neq -1$ or 0 (not in domain of f).

31. $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$f'(x) = 0$ for $x = \pm 1$.

(a) On $[0, 2]$, 1 is the only critical number. We calculate:
$f(0) = 1$
$f(1) = -1$ is the abs min.
$f(2) = 3$ is the abs max.

(b) On the interval $[-3, 2]$, we have both 1 and -1 as critical numbers. We calculate:
$f(-3) = -17$ is the abs min.
$f(-1) = 3$ is the abs max.
$f(1) = -1$
$f(2) = 3$ is also the abs max.

32. $f(x) = x^4 - 8x^2 + 2$

$f'(x) = 4x^3 - 16x = 0$ when $x = 0$ and $x = \pm 2$.

(a) On $[-3, 1]$:
$f(-3) = 11$, $f(-2) = -14$, $f(0) = 2$, and $f(1) = -5$.
The abs min on this interval is $f(-2) = -14$ and the abs max is $f(-3) = 11$.

(b) On $[-1, 3]$:
$f(-1) = -5$, $f(2) = -14$, and $f(3) = 11$.
The abs min on this interval is

$f(2) = -14$ and the abs max is $f(3) = 11$.

33. $f(x) = x^{2/3}$
$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$
$f'(x) \neq 0$ for any x, but $f'(x)$ undefined for $x = 0$, so $x = 0$ is critical number.

(a) On $[-4, -2]$:
$0 \notin [-4, -2]$ so we only look at endpoints.
$f(-4) = \sqrt[3]{16} \approx 2.52$
$f(-2) = \sqrt[3]{4} \approx 1.59$
So $f(-4) = \sqrt[3]{16}$ is the abs max and $f(-2) = \sqrt[3]{4}$ is the abs min.

(b) On $[-1, 3]$, we have 0 as a critical number.
$f(-1) = 1$
$f(0) = 0$ is the abs min.
$f(3) = 3^{2/3}$ is the abs max.

34. $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x = 0$ when $x = \frac{\pi}{4} + k\pi$ for integers k.

(a) On $[0, 2\pi]$:
$f(0) = 1$, $f(\pi/4) = \sqrt{2}$, $f(5\pi/4) = -\sqrt{2}$, and $f(2\pi) = 1$.
The abs min on this interval is $f(5\pi/4) = -\sqrt{2}$ and the abs max is $f(\pi/4) = \sqrt{2}$.

(b) On $[\pi/2, \pi]$:
$f(\pi/2) = 1$, $f(\pi) = -1$.
The abs min on this interval is $f(\pi) = -1$ and the abs max is $f(\pi/2) = 1$.

35. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
Hence $x = 0$ is the only critical number.

(a) On $[0, 2]$:
$f(0) = 1$ is the abs max.
$f(2) = e^{-4}$ is the abs min.

(b) On $[-3, 2]$:
$f(-3) = e^{-9}$ is the abs min.
$f(0) = 1$ is the abs max.
$f(2) = e^{-4}$

36. $f(x) = x^2e^{-4x}$
$f'(x) = 2xe^{-4x} - 4x^2e^{-4x} = 0$ when $x = 0$ and $x = 1/2$.

(a) On $[-2, 0]$:
$f(-2) = 4e^8$, $f(0) = 0$.
The abs min is $f(0) = 0$ and the abs max is $f(-2) = 4e^8$.

(b) On $[0, 4]$:
$f(1/2) = e^{-2}/4$, $f(4) = 16e^{-16}$.
The abs min is $f(0) = 0$ and the abs max is $f(1/2) = e^{-2}/4$.

37. $f(x) = \dfrac{3x^2}{x-3}$
Note that $x = 3$ is not in the domain of f.

$$\begin{aligned} f'(x) &= \frac{6x(x-3) - 3x^2(1)}{(x-3)^2} \\ &= \frac{6x^2 - 18x - 3x^2}{(x-3)^2} \\ &= \frac{3x^2 - 18x}{(x-3)^2} \\ &= \frac{3x(x-6)}{(x-3)^2} \end{aligned}$$

The critical points are $x = 0$, $x = 6$.

(a) On $[-2, 2]$:
$f(-2) = -12/5$
$f(2) = -12$
$f(0) = 0$
Hence abs max is $f(0) = 0$ and abs min is $f(2) = -12$.

(b) On $[2, 8]$, the function is not continuous and in fact has no absolute max or min.

38. $f(x) = \tan^{-1}(x^2)$
$f'(x) = \frac{2x}{1+x^4} = 0$ when $x = 0$.

(a) On $[0, 1]$:
$f(0) = 0$ and $f(1) = \pi/4$.

The abs min is $f(0) = 0$ and the abs max is $f(1) = \pi/4$.

(b) On $[-3, 4]$:
$f(-3) \approx 1.46$, $f(0) = 0$, and $f(4) \approx 1.51$.
The abs min is $f(0) = 0$ and the abs max is $f(4) = \tan^{-1} 16$.

39. $f'(x) = 4x^3 - 6x + 2 = 0$ at about $x = 0.3660$, -1.3660 and at $x = 1$.

(a) $f(-1) = 3$, $f(1) = 1$.
The absolute min is $(-1, 3)$ and the absolute max is approximately $(0.3660, 1.3481)$.

(b) The absolute min is approximately $(-1.3660, -3.8481)$ and the absolute max is $(-3, 49)$.

40. $f'(x) = 6x^5 - 12x - 2 = 0$ at about -1.3673, -0.5860 and 1.4522.

(a) $f(-1) = 1$, $f(1) = -3$. $f(-0.5860) = 1.8587$.
The absolute min is $f(1) = -3$ and the absolute max is approximately $f(-0.5860) = 1.8587$.

(b) $f(-2) = 21$ and $f(2) = 13$. $f(-1.3673) = -.2165$ and $f(1.4522) = -5.8675$.
The absolute min is approximately $f(1.4522) = -5.8675$ and the absolute max is $f(-2) = 21$.

41. $f'(x) = 2x - 3\cos x + 3x \sin x = 0$ at about $x = 0.6371$, -1.2269 and -2.8051.

(a) The absolute min is approximately $(0.6371, -1.1305)$ and the absolute max is approximately $(-1.2269, 2.7463)$.

(b) The absolute min is approximately $(-2.8051, -0.0748)$ and the absolute max is approximately $(-5, 29.2549)$.

42. $f'(x) = e^{\cos 2x} - 2x(\sin 2x)e^{\cos 2x} = 0$ at approximately $x = -1.3863$, -0.5571, 0.5571, 1.3863, $x = 3.2196$ and 4.6586.

(a) $f(-2) \approx -1.0403$ and $f(2) \approx 1.0403$. $f(x)$ evaluated at these values are not the absolute extrema. The absolute min is $f(-2) \approx -1.0403$ and the absolute max is $f(2) \approx 1.0403$.

(b) $f(2) \approx 1.0403$ and $f(5) \approx 2.1606$.
$f(3.2196) = 8.6461$ and $f(4.6586) = 1.7237$.
The absolute min is $f(2) \approx 1.0403$ and the absolute max is approximately $f(3.2196) = 8.6461$.

43. $f'(x) = \sin x + x \cos x = 0$ at $x = 0$ and about 2.0288 and 4.9132.

(a) The absolute min is $(0, 3)$ and the absolute max is $(\pm\pi/2, 3 + \pi/2)$.

(b) The absolute min is approximately $(4.9132, -1.814)$ and the absolute max is approximately $(2.0288, 4.820)$.

44. $f'(x) = 2x + e^x = 0$ at approximately $x = -0.3517$.

(a) $f(0) = 1$ and $f(1) = 1 + e \approx 3.71828$. $f'(x) \neq 0$ on this interval, so the absolute min is $f(0) = 1$ and the absolute max is $f(1) = 1 + e \approx 3.71828$.

(b) $f(-2) \approx 4.1353$ and $f(2) \approx 11.3891$. $f(-0.3517) = 0.8272$.
The absolute min is approximately $f(-0.3517) = 0.8272$

and the absolute max is approximately $f(2) = 11.3891$.

45. If an absolute max or min occurs only at the endpoint of a closed interval, then there will be no absolute max or min on the open interval.

31) on $(0, 2)$, $f(1) = -1$ is min, no max.
on $(-3, 2)$, $f(-1) = 3$ is max, no min.

32) on $(-3, 1)$, $f(-2) = -14$ is min, no max.
on $(-1, 3)$, $f(2) = -14$ is min, no max.

33) on $(-4, -2)$, no max or min.
on $(-1, 3)$, $f(0) = 0$ is min, no max.

34) on $(0, 2\pi)$, $f(5\pi/4) = -\sqrt{2}$ is min, $f(\pi/4) = \sqrt{2}$ is max.
on $(\pi/2, \pi)$, no max or min.

35) on $(0, 2)$, no max or min
on $(-3, 2)$, $f(0) = 1$ is max; no min

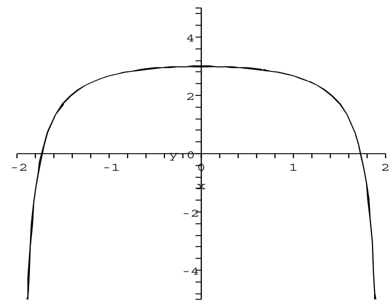
36) on $(-2, 0)$, no min or max.
on $(0, 4)$, $f(1/2) = e^{-2}/4$ is max, no min.

37) on $(-2, 2)$, $f(0) = 0$ is max; no min
on $(2, 8)$, no max or min

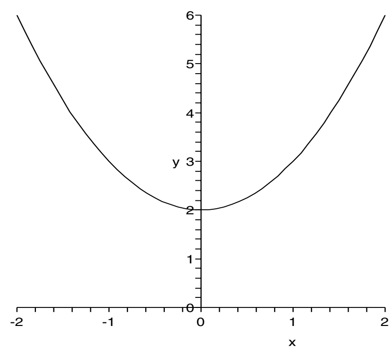
38) on $(0, 1)$, no min or max.
on $(-3, 4)$, no max, $f(0) = 0$ is min.

46. To find extrema in the open interval (a, b), or the half-open intervals $(a, b]$ or $[a, b)$, look at the graph to get an idea of where the extrema will be located, or if they exist. Evaluate the function at the critical points, and the included endpoints (if any). These are the only places extrema can exist.

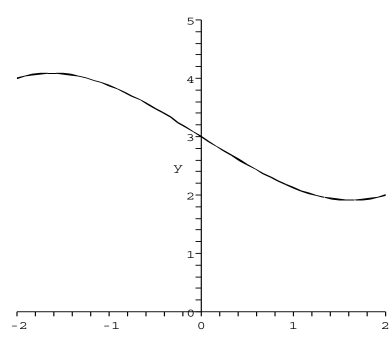
47. On $[-2, 2]$, the absolute maximum is 3 and the absolute minimum doesn't exist.



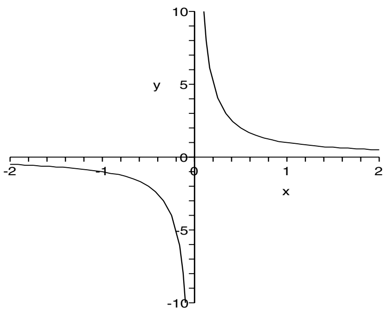
48. On $(-2, 2)$ minimum is 2 and the maximum does not exist. (The maximum would exist at the endpoints which are not included in the interval.)



49. On $(-2, 2)$ the absolute maximum is 4 and the absolute minimum is 2.



50. Absolute extrema do not exist because of the vertical asymptote.



51. You will not be able to construct an example with a continuous function, but there are many examples using a function with a discontinuity, for example $f(x) = \sec^2 x$.

52. $f(p) = p^m(1-p)^{n-m}$
$f'(p) = mp^{m-1}(1-p)^{n-m}$
$\quad - p^m(n-m)(1-p)^{n-m-1}$
To find the critical numbers, we set $f'(p) = 0$ which gives
$mp^{m-1}(1-p)^{n-m}$
$\quad - p^m(n-m)(1-p)^{n-m-1} = 0$
$mp^{m-1}(1-p)^{n-m}$
$\quad = p^m(n-m)(1-p)^{n-m-1}$
$m(1-p) = p(n-m)$
$m - mp = pn - pm$
$p = m/n.$
Since this is the only critical number, $f(p)$ is continuous, $f(0) = f(1) = 0$ and $f(m/n) > 0$, $p = m/n$ must maximize $f(p)$.

53. $f(x) = x^3 + cx + 1$
$f'(x) = 3x^2 + c$
We know (perhaps from a precalculus course) that for any cubic polynomial with positive leading coefficient, when x is large and positive the value of the polynomial is very large and positive, and when x is large and negative, the value of the polynomial is very large and negative.

Type 1: $c > 0$. There are no critical numbers. As you move from left to right, the graph of f is always rising.

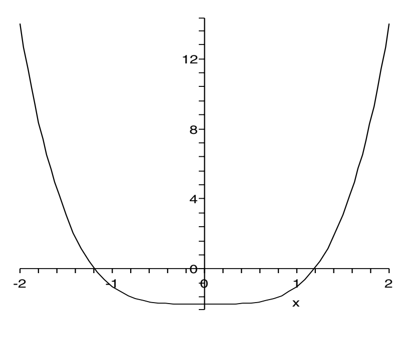
Type 2: $c < 0$ There are two critical numbers $x = \pm\sqrt{-c/3}$. As you move from left to right, the graph rises until we get to the first critical number, then the graph must fall until we get to the second critical number, and then the graph rises again. So the critical number on the left is a local maximum and the critical number on the right is a local minimum.

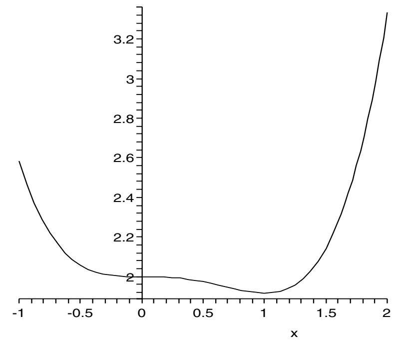
Type 3: $c = 0$. There is only one critical number, which is neither a local max nor a local min.

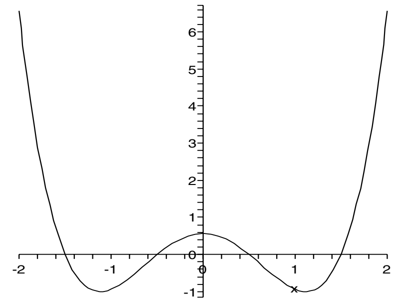
54. The derivative of a fourth-order polynomial is a cubic polynomial. We know that cubic polynomials must have one root, and can have up to three roots. If $p(x)$ is a fourth-order polynomial, we know that

$$\lim_{x\to\infty} p(x) = \lim_{x\to-\infty} p(x) = \infty$$

if the coefficient of x^4 is positive, and is $-\infty$ if the coefficient of x^4 is negative. This guarantees that at least one of the critical numbers will be an extremum.







55. $f(x) = x^3 + bx^2 + cx + d$
$f'(x) = 3x^2 + 2bx + c$
The quadratic formula says that the critical numbers are

$$x = \frac{-2b \pm \sqrt{4b^2 - 12c}}{6} = \frac{-b \pm \sqrt{b^2 - 3c}}{3}.$$

So if $c < 0$, the quantity under the square root is positive and there are two critical numbers. This is like the Type 2 cubics in Exercise 53. We know that as x goes to infinity, the polynomial $x^3 + bx^2 + cx + d$ gets very large and positive, and when x goes to minus infinity, the polynomial is very large but negative. Therefore, the critical number on the left must be a local max, and the critical number on the right must be a local min.

56. $f'(x) = 3x^2 + 2bx + c = 0$ when $x = \frac{-2b \pm \sqrt{4b^2 - 12c}}{6}$. Adding these values together yields $-2b/3$.

57. $f(x) = x^4 + cx^2 + 1$
$f'(x) = 4x^3 + 2cx = 2x(2x^2 + c)$
So $x = 0$ is always a critical number.

Case 1: $c \geq 0$. The only solution to $2x(2x^2 + c) = 0$ is $x = 0$, so $x = 0$ is the only critical number. This must be a minimum, since we know that the function $x^4 + cx^2 + 1$ is large and positive when $|x|$ is large (so the graph is roughly U-shaped). We could also note that $f(0) = 1$, and 1 is clearly the absolute minimum of this function if $c \geq 0$.

Case 2: $c < 0$. Then there are two other critical numbers $x = \pm\sqrt{-c/2}$. Now $f(0)$ is still equal to 1, but the value of f at both new critical numbers is less than 1. Hence $f(0)$ is a local max, and both new critical numbers are local minimums.

58. $f'(x) = 4x^3 + 3cx^2 = 0$ when $x = 0$ and $x = -3c/4$. Only $x = -3c/4$ will be an extreme point (an absolute minimum). $x = 0$ will be an inflection point.

59. With $t = 90$ and $r = 1/30$, we have

$$P(n) = \frac{3^n}{n!}e^{-3}.$$

We compute P for the first few values of n:

n	P
0	e^{-3}
1	$3e^{-3}$
2	$4.5e^{-3}$
3	$4.5e^{-3}$
4	$3.375e^{-3}$

Once $n > 3$, the values of P will decrease as n increases. This is due to the fact that to get $P(n+1)$ from $P(n)$, we multiply $P(n)$ by $3/(n+1)$. Since $n > 3$, $3/(n+1) < 1$ and so $P(n+1) < P(n)$. Thus we see from the table that P is maximized at $n = 3$ (it is also maximized at $n = 2$). It makes sense that P would be maximized at $n = 3$ because

$$(90 \text{ mins})\left(\frac{1}{30}\text{ goals/min}\right) = 3 \text{ goals}.$$

60. With $r = 1/30$ and $n = 1$ we have

$$P(t) = \frac{t}{30}e^{-t/30}.$$

To maximize $P(t)$, we take the derivative and set it equal to 0:

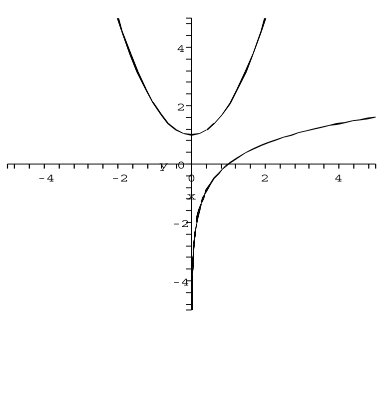
$$\begin{aligned} P'(t) &= \frac{1}{30}e^{-t/30} + \frac{t}{30}e^{-t/30}\frac{-1}{30} \\ &= e^{-t/30}\left(\frac{1}{30} - \frac{t}{900}\right) \end{aligned}$$

Since $e^{-t/30}$ is never 0, we see that $P'(t) = 0$ only when $t = 30$. This makes sense since

$$(30 \text{ mins})\left(\frac{1}{30}\text{ goals/min}\right) = 1 \text{ goal}.$$

61. Since f is differentiable on (a, b), it is continuous on the same interval. Since f is decreasing at a and increasing at b, f must have a local minimum for some value c, where $a < c < b$. By Fermat's theorem, c is a critical number for f. Since f is differentiable at c, $f'(c)$ exists, and therefore $f'(c) = 0$.

62. Graph of $f(x) = x^2 + 1$ and $g(x) = \ln x$:

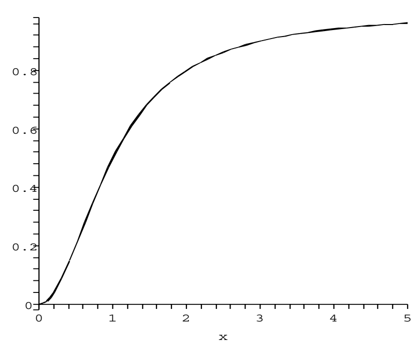


$h(x) = f(x) - g(x) = x^2 + 1 - \ln x$
$h'(x) = 2x - 1/x = 0$
$2x^2 = 1$
$x = \pm\sqrt{1/2}$

$x = \sqrt{1/2}$ is min
$f'(x) = 2x$
$g'(x) = 1/x$
$f'\left(\sqrt{1/2}\right) = 2\sqrt{1/2} = \sqrt{2}$
$g'\left(\sqrt{1/2}\right) = \frac{1}{\sqrt{1/2}} = \sqrt{2}$

So the tangents are parallel. If the tangent lines were not parallel, then they would be getting closer together in one direction. Since the tangent lines approximate the curves, this should mean the curves are also getting closer together in that direction.

63. Graph of $f(x) = \dfrac{x^2}{x^2+1}$:



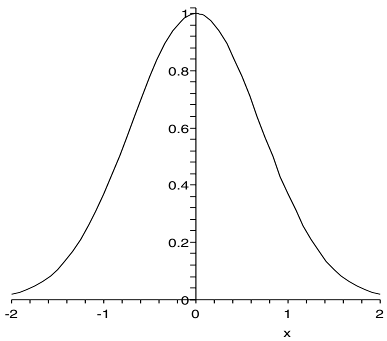
$$\begin{aligned} f'(x) &= \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} \\ &= \frac{2x}{(x^2+1)^2} \\ f''(x) &= \frac{2(x^2+1)^2 - 2x \cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} \\ &= \frac{2(x^2+1)\left[(x^2+1) - 4x^2\right]}{(x^2+1)^4} \\ &= \frac{2\left[1 - 3x^2\right]}{(x^2+1)^3} \end{aligned}$$

$f''(x) = 0$ for $x = \pm\dfrac{1}{\sqrt{3}}$,

$x = -\dfrac{1}{\sqrt{3}} \notin (0, \infty)$

$x = \dfrac{1}{\sqrt{3}}$ is steepest point.

64. Graph of $f(x) = e^{-x^2}$:



$f(x)$ is steepest where $f'(x) = -2xe^{-x^2}$ is maximum.
$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 0$ when $x = \pm\sqrt{2}/2$. This is where $f(x)$ is steepest.

65. $y = x^5 - 4x^3 - x + 10$, $x \in [-2, 2]$
$y' = 5x^4 - 12x^2 - 1$
$x = -1.575$, 1.575 are critical numbers of y. There is a local max at $x = -1.575$, local min at $x = 1.575$.
$x = -1.575$ represents the top and $x = 1.575$ represents the bottom of the roller coaster.
$y''(x) = 20x^3 - 24x = 4x(5x^2 - 6) = 0$
$x = 0$, $\pm\sqrt{6/5}$ are critical numbers of y'. We calculate y' at the critical numbers and at the endpoints $x = \pm 2$:
$y'(0) = -1$
$y'\left(\pm\sqrt{6/5}\right) = -41/5$
$y'(\pm 2) = 31$
So the points where the roller coaster is making the steepest descent are $x = \pm\sqrt{6/5}$, but the steepest part of the roller coast is during the ascents at ± 2.

66. To maximize entropy, we find the critical numbers of H.
$H'(x) = -\ln x - 1 + \ln(1-x) + 1 = 0$
where $\ln x = \ln(1-x)$, or where $x = 1 - x$. That is $x = 1/2$. This maximizes unpredictablility since for this value, errors and non-errors are equally likely.

67. $W(t) = a \cdot e^{-be^{-t}}$
as $t \to \infty$, $-be^{-t} \to 0$, so $W(t) \to a$.
$W'(t) = a \cdot e^{-be^{-t}} \cdot be^{-t}$
as $t \to \infty, be^{-t} \to 0$, so $W'(t) \to 0$.

$$\begin{aligned} W''(t) &= (a \cdot e^{-be^{-t}} \cdot be^{-t}) \cdot be^{-t} \\ &\quad + (a \cdot e^{-be^{-t}}) \cdot (-be^{-t}) \\ &= a \cdot e^{-be^{-t}} \cdot be^{-t}\left[be^{-t} - 1\right] \end{aligned}$$

$W''(t) = 0$ when $be^{-t} = 1$
$e^{-t} = b^{-1}$
$-t = \ln b^{-1}$
$t = \ln b$

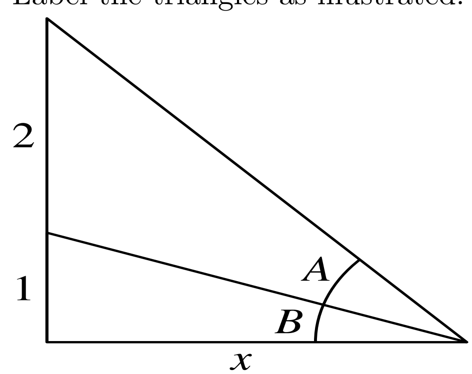
$$\begin{aligned} W'(\ln b) &= a \cdot e^{-be^{-\ln b}} \cdot be^{-\ln b} \\ &= a \cdot e^{-b\left(\frac{1}{b}\right)} \cdot b \cdot \tfrac{1}{b} = ae^{-1} \end{aligned}$$

Maximum growth rate is ae^{-1} when $t = \ln b$.

68. $R'([S]) = \dfrac{(K_m + [S])R_m - [S]R_m}{(K_m + [S])^2} \neq 0.$

The function doesn't have a true maximum, but $\lim\limits_{[S]\to\infty} R = R_m$. The rate of reaction approaches R_m but never reaches it.

69. Label the triangles as illustrated.



$\tan(A + B) = 3/x$
$A + B = \tan^{-1}(3/x)$
$\tan B = 1/x$
$B = \tan^{-1}(1/x)$

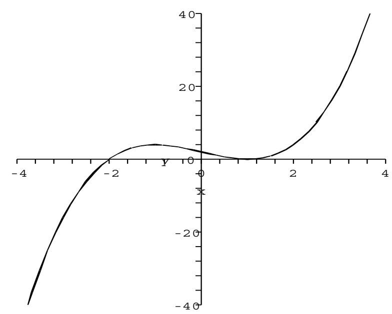
Therefore,
$A = (A + B) - B$
$A = \tan^{-1}(3/x) - \tan^{-1}(1/x)$
$$\frac{dA}{dx} = \frac{-3/x^2}{1 + (3/x)^2} - \frac{-1/x^2}{1 + (1/x)^2}$$
$$= \frac{1}{x^2+1} - \frac{3}{x^2+9}$$
The maximum viewing angle will occur at a critical value.
$$\frac{dA}{dx} = 0$$
$$\frac{1}{x^2+1} = \frac{3}{x^2+9}$$
$x^2 + 9 = 3x^2 + 3$
$2x^2 = 6$
$x^2 = 3$
$x = \sqrt{3}$ ft ≈ 1.73 ft
This is a maximum because when x is large and when x is a little bigger than 0, the angle is small.

70. If the person's eyes are at 6 feet, then the angle A is given by $A = \tan^{-1}(2/x)$.
$$A'(x) = \frac{-2}{x^2} \cdot \frac{1}{1 + (2/x)^2} \neq 0$$
The angle approaches a maximum of $\pi/2$ as x approaches 0.

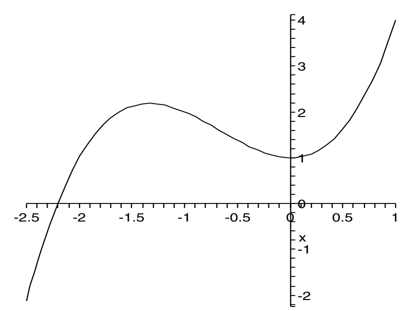
3.4 Increasing and Decreasing Functions

1. $y = x^3 - 3x + 2$
$y' = 3x^2 - 3 = 3(x^2 - 1)$
$= 3(x+1)(x-1)$

$x = \pm 1$ are critical numbers.
$(x+1) > 0$ on $(-1, \infty)$, $(x+1) < 0$ on $(-\infty, -1)$
$(x-1) > 0$ on $(1, \infty)$, $(x-1) < 0$ on $(-\infty, -1)$
$3(x+1)(x-1) > 0$ on $(1, \infty) \cup (-\infty, -1)$ so y is increasing on $(1, \infty)$ and on $(-\infty, -1)$
$3(x+1)(x-1) < 0$ on $(-1, 1)$, so y is decreasing on $(-1, 1)$.



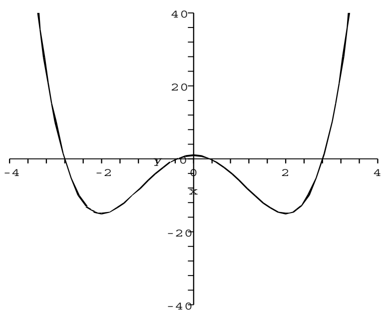
2. $y = x^3 + 2x^2 + 1$
$y' = 3x^2 + 4x = x(3x + 4)$
The function is increasing when $x < -\frac{4}{3}$, decreasing when $-\frac{4}{3} < x < 0$, and increasing when $x > 0$.



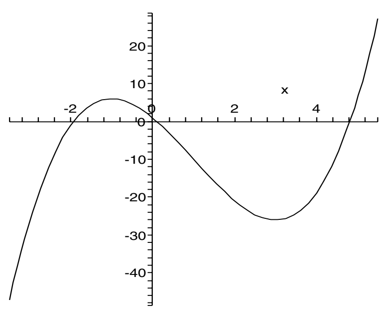
3. $y = x^4 - 8x^2 + 1$
$y' = 4x^3 - 16x = 4x(x^2 - 4)$
$= 4x(x-2)(x+2)$
$x = 0, 2, -2$

$4x > 0$ on $(0, \infty)$, $4x < 0$ on $(-\infty, 0)$
$(x-2) > 0$ on $(2, \infty)$, $(x-2) < 0$ on $(-\infty, 2)$
$(x+2) > 0$ on $(-2, \infty)$, $(x+2) < 0$ on $(-\infty, -2)$
$4(x-2)(x+2) > 0$ on $(-2, 0) \cup (2, \infty)$, so the function is increasing on $(-2, 0)$ and on $(2, \infty)$.
$4(x-2)(x+2) < 0$ on $(-\infty, -2) \cup (0, 2)$, so y is decreasing on $(-\infty, -2)$

and on $(0, 2)$.

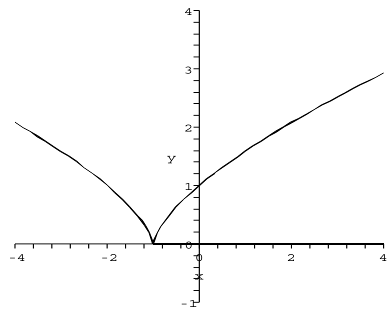


4. $y = x^3 - 3x^2 - 9x + 1$
$y' = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$.
The function is increasing when $x < -1$, decreasing when $-1 < x < 3$, and increasing when $x > 3$.

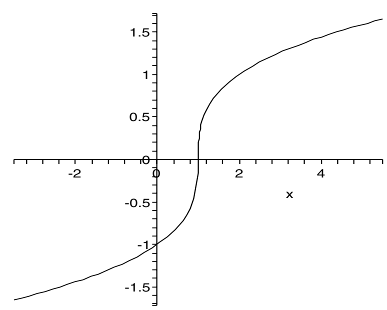


5. $y = (x + 1)^{2/3}$
$$y' = \frac{2}{3}(x + 1)^{-1/3} = \frac{2}{3\sqrt[3]{x + 1}}$$

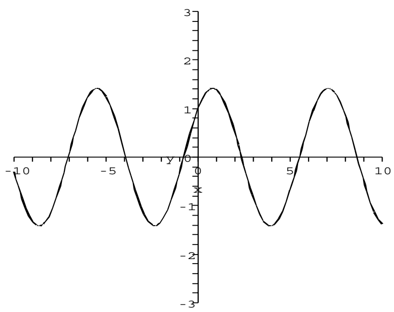
y' is not defined for $x = -1$
$\dfrac{2}{3\sqrt[3]{x+1}} > 0$ on $(-1, \infty)$, y is increasing
$\dfrac{2}{3\sqrt[3]{x+1}} < 0$ on $(-\infty, -1)$, y is decreasing



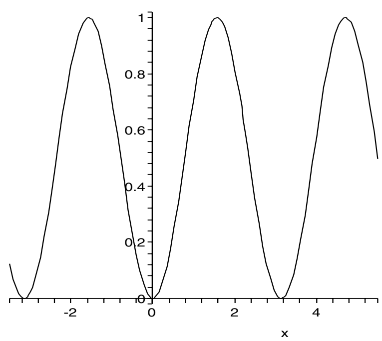
6. $y = (x - 1)^{1/3}$
$y' = \frac{1}{3}(x - 1)^{-2/3}$.
The function is increasing for all x. The slope approaches vertical as x approaches 1.



7. $y = \sin x + \cos x$
$y' = \cos x - \sin x = 0$
$\cos x = \sin x$
$x = \pi/4,\ 5\pi/4,\ 9\pi/4$, etc. $\cos x - \sin x > 0$ on $(-3\pi/4, \pi/4) \cup (5\pi/4, 9\pi/4) \cup \ldots$
$\cos x - \sin x < 0$ on $(\pi/4, 5\pi/4) \cup (9\pi/4, 13\pi/4) \cup \ldots$
So $y = \sin x + \cos x$ is decreasing on $(\pi/4, 5\pi/4), (9\pi/4, 13\pi/4)$, etc., and is increasing on $(-3\pi/4, \pi/4), (5\pi/4, 9\pi/4)$, etc.

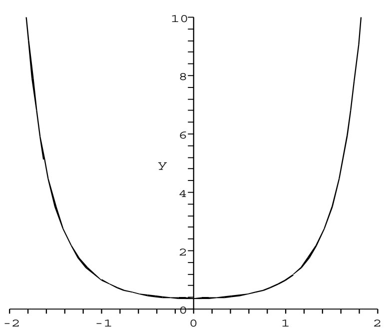


8. $y = \sin^2 x$
$y' = 2 \sin x \cos x.$
The function is increasing for $0 < x < \frac{\pi}{2}$, and decreasing for $\frac{\pi}{2} < x < \pi$, and this pattern repeats with period π.

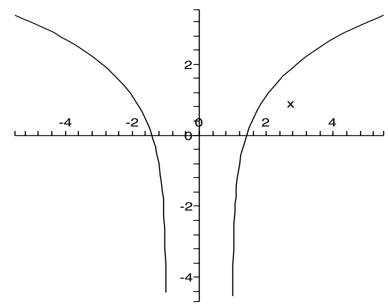


9. $y = e^{x^2-1}$
$y' = e^{x^2-1} \cdot 2x = 2xe^{x^2-1}$
$x = 0$

$2xe^{x^2-1} > 0$ on $(0, \infty)$, y is increasing
$2xe^{x^2-1} < 0$ on $(-\infty, 0)$, y is decreasing



10. $y = \ln(x^2 - 1)$
$y' = \frac{2x}{x^2-1}.$
The function is defined for $|x| > 1$. The function is decreasing for $x < -1$ and increasing for $x > 1$.



11. $y = x^4 + 4x^3 - 2$
$y' = 4x^3 + 12x^2 = 4x^2(x + 3)$
Critical numbers are $x = 0$, $x = -3$.
$4x^2(x + 3) > 0$ on $(-3, 0) \cup (0, \infty)$
$4x^2(x + 3) < 0$ on $(-\infty, -3)$
Hence $x = -3$ is a local minimum and $x = 0$ is not an extremum.

12. $y = x^5 - 5x^2 + 1$
$y' = 5x^4 - 10x = 5x(x^3 - 2).$
At $x = 0$ the slope changes from positive to negative indicating a local maximum. At $x = \sqrt[3]{2}$ the slope changes from negative to positive indicating a local minimum.

13. $y = xe^{-2x}$
$y' = 1 \cdot e^{-2x} + x \cdot e^{-2x}(-2)$
$\quad = e^{-2x} - 2xe^{-2x}$
$\quad = e^{-2x}(1 - 2x)$
$x = \frac{1}{2}$
$e^{-2x}(1 - 2x) > 0$ on $(-\infty, 1/2)$
$e^{-2x}(1 - 2x) < 0$ on $(1/2, \infty)$
So $y = xe^{-2x}$ has a local maximum at $x = 1/2$.

14. $y = x^2e^{-x}$
$y' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x).$
At $x = 0$ the slope changes from negative to positive indicating a local minimum. At $x = 2$ the slope changes

from positive to negative indicating a local maximum.

15. $y = \tan^{-1}(x^2)$
$y' = \dfrac{2x}{1+x^4}$
Critical number is $x = 0$.
$\dfrac{2x}{1+x^4} > 0$ for $x > 0$
$\dfrac{2x}{1+x^4} < 0$ for $x < 0$.
Hence $x = 0$ is a local minimum.

16. $y = \sin^{-1}\left(1 - \dfrac{1}{x^2}\right)$
$y' = \dfrac{2}{x^3} \cdot \dfrac{1}{\sqrt{1-(1-\frac{1}{x^2})^2}}.$
The derivative is never 0 and is defined where the function is defined, so there are no critical points.

17. $y = \dfrac{x}{1+x^3}$
Note that the function is not defined for $x = -1$.
$$y' = \frac{1(1+x^3) - x(3x^2)}{(1+x^3)} = \frac{1+x^3-3x^3}{(1+x^3)^2} = \frac{1-2x^3}{(1+x^3)^2}$$
Critical number is $x = \sqrt[3]{1/2}$
$y' > 0$ on $(-\infty, -1) \cup (-1, -\sqrt[3]{1/2})$
$y' < 0$ on $(\sqrt[3]{1/2}, \infty)$
Hence $x = \sqrt[3]{1/2}$ is a local max.

18. $y = \dfrac{x}{1+x^4}$
$y' = \dfrac{(1+x^4) - 4x^4}{(1+x^4)^2} = \dfrac{1-3x^4}{(1+x^4)^2}.$
At $x = -\sqrt[4]{1/3}$ the slope changes from negative to positive indicating a local minimum. At $x = \sqrt[4]{1/3}$ the slope changes from positive to negative incicating a local maximum.

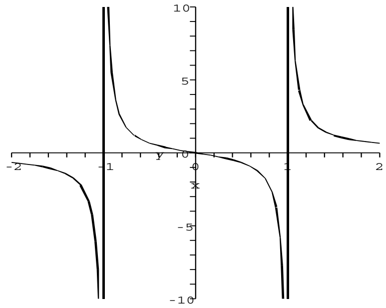
19. $y = \sqrt{x^3+3x^2} = (x^3+3x^2)^{1/2}$
Domain is all $x \geq -3$.
$$y' = \frac{1}{2}(x^3+3x^2)^{-1/2}(3x^2+6x) = \frac{3x^2+6x}{2\sqrt{x^3+3x^2}} = \frac{3x(x+2)}{2\sqrt{x^3+3x^2}}$$
$x = 0, -2, -3$ are critical numbers.
y' undefined at $x = 0, -3$
$y' > 0$ on $(-3,-2) \cup (0, \infty)$
$y' < 0$ on $(-2, 0)$
So $y = \sqrt{x^3+3x^2}$ has local max at $x = -2$, local min at $x = 0$. $x = -3$ is an endpoint, and so is not a local extremum.

20. $y = x^{4/3} + 4x^{1/3}$
$y' = \dfrac{4}{3}x^{1/3} + \dfrac{4}{3x^{2/3}} = \dfrac{4}{3} \cdot \dfrac{x+1}{x^{2/3}}.$
At $x = -1$ the slope changes from negative to positive indicating a local minimum. At $x = 0$ the slope is vertical and is positive on both sides, so this is neither a minimum nor a maximum.

21. $y = \dfrac{x}{x^2-1}$
$$y' = \frac{x^2-1-x(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$
There are no values of x for which $y' = 0$. There are no critical points, because the values for which y' does not exist (that is, $x = \pm 1$) are not in the domain.

There are vertical asymptotes at $x = \pm 1$, and a horizontal asymptote at $y = 0$. This can be verified by calculating the following limits:
$$\lim_{x\to\pm\infty} \frac{x}{x^2-1} = 0$$
$$\lim_{x\to -1} \frac{x}{x^2-1} = \infty$$

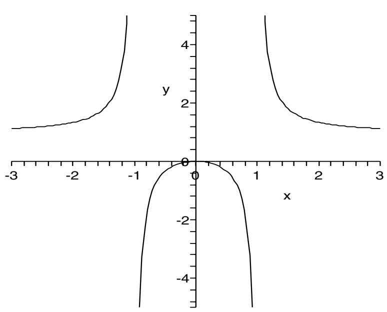
$$\lim_{x\to 1}\frac{x}{x^2-1}=-\infty$$



22. $y=\dfrac{x^2}{x^2-1}$ has vertical asymptotes at $x=\pm 1$ and horizontal asymptote $y=1$.

$$y'=\frac{(x^2-1)2x-2x(x^2)}{(x^2-1)^2}=\frac{-2x}{(x^2-1)^2}.$$

At $x=0$ the slope changes from positive to negative indicating a local maximum.

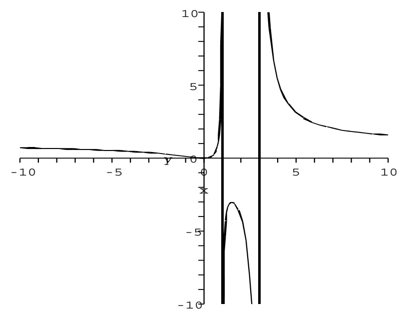


23. $y=\dfrac{x^2}{x^2-4x+3}=\dfrac{x^2}{(x-1)(x-3)}$

Vertical asymptotes $x=1$, $x=3$. When $|x|$ is large, the function approaches the value 1, so $y=1$ is a horizontal asymptote.

$$\begin{aligned} y' &= \frac{2x(x^2-4x+3)-x^2(2x-4)}{(x^2-4x+3)^2} \\ &= \frac{2x^3-8x^2+6x-2x^3+4x^2}{(x^2-4x+3)^2} \\ &= \frac{-4x^2+6x}{(x^2-4x+3)^2} \\ &= \frac{2x(-2x+3)}{(x^2-4x+3)^2} \\ &= \frac{2x(-2x+3)}{[(x-3)(x-1)]^2} \end{aligned}$$

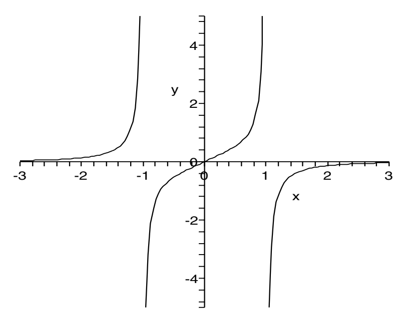
Critical numbers are $x=0$ (local min) and $x=3/2$ (local max).



24. $y=\dfrac{x}{1-x^4}$ has vertical asymptotes at $x=\pm 1$ and horizontal asymptote $y=0$.

$$y'=\frac{(1-x^4)+4x^4}{(1-x^4)^2}=\frac{1+3x^4}{(1-x^4)^2}\neq 0$$

for any x and is defined where the function is defined.



25. $y=\dfrac{x}{\sqrt{x^2+1}}$

$$\begin{aligned} y' &= \frac{\sqrt{x^2+1}-x^2/\sqrt{x^2+1}}{x^2+1} \\ &= \frac{1}{(x^2+1)^{3/2}} \end{aligned}$$

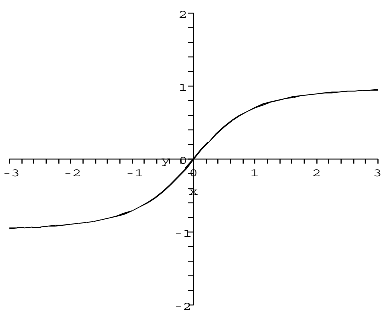
The derivative is never zero, so there are no critical points. To verify that there are horizontal asymptotes at $y=\pm 1$:

$$
\begin{aligned}
y &= \frac{x}{\sqrt{x^2+1}} \\
&= \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} \\
&= \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}
\end{aligned}
$$

Thus,

$$\lim_{x\to\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = 1$$

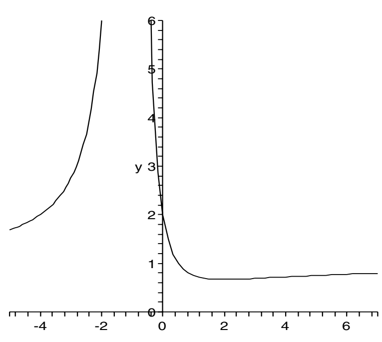
$$\lim_{x\to-\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = -1$$



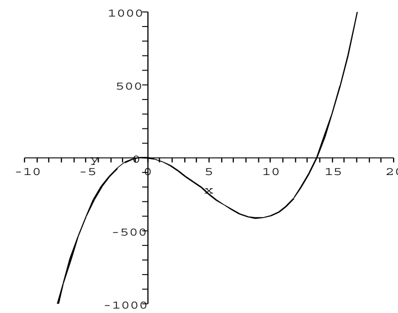
26. $y = \dfrac{x^2+2}{(x+1)^2}$ has a vertical asymptote at $x = -1$, and a horizontal asymptote at $y = 1$.

$$
\begin{aligned}
y' &= \frac{2x(x+1)^2 - (x^2+2)2(x+1)}{(x+1)^4} \\
&= \frac{2(x-2)(x+1)}{(x+1)^4}
\end{aligned}
$$

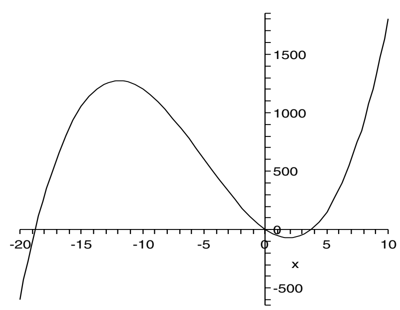
$x = 2$ is the only critical number. Since $f'(0) < 0$ and $f'(3) > 0$, we see that $f(2)$ is a local minimum.



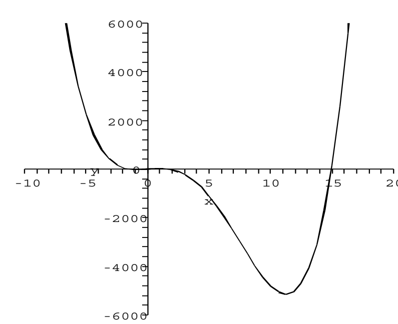
27. $y' = 3x^2 - 26x - 10 = 0$ when $x = \dfrac{26 \pm \sqrt{796}}{6}$. Local max at $x = -0.3689$; local min at $x = 9.0356$.



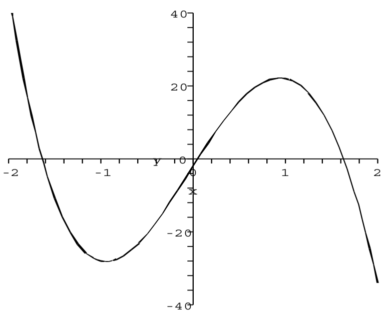
28. $y' = 3x^2 + 30x - 70 = 0$ when $x = \dfrac{-15 \pm \sqrt{435}}{3}$. At $x \approx -11.9522$ the slope changes from positive to negative indicating a local maximum, and at $x \approx 1.9522$ the slope changes from negative to positive indicating a local minimum.



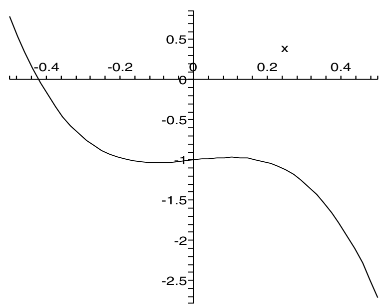
29. $y' = 4x^3 - 45x^2 - 4x + 40$
Local minima at $x = -0.9474, 11.2599$; local max at 0.9374.
Local behavior near $x = 0$ looks like



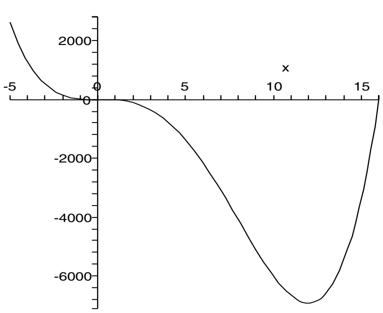
Global behavior of the function looks like



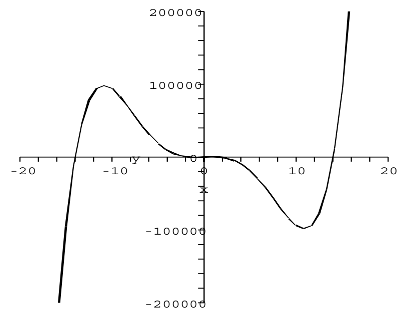
30. $y' = 4x^3 - 48x^2 - 0.2x + 0.5 = 0$ at approximately $x = -0.1037$ (local minimum), $x = 0.1004$ (local maximum), and $x = 12.003$ (local minimum). Local behavior near $x = 0$ looks like



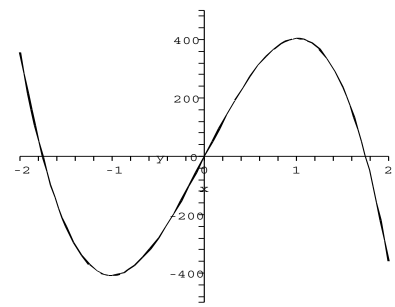
Global behavior of the function looks like



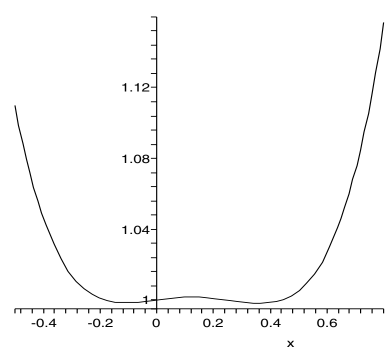
31. $y' = 5x^4 - 600x + 605$
Local minima at $x = -1.0084, 10.9079$; local maxima at $x = -10.9079, 1.0084$. Local behavior near $x = 0$ looks like



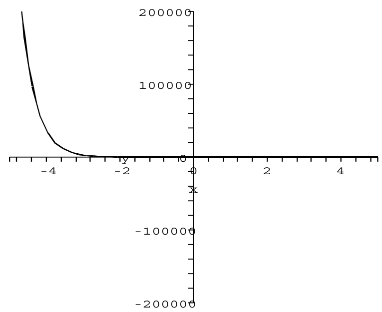
Global behavior of the function looks like



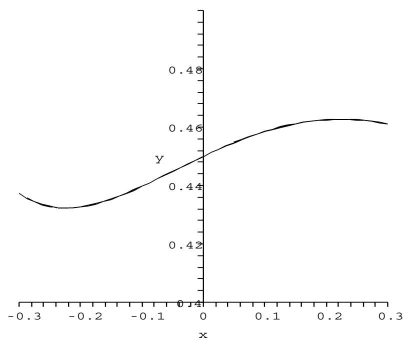
32. $y' = 4x^3 - 1.5x^2 - 0.04x + 0.02 = 0$ at approximately $x = -0.1121$ (local minimum), $x = 0.1223$ (local maximum), and $x = 0.3648$ (local minimum).



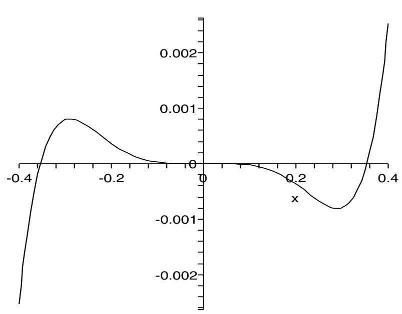
33. $y' = (2x + 1)e^{-2x}$
$\qquad + (x^2 + x + 0.45)(-2)e^{-2x}$
Local min at $x = -0.2236$; local max at $x = 0.2236$. Local behavior near $x = 0$ looks like



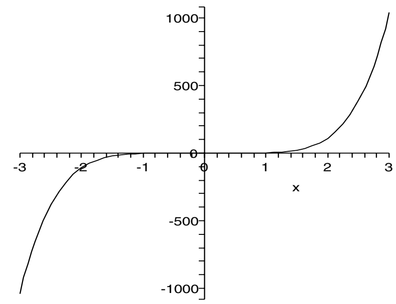
Global behavior of the function looks like



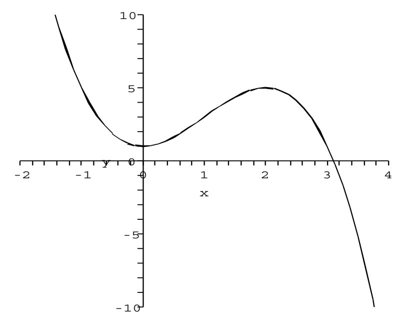
34. $y' = 5x^4 \ln(8x^2) + x^5 \frac{16x}{8x^2}$
$= x^4(5\ln(8x^2) + 2) = 0$ at approximately $x = \pm 0.2895$ (a local maximum and local minimum). The derivative and the function are undefined at $x = 0$, but the slope is negative on both sides (neither a minimum nor a maximum). Locally, near $x = \pm 0.2895$, the function looks like



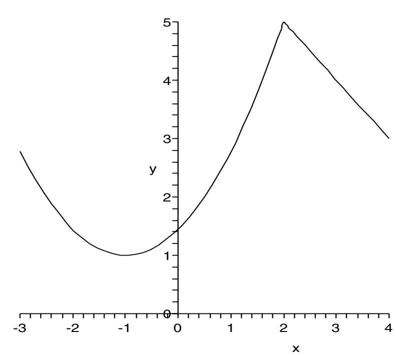
Globally, the function looks like a quintic



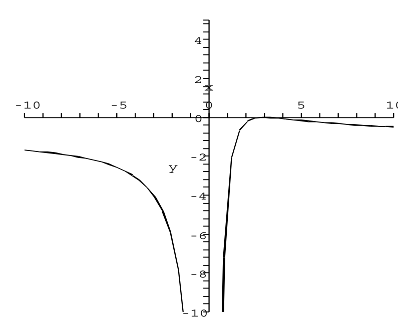
35. One possible graph:



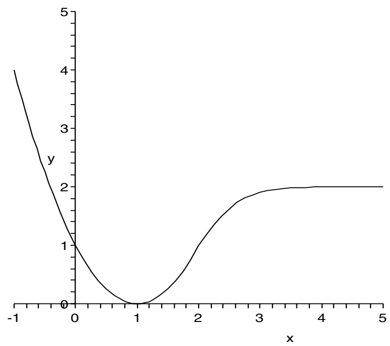
36. One possible graph:

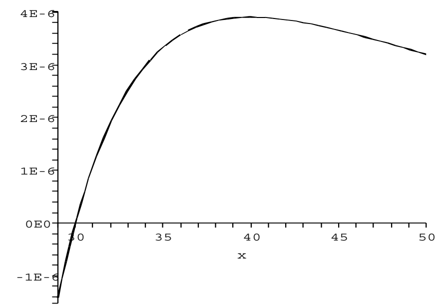


37. One possible graph:



38. One possible graph:

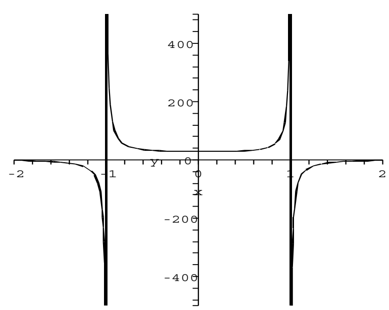




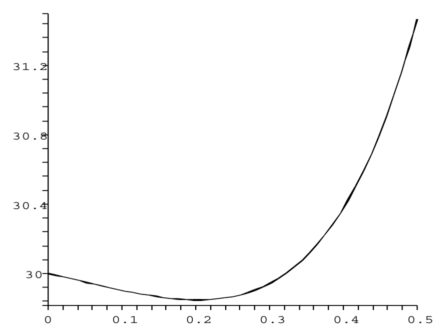
39. The derivative is

$$y' = \frac{-3x^4 + 120x^3 - 1}{(x^4 - 1)^2}.$$

We estimate the critical numbers to be approximately 0.2031 and 39.999.
The following graph shows global behavior:



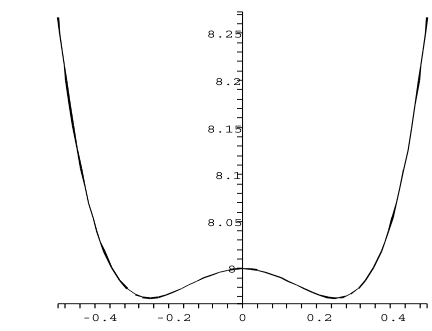
The following graphs show local behavior:



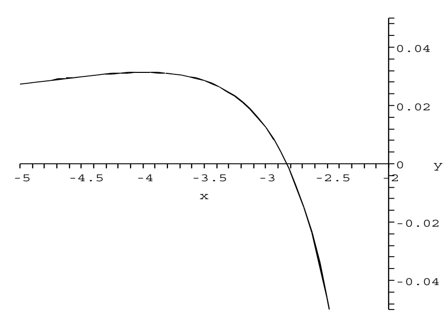
40. The derivative is

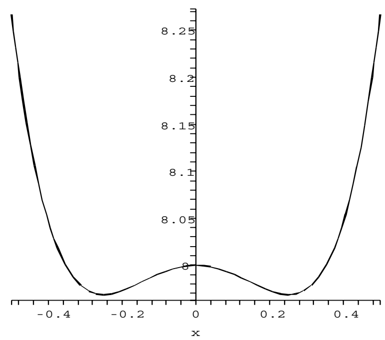
$$y' = \frac{-2x^5 + 32x^3 - 2x}{(x^4 - 1)^2}.$$

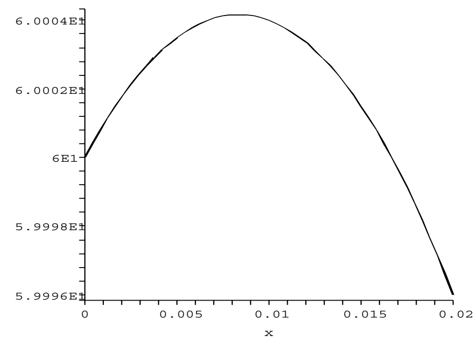
We estimate the critical numbers to be approximately ± 0.251, ± 3.992 and $x = 0$.
The following graph shows global behavior:



The following graphs show local behavior:



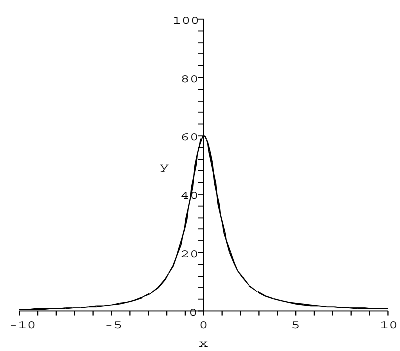




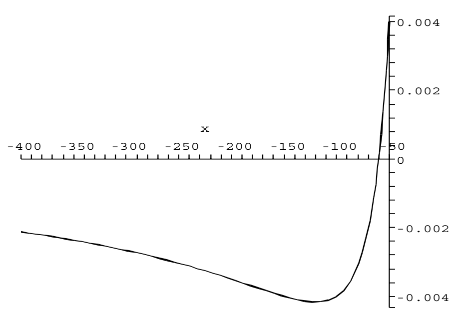
41. The derivative is

$$y' = \frac{-x^2 - 120x + 1}{(x^2+1)^2}.$$

We estimate the critical numbers to be approximately 0.008 and -120.008.
The following graph shows global behavior:



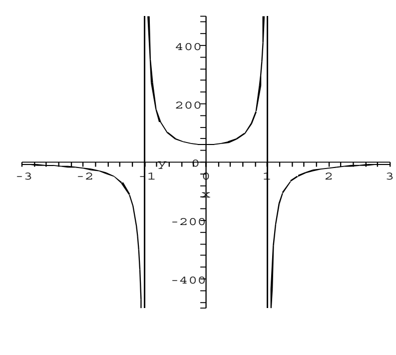
The following graphs show local behavior:



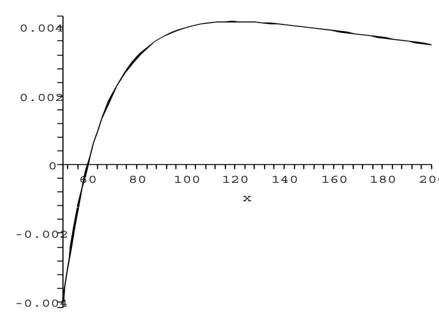
42. The derivative is

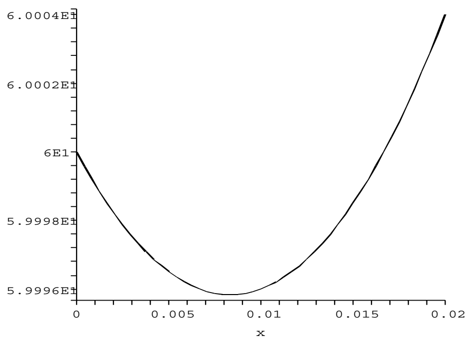
$$y' = \frac{-x^2 + 120x - 1}{(x^2-1)^2}.$$

We estimate the critical numbers to be approximately 0.008 and 119.992.
The following graph shows global behavior:



The following graphs show local behavior:





43. $f'(0) = \lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$
$= \lim_{x\to 0} \dfrac{f(x)}{x}$
$= \lim_{x\to 0} \left[1 + 2x\sin\left(\dfrac{1}{x}\right)\right] = 1$

For $x \neq 0$,
$f'(x) =$
$1+2\left[2x\sin\left(\dfrac{1}{x}\right) + x^2\left(\dfrac{-1}{x^2}\right)\cos\left(\dfrac{1}{x}\right)\right]$
$= 1 + 4x\sin\left(\dfrac{1}{x}\right) - 2\cos\left(\dfrac{1}{x}\right)$

For values of x close to the origin, the middle term of the derivative is small, and since the last term $-2\cos(1/x)$ reaches its minimum value of -2 in every neighborhood of the origin, f' has negative values on every neighborhood of the origin. Thus, f is not increasing on any neighborhood of the origin.

This conclusion does not contradict Theorem 4.1 because the theorem states that if a function's derivative is positive for all values in an interval, then it is increasing in that interval. In this example, the derivative is not positive throughout any interval containing the origin.

44. We have $f'(x) = 3x^2$, so $f'(x) > 0$ for all $x \neq 0$, but $f'(0) = 0$. Since $f'(x) > 0$ for all $x \neq 0$, we know $f(x)$ is increasing on any interval not containing 0. We know that if $x_1 < 0$ then $x_1^3 < 0$ and if $x_2 > 0$ then $x_2^3 > 0$. If $x_1 < 0$ and $x_2 = 0$ then $x_1^3 < 0^3 = 0$, so $f(x)$ is increasing on intervals of the form $(x_1, 0)$. Similarly, $f(x)$ is increasing on intervals of the form $(0, x_2)$. Finally, on intervals of the form (x_1, x_2) where $x_1 < 0 < x_2$, we have $x_1^3 < 0 < x_2^3$ so $f(x)$ is again increasing on these intervals. Thus $f(x)$ is increasing on any interval.

This does not contradict Theorem 4.1 because Theorem 4.1 is not an "if and only if" statement. It says that if $f'(x) > 0$, then f is increasing (on that interval) but it does not say that if $f'(x)$ is not strictly positive that f is not increasing.

45. f is continuous on $[a, b]$, and $c \in (a, b)$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, by Theorem 3.1, f is increasing on (a, c) and decreasing on (c, b), so $f(c) > f(x)$ for all $x \in (a, c)$ and $x \in (c, b)$. Thus $f(c)$ is a local max.

(ii) If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, by Theorem 3.1, f is decreasing on (a, c) and increasing on (c, b). So $f(c) < f(x)$ for all $x \in (a, c)$ and $x \in (c, b)$. Thus $f(c)$ is a local min.

(iii) If $f'(x) > 0$ on (a, c) and (c, b), then $f(c) > f(x)$ for all $x \in (a, c)$ and $f(c) < f(x)$ for all $x \in (c, b)$, so c is not a local extremum.

If $f'(x) < 0$ on (a, c) and (c, b), then $f(c) < f(x)$ for all $x \in$

(a, c) and $f(c) > f(x)$ for all $x \in (c, b)$, so c is not a local extremum.

46. If $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$. Graphically, this makes sense: f and g start at the same place, but f is increasing faster, therefore f should be larger than g for all $x > a$.

To prove this, apply the Mean Value Theorem to the function $f(x) - g(x)$. If $x > a$ then there exists a number c between a and x with
$f'(c) - g'(c)$
$= \dfrac{(f(x) - g(x)) - (f(a) - g(a))}{x - a}$.
Multiply by $(x-a)$ (and recall $f(a) = g(a)$) to get $(x - a)(f'(c) - g'(c)) = f(x) - g(x)$. The lefthand side of this equation is positive, therefore $f(x)$ is greater than $g(x)$.

47. Let $f(x) = 2\sqrt{x}$, $g(x) = 3 - 1/x$.
Then $f(1) = 2\sqrt{1} = 2$, and $g(1) = 3 - 1 = 2$, so $f(1) = g(1)$.
$f'(x) = \dfrac{1}{\sqrt{x}}$
$g'(x) = \dfrac{1}{x^2}$
So $f'(x) > g'(x)$ for all $x > 1$, and
$f(x) = 2\sqrt{x} > 3 - \dfrac{1}{x} = g(x)$
for all $x > 1$.

48. Let $f(x) = x$ and $g(x) = \sin x$.
Then $f(0) = g(0)$. $f'(x) = 1$. $g'(x) = \cos x$.
$\cos x \le 1$ for all x, therefore exercise 46 implies that $x > \sin x$ for all $x > 0$.

49. Let $f(x) = e^x$, $g(x) = x + 1$.
Then $f(0) = e^0 = 1$, $g(0) = 0+1 = 1$, so $f(0) = g(0)$.
$f'(x) = e^x$, $g'(x) = 1$
So $f'(x) > g'(x)$ for $x > 0$.
Thus $f(x) = e^x > x + 1 = g(x)$ for $x > 0$.

50. Let $f(x) = x - 1$ and $g(x) = \ln x$.
Then $f(1) = g(1)$. $f'(x) = 1$. $g'(x) = \frac{1}{x}$.
$1/x \le 1$ for all $x > 1$, therefore exercise 46 implies that $x - 1 > \ln x$ for all $x > 1$.

51. Let $f(x) = 3 + e^{-x}$; then $f(0) = 4$, $f'(x) = -e^{-x} < 0$, so f is decreasing. But $f(x) = 3 + e^{-x} = 0$ has no solution.

52. Let y_1 and y_2 be two points in the domain of f^{-1} with $y_1 < y_2$. Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. We want to show $x_1 < x_2$. Suppose not. Then $x_2 \le x_1$. But then, since f is increasing, $f(x_2) \le f(x_1)$. That is $y_2 \le y_1$, which contradicts our choice of y_1 and y_2.

53. The domain of $\sin^{-1} x$ is the interval $[-1, 1]$. The function is increasing on the entire domain.

54. $\sin^{-1}\left(\dfrac{2}{\pi}\tan^{-1} x\right)$ is defined for all x.
The derivative,

$$\frac{2}{\pi(1 + x^2)\sqrt{1 - (\frac{2}{\pi}\tan^{-1} x)^2}} > 0$$

for all x. The function is increasing everywhere.

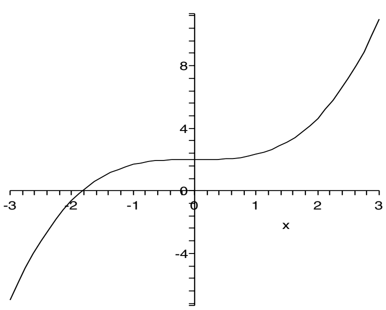
55. TRUE. If $x_1 < x_2$, then $g(x_1) < g(x_2)$ since g is increasing, and then $f(g(x_1)) < f(g(x_2))$ since f is increasing.

56. We can say that $g(1) < g(4)$ and $g(f(1)) < g(f(4))$, but it is not possible to determine the maximum and minimum values without more information.

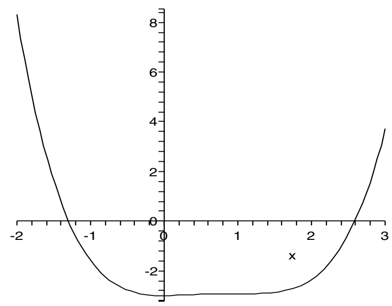
57. $s(t) = \sqrt{t+4} = (t+4)^{1/2}$
$s'(t) = \dfrac{1}{2}(t+4)^{-1/2} = \dfrac{1}{2\sqrt{t+4}} > 0$
So total sales are always increasing at the rate of $\dfrac{1}{2\sqrt{t+4}}$ thousand dollars per month.

58. $s'(t) = \dfrac{1}{2\sqrt{t+4}} > 0$ for all $t > 0$. If s represents the total sales so far, then s cannot decrease. The rate of new sales can decrease, but we cannot lose sales that already have occurred.

59. If the roots of the derivative are very close together, then the extrema will be very close together and difficult to see on a graph showing global behavior of the function. One function with the given derivative is
$f(x) = \dfrac{1}{3}x^3 - 0.01x + 2$



The two extreme points near $x = 0$ are impossible to detect from the graph using a usual scale.
To construct a degree 4 polynomial with two hidden extrema near $x = 1$ and another extrema (not hidden) near $x = 0$, we start with a derivative,
$g'(x) = x(x-0.9)(x-1.1)$
$\quad = x^3 - 2x^2 + 0.99x.$
A function with this derivative is
$g(x) = \dfrac{1}{4}x^4 - \dfrac{2}{3}x^3 + \dfrac{0.99}{2}x^2 - 3$



60. TRUE. $(f \circ g)'(c) = f'(g(c))g'(c) = 0$, since c is a critical number of g.

61. $f(x) = x^3 + bx^2 + cx + d$
$f'(x) = 3x^2 + 2bx + c$
$f'(x) \geq 0$ for all x if and only if
$(2b)^2 - 4(3)(c) \leq 0$
if and only if $4b^2 \leq 12c$
if and only if $b^2 \leq 3c$.

62. $f(x) = x^5 + bx^3 + cx + d$
$f'(x) = 5x^4 + 3bx^2 + c$
Using the quadratic formula, we find

$$x^2 = \frac{-3b \pm \sqrt{9b^2 - 20c}}{10}.$$

Thus, if $9b^2 < 20c$, then the roots are imaginary and so $f'(x) \geq 0$ for all x. If this is not the case, then we need to consider

$$x = \pm\sqrt{\frac{-3b \pm \sqrt{9b^2 - 20c}}{10}}.$$

Now we need the expression inside the square root to be less than or equal to 0, which is the same as requiring the numerator of the expression inside the square root to be less than or equal to 0. So we need both
$-3b < \sqrt{9b^2 - 20c}$ and
$-3b < -\sqrt{9b^2 - 20c}$.
Of course, both are true if and only if the latter is true. In conclusion, $f(x)$ is an increasing function if $9b^2 < 20c$ or $-3b < -\sqrt{9b^2 - 20c}$.

63. (a)

$$\begin{aligned}\mu'(-10) &\approx \frac{0.0048 - 0.0043}{-12 - (-8)}\\ &= \frac{0.0005}{-4}\\ &= -0.000125\end{aligned}$$

(b)

$$\begin{aligned}\mu'(-6) &\approx \frac{0.0048 - 0.0043}{-4 - (-8)}\\ &= \frac{0.0005}{4}\\ &= 0.000125\end{aligned}$$

Whether the warming of the ice due to skating makes it easier or harder depends on the current temperature of the ice. As seen from these examples, the coefficient of friction μ is decreasing when the temperature is $-10°$ and increasing when the temperature is $-6°$.

64. We find the derivative of $f(t)$:

$$\begin{aligned}f'(t) &= \frac{a^2 + t^2 - t(2t)}{(a^2+t^2)^2}\\ &= \frac{a^2 - t^2}{(a^2+t^2)^2}.\end{aligned}$$

The denominator is always positive, while the numerator is positive when $a^2 > t^2$, i.e., when $a > t$. We now find the derivative of $\theta(x)$:

$$\begin{aligned}\theta'(x) &= \frac{1}{1 + \left(\frac{29.25}{x}\right)^2}\left(\frac{-29.25}{x^2}\right)\\ &\quad - \frac{1}{1 + \left(\frac{10.75}{x}\right)^2}\left(\frac{-10.75}{x^2}\right)\\ &= \frac{-29.25}{x^2 + (29.25)^2} + \frac{10.75}{x^2 + (10.75)^2}.\end{aligned}$$

We consider each of the two terms of the last line above as instances of $f(t)$, the first as $-f(29.25)$ and the second as $f(10.75)$. Now, for any given x where $x \geq 30$, this x is our a in $f(t)$ and since $a = x$ is greater than 29.25 and greater than 10.75, $f(t)$ is increasing for these two t values and this value of a. Thus $f(29.25) > f(10.75)$. This means that

$$\theta'(x) = -f(29.25) + f(10.75) < 0$$

(where $a = x$) and so $\theta(x)$ is decreasing for $x \geq 30$. Since $\theta(x)$ is increasing for $x \geq 30$, the announcers would be wrong to suggest that the angle increases by backing up 5 yards when the team is between 50 and 60 feet away from the goal post.

3.5 Concavity and the Second Derivative Test

1. $f'(x) = 3x^2 - 6x + 4$
$f''(x) = 6x - 6 = 6(x-1)$
$f''(x) > 0$ on $(1, \infty)$
$f''(x) < 0$ on $(-\infty, 1)$
So f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

2. $f'(x) = 4x^3 - 12x + 2$ and $f''(x) = 12x^2 - 12$. The graph is concave up where $f''(x)$ is positive, and concave down where f'' is negative. Concave up for $x < -1$ and $x > 1$, and concave down for $-1 < x < 1$.

3. $f(x) = x + \frac{1}{x} = x + x^{-1}$
$f'(x) = 1 - x^{-2}$
$f''(x) = 2x^{-3}$
$f''(x) > 0$ on $(0, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$
So f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

4. $y' = 1 - (1-x)^{-2/3}$ and $y'' = \frac{-2}{3}(1-x)^{-5/3}$. Concave up for $x > 1$ and concave down for $x < 1$.

5. $f'(x) = \cos x + \sin x$
$f''(x) = -\sin x + \cos x$
$f''(x) < 0$ on $\ldots \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right) \ldots$
$f''(x) > 0$ on $\ldots \left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right) \ldots$
f is concave down on $\ldots \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right) \ldots$,
concave up on $\ldots \left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right) \ldots$

6. $f'(x) = \dfrac{2x}{1+x^4}$ and $f''(x) = \dfrac{2-6x^4}{(1+x^4)^2}$.
Concave up for $-\sqrt[4]{\frac{1}{3}} < x < \sqrt[4]{\frac{1}{3}}$, and concave down for $x < -\sqrt[4]{\frac{1}{3}}$ and $x > \sqrt[4]{\frac{1}{3}}$.

7. $f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$
$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$
$= \frac{4}{9x^{2/3}}\left(1 - \frac{2}{x}\right)$
The quantity $\dfrac{4}{9x^{2/3}}$ is never negative, so the sign of the second derivative is the same as the sign of $1 - \dfrac{2}{x}$. Hence the function is concave up for $x > 2$ and $x < 0$, and is concave down for $0 < x < 2$.

8. $f'(x) = e^{-4x} - 4xe^{-4x}$ and $f''(x) = 8e^{-4x}(2x-1)$.
Concave up for $x > 1/2$, and concave down for $x < 1/2$.

9. $f(x) = x^4 + 4x^3 - 1$
$f'(x) = 4x^3 + 12x^2 = x^2(4x+12)$
So the critical numbers are $x = 0$ and $x = -3$.
$f''(x) = 12x^2 + 24x$
$f''(0) = 0$ so the second derivative test for $x = 0$ is inconclusive.
$f''(-3) = 36 > 0$ so $x = -3$ is a local minimum.

10. $f(x) = x^4 + 4x^2 + 1$
$f'(x) = 4x^3 + 8x$
So the only critical number is $x = 0$.
$f''(x) = 12x^2 + 8$
$f''(0) = 8 > 0$ so $x = 0$ is a local minimum.

11. $f(x) = xe^{-x}$
$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$
So the only critical number is $x = 1$.
$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(-2+x)$
$f''(1) = e^{-1}(-1) < 0$ so $x = 1$ is a local maximum.

12. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
So the only critical number is $x = 0$.
$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$
$f''(0) = -2+0 < 0$ so $x = 0$ is a local maximum.

13. $f(x) = \dfrac{x^2 - 5x + 4}{x}$
$$f'(x) = \frac{(2x-5)x - (x^2-5x+4)(1)}{x^2} = \frac{x^2-4}{x^2}$$
So the critical numbers are $x = \pm 2$.
$$f''(x) = \frac{(2x)(x^2) - (x^2-4)(2x)}{x^4} = \frac{8x}{x^4}$$
$f''(2) = 1 > 0$ so $x = 2$ is a local minimum.
$f''(-2) = -1 < 0$ so $x = -2$ is a local maximum.

14. $f(x) = \dfrac{x^2-1}{x}$
$$f'(x) = \frac{(2x)(x) - (x^2-1)(1)}{x^2} = \frac{x^2+1}{x^2}$$
There are no critical numbers and so there are no local extrema.

15. $y = (x^2+1)^{2/3}$

$y' = \dfrac{2}{3}(x^2+1)^{-1/3}(2x)$

$= \dfrac{4x(x^2+1)^{-1/3}}{3}$

So the only critical number is $x = 0$.

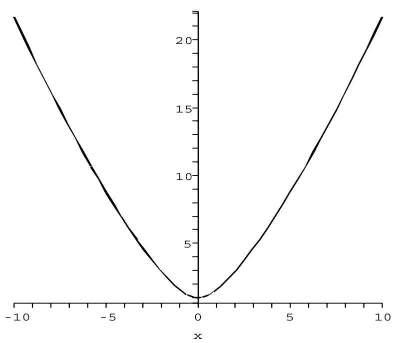
$y'' =$

$\dfrac{4}{3}\left[(x^2+1)^{-1/3} + \left(\dfrac{-2x^2}{3}\right)(x^2+1)^{-4/3}\right]$

$= \dfrac{4}{3}\dfrac{(x^2+1-\frac{2x^2}{3})}{(x^2+1)^{4/3}} = \dfrac{4}{9}\dfrac{(3x^2+3-2x^2)}{(x^2+1)^{4/3}}$

$= \dfrac{4}{9}\dfrac{(x^2+3)}{(x^2+1)^{4/3}}$

So the function is concave up everywhere, decreasing for $x < 0$, and increasing for $x > 0$. Also $x = 0$ is a local min.



16. $f(x) = x \ln x$

$f'(x) = \ln x + 1$

So the only critical number is e^{-1}.

$f''(x) = 1/x$

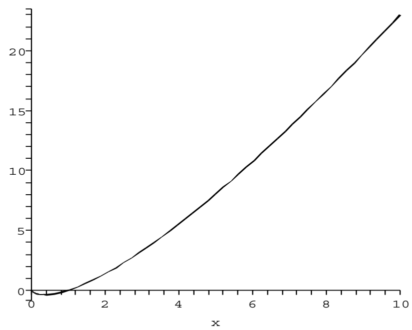
$f''(e^{-1}) = e > 0$ so $f(x)$ has a local minimum at $x = e^{-1}$.

The domain of $f(x)$ is $(0, \infty)$.

$f'(x) < 0$ on $(0, e^{-1})$ so $f(x)$ is decreasing on this interval. $f'(x) > 0$ on (e^{-1}, ∞), so $f(x)$ is increasing on this interval.

$f''(x) > 0$ for all x in the domain of $f(x)$, so $f(x)$ is concave up for all $x > 0$.

Finally, $f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as $x \to 0^+$.



17. $f(x) = \dfrac{x^2}{x^2-9}$

$f'(x) = \dfrac{2x(x^2-9) - x^2(2x)}{(x^2-9)^2}$

$= \dfrac{-18x}{(x^2-9)^2}$

$= \dfrac{-18x}{\{(x+3)(x-3)\}^2}$

$f''(x) =$

$\dfrac{-18(x^2-9)^2 + 18x \cdot 2(x^2-9)\cdot 2x}{(x^2-9)^4}$

$= \dfrac{54x^2+162}{(x^2-9)^3}$

$= \dfrac{54(x^2+3)}{(x^2-9)^3}$

$f'(x) > 0$ on $(-\infty,-3) \cup (-3,0)$

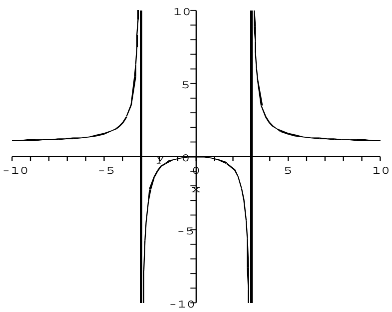
$f'(x) < 0$ on $(0,3) \cup (3,\infty)$

$f''(x) > 0$ on $(-\infty,-3) \cup (3,\infty)$

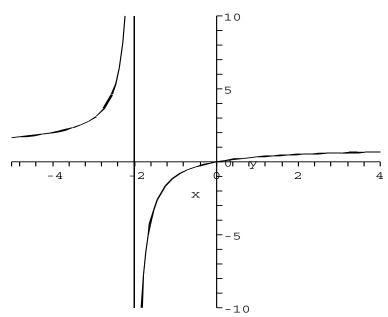
$f''(x) < 0$ on $(-3,3)$

$f''(0) = \dfrac{162}{(-9)^3}$

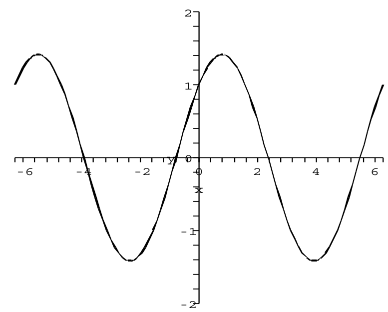
f is increasing on $(-\infty,-3)\cup(-3,0)$, decreasing on $(0,3)\cup(3,\infty)$, concave up on $(-\infty,-3)\cup(3,\infty)$, concave down on $(-3,3)$, $x = 0$ is a local max. f has a horizontal asymptote of $y = 1$ and vertical asymptotes at $x = \pm 3$.



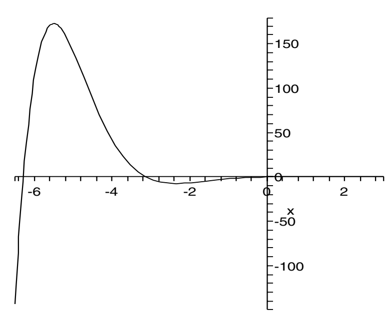
18. $f(x) = \dfrac{x}{x+2}$
The domain of $f(x)$ is $\{x|x \neq -2\}$.
There is a vertical asymptote at $x = -2$ such that $f(x) \to \infty$ as $x \to -2^-$ and $f(x) \to -\infty$ as $x \to -2^+$.
$f'(x) = \dfrac{x+2-x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$
So there are no critical numbers. Furthermore, $f'(x) > 0$ for all $x \neq -2$, so $f(x)$ is increasing everywhere.
$f''(x) = -4(x+2)^{-3}$
$f''(x) > 0$ on $(-\infty, -2)$ (so $f(x)$ is concave up on this interval)
$f''(x) > 0$ on $(-2, \infty$ (so $f(x)$ is concave down on this interval)



19. $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x$
$f''(x) = -\sin x - \cos x$
$f'(x) = 0$ when $x = \pi/4 + k\pi$ for all integers k. When k is even, $f''(\pi/4 + k\pi) = -\sqrt{2} < 0$ so $f(x)$ has a local maximum. When k is odd, $f''(\pi/4 + k\pi) = \sqrt{2} > 0$ so $f(x)$ has a local minimum.
$f'(x) < 0$ on the intervals of the form $(\pi/4 + 2k\pi, \pi/4 + (2k+1)\pi)$, so $f(x)$ is decreasing on these intervals.
$f'(x) > 0$ on the intervals of the form $(\pi/4 + (2k+1)\pi, \pi/4 + (2k+2)\pi)$, so $f(x)$ is increasing on these intervals.
$f''(x) > 0$ on the intervals of the form $(3\pi/4 + 2k\pi, 3\pi/4 + (2k+1)\pi)$ so $f(x)$ is concave up on these intervals.
$f''(x) < 0$ on the intervals of the form $(3\pi/4 + (2k+1)\pi, 3\pi/4 + (2k+2)\pi)$ so $f(x)$ is concave down on these intervals.



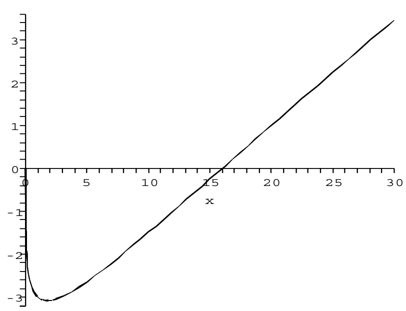
20. $y = e^{-x} \sin x$
$y' = -e^{-x} \sin x + e^{-x} \cos x = 0$ when $x = \pi/4 + k\pi$ for integers k.
$y'' = -2e^{-x} \cos x = 0$ at $\pi/2 + 2k\pi$ for integers k. These are inflection points. The function is concave up for $-\pi/2 < x < \pi/2$ and concave down for $\pi/2 < x < 3\pi/2$, and the pattern repeats with period 2π. The critical values are all extrema, and they alternate between maxima and minima.



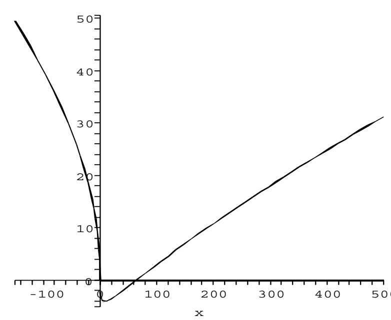
21. $f(x) = x^{3/4} - 4x^{1/4}$
Domain of $f(x)$ is $\{x|x \geq 0\}$.

$$f'(x) = \frac{3}{4}x^{-1/4} - x^{-3/4} = \frac{\frac{3}{4}\sqrt{x} - 1}{x^{3/4}}$$

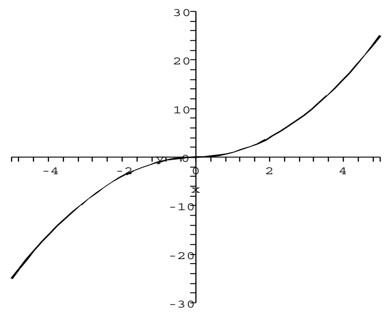
So $x = 0$ and $x = 16/9$ are critical points, but because of the domain we only need to really consider the latter.
$f'(1) = -1/4$ so $f(x)$ is decreasing on $(0, 16/9)$.
$f'(4) = \dfrac{0.5}{4^{3/4}} > 0$ so $f(x)$ is increasing on $(16/9, \infty)$.
Thus $x = 16/9$ is the location of a local minimum for $f(x)$.
$f''(x) = \frac{-3}{16}x^{-5/4} + \frac{3}{4}x^{-7/4}$
$= \dfrac{\frac{-3}{16}\sqrt{x} + \frac{3}{4}}{x^{7/4}}$
The critical number here is $x = 16$. We find that $f''(x) > 0$ on the interval $(0, 16)$ (so $f(x)$ is concave up on this interval) and $f''(x) < 0$ on the interval $(16, \infty)$ (so $f(x)$ is concave down on this interval).



22. $f(x) = x^{2/3} - 4x^{1/3}$
$f'(x) = \frac{2}{3}x^{-1/3} - \frac{4}{3}x^{-2/3}$
$= \dfrac{\frac{2}{3}x^{1/3} - \frac{4}{3}}{x^{2/3}}$
So $x = 0$ and $x = 8$ are critical numbers.
$f'(-1) < 0$ so $f(x)$ is decreasing for $x < 0$.
$f'(1) < 0$ so $f(x)$ is decreasing for $0 < x < 8$.
$f'(27) > 0$ so $f(x)$ is increasing on $8 < x$.
$f''(x) = -\frac{2}{9}x^{-4/3} + \frac{8}{9}x^{-5/3}$
$= \dfrac{-\frac{2}{9}x^{1/3} + \frac{8}{9}}{x^{5/3}}$
The critical numbers here are $x = 0$ and $x = 64$.
$f''(-1) < 0$ so $f(x)$ is concave down on $(-\infty, 0)$.
$f''(1) > 0$ so $f(x)$ is concave up on $(0, 64)$.
$f''(125) < 0$ so $f(x)$ is concave down on $(64, \infty)$.



23. The easiest way to sketch this graph is to notice that
$$f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$
Since
$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$
there is a critical point at $x = 0$. However, it is neither a local maximum nor a local minimum. Since
$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$
there is an inflection point at the origin. Note that the second derivative does not exist at $x = 0$.



24. The easiest way to sketch this graph is to notice that

$$f(x) = x^2|x| = \begin{cases} -x^3 & x < 0 \\ x^3 & x \geq 0 \end{cases}$$

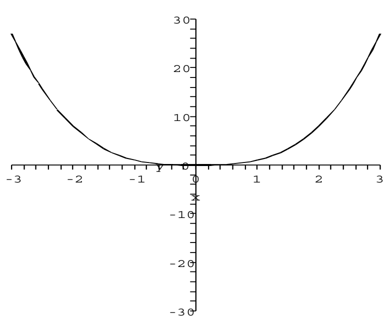
Since

$$f'(x) = \begin{cases} -3x^2 & x < 0 \\ 3x^2 & x \geq 0 \end{cases}$$

there is a critical point (and local minimum) at $x = 0$. Since

$$f''(x) = \begin{cases} -6x & x < 0 \\ 6x & x \geq 0 \end{cases}$$

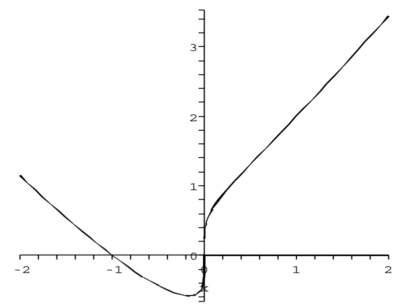
there is a critical point at the origin but this is not an inflection point.



25. $f(x) = x^{1/5}(x+1) = x^{6/5} + x^{1/5}$

$$f'(x) = \tfrac{6}{5}x^{1/5} + \tfrac{1}{5}x^{-4/5}$$
$$= \tfrac{1}{5}x^{-4/5}(6x+1)$$
$$f''(x) = \tfrac{6}{25}x^{-4/5} - \tfrac{4}{25}x^{-9/5}$$
$$= \tfrac{2}{25}x^{-9/5}(3x-2)$$

Note that $f(0) = 0$, and yet the derivatives do not exist at $x = 0$. This means that there is a vertical tangent line at $x = 0$. The first derivative is negative for $x < -1/6$ and positive for $-1/6 < x < 0$ and $x > 0$. The second derivative is positive for $x < 0$ and $x > 2/3$, and negative for $0 < x < 2/3$. Thus, there is a local minimum at $x = -1/6$ and inflection points at $x = 0$ and $x = 2/3$.



26. $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$

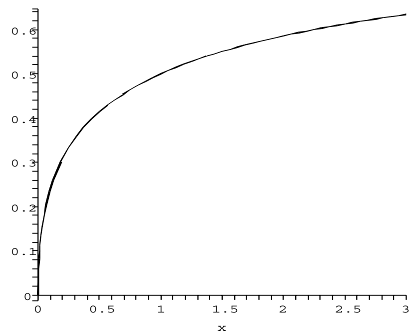
The domain of $f(x)$ is $\{x|x \geq 0\}$.

$$f'(x) = \frac{\frac{1}{2}x^{-1/2}(1+\sqrt{x}) - \sqrt{x}(\frac{1}{2}x^{-1/2})}{(x+\sqrt{x})^2}$$
$$= \frac{x^{-1/2}}{2(1+\sqrt{x})^2}$$

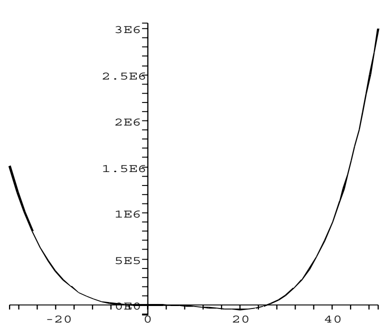
The only critical point is $x = 0$, which we need not consider because of the domain. Since $f'(1) > 0$, $f(x)$ is increasing on $(0, \infty)$.

$$f''(x) = \frac{-x^{-3/2}(1+\sqrt{x})^2 - 2x^{-1/2}(1+\sqrt{x})x^{-1/2}}{4(1+\sqrt{x})^4}$$
$$= \frac{-(x^{-1/2}+3)}{4x(1+\sqrt{x})^3}$$

The critical numbers are $x = 0$ (which we again ignore) and $x = 1/9$. Since $f''(1) < 0$ and $f''(1/16) < 0$, $f(x)$ is concave down on $(0, \infty)$.



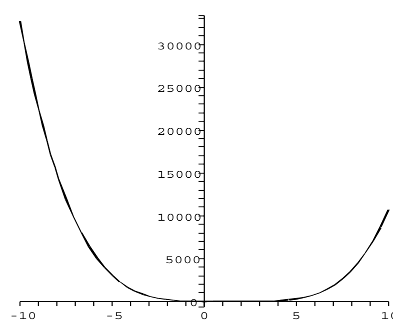
27. $f(x) = x^4 - 26x^3 + x$
$f'(x) = 4x^3 - 78x^2 + 1$
The critical numbers are approximately -0.1129, 0.1136 and 19.4993.
$f'(-1) < 0$ implies $f(x)$ is decreasing on $(-\infty, -0.1129)$.
$f'(0) > 0$ implies $f(x)$ is increasing on $(-0.1129, 0.1136)$.
$f'(1) < 0$ implies $f(x)$ is decreasing on $(0.1136, 19.4993)$.
$f'(20) > 0$ implies $f(x)$ is increasing on $(19.4993, \infty)$.
Thus $f(x)$ has local minimums at $x = -0.1129$ and $x = 19.4993$ and a local maximum at $x = 0.1136$.
$f''(x) = 12x^2 - 156x = x(12x - 156)$
The critical numbers are $x = 0$ and $x = 13$.
$f''(-1) > 0$ implies $f(x)$ is concave up on $(-\infty, 0)$.
$f''(1) < 0$ implies $f(x)$ is concave down on $(0, 13)$.
$f''(20) > 0$ implies $f(x)$ is concave up on $(13, \infty)$.



28. $f(x) = 2x^4 - 11x^3 + 17x^2$
$f'(x) = 8x^3 - 33x^2 + 34x$
$= x(8x - 17)(x - 2)$
The critical numbers are $x = 0$, $x = 2$ and $x = 17/8$.
$f''(x) = 24x^2 - 66x + 34$
$f''(0) > 0$ implies $f(x)$ is concave up at $x = 0$ so $f(x)$ has a local minimum here and $f(x)$ is decreasing on $(-\infty, 0)$.
$f''(2) < 0$ implies $f(x)$ is concave down at $x = 2$ so $f(x)$ has a local maximum here and $f(x)$ is increasing on $(0, 2)$.
$f''(17/8) > 0$ implies $f(x)$ is concave up at $x = 17/8$ so $f(x)$ has a local minimum here and $f(x)$ is decreasing on $(2, 17/8)$ and increasing on $(17/8, \infty)$.
$f''(x) = 2(12x^2 - 33x + 17)$
The critical numbers are

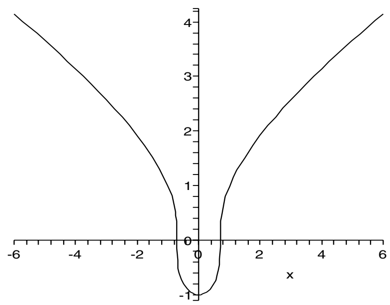
$$x = \frac{33 \pm \sqrt{273}}{24} = 2.0635,\ 0.6866.$$

So $f(x)$ is concave up on $(-\infty, 0.6866)$ and $(2.0635, \infty)$ and $f(x)$ is concave down on $(0.6866, 2.0635)$.



29. $y = \sqrt[3]{x^2 - 1}$
$y' = \dfrac{4x}{3(2x^2 - 1)^{2/3}} = 0$ at $x = 0$ and is undefined at $x = \pm\sqrt{1/2}$.
$y'' = \dfrac{-4(2x^2 + 3)}{9(2x^2 - 1)^{5/3}}$ is never 0, and is undefined where y' is.

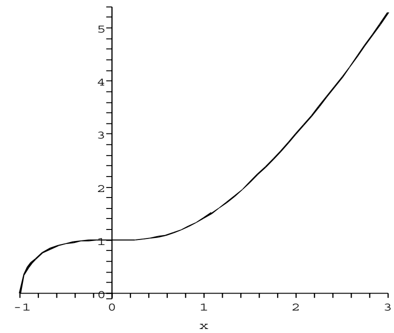
The function changes concavity at $x = \pm\sqrt{1/2}$, so these are inflection points. The slope does not change at these values, so they are not extrema. The Second Derivative Test shows that $x = 0$ is a minimum.



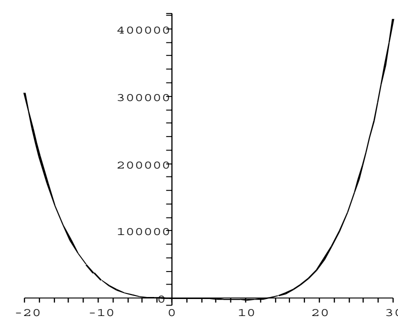
30. $f(x) = \sqrt{x^3+1}$
$f(x)$ is defined for $x \geq -1$.
$f'(x) = \frac{1}{2}(x^3+1)^{-1/2}(3x^2)$.
The critical numbers are $x = -1$ (which we ignore because of the domain) and $x = 0$.
$f'(-1/2) > 0$ so $f(x)$ is increasing on $(-1, 0)$. $f'(1) > 0$ so $f(x)$ is also increasing on $(0, \infty)$ so $f(x)$ has no relative extrema.

$$f''(x) = \frac{3}{2}\cdot\frac{2x(x^3+1)^{1/2} - x^2\frac{1}{2}(x^3+1)^{-1/2}3x^2}{x^3+1}$$
$$= \frac{2x(x^3+1) - \frac{3}{2}x^4}{(x^3+1)^{3/2}}$$
$$= \frac{-\frac{1}{2}x^4 + 2x}{(x^3+1)^{3/2}}$$

The critical numbers are $x = 0$ and $x = 4^{1/3}$ (and $x = -1$, which we need not consider).
$f''(-1/2) < 0$ so $f(x)$ is concave down on $(-1, 0)$. $f''(1) > 0$ so $f(x)$ is concave up on $(0, 4^{1/3})$. $f''(2) > 0$ so $f(x)$ is concave up on $(4^{1/3}, \infty)$.

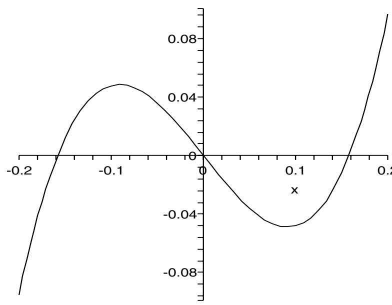


31. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
$f'(x) = 4x^3 - 48x^2 + 84x - 39.6$
$f''(x) = 12x^2 - 96x + 84$
$= 12(x^2 - 8x + 7)$
$= 12(x-7)(x-1)$
$f'(x) > 0$ on $(.8952, 1.106) \cup (9.9987, \infty)$
$f'(x) < 0$ on $(-\infty, .8952) \cup (1.106, 9.9987)$
$f''(x) > 0$ on $(-\infty, 1) \cup (7, \infty)$
$f''(x) < 0$ on $(1, 7)$
f is increasing on $(.8952, 1.106)$ and on $(9.9987, \infty)$, decreasing on $(-\infty, .8952)$ and on $(1.106, 9.9987)$, concave up on $(-\infty, 1) \cup (7, \infty)$, concave down on $(1, 7)$, $x = .8952$, 9.9987 are local min, $x = 1.106$ is local max, $x = 1, 7$ are inflection points.

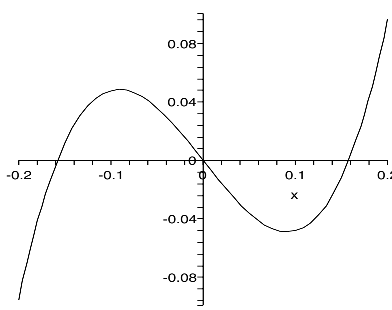


32. $y = x^4 + 32x^3 - 0.02x^2 - 0.8x$
$y' = 4x^3 + 96x^2 - 0.04x - 0.8 = 0$ at approximately $x = -24$, -0.09125, and 0.09132.
$y'' = 12x^2 + 192x - 0.04 = 0$ at approximately $x = 16.0002$ and 0.0002, and

changes sign at these values, so these are inflection points. The Second Derivative Test shows that $x = -24$ and 0.09132 are minima, and that $x = -0.09125$ is a maxima. The extrema near $x = 0$ look like this:



The global behavior looks like this:

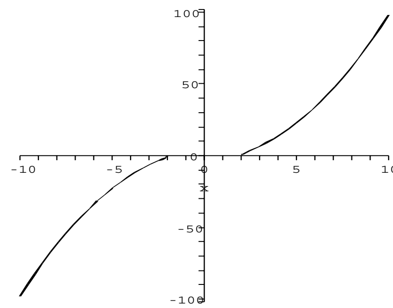


33. $f(x) = x\sqrt{x^2-4}$; f undefined on $(-2, 2)$

$$f'(x) = \sqrt{x^2-4} + x\left(\tfrac{1}{2}\right)(x^2-4)^{-1/2}(2x) = \sqrt{x^2-4} + \frac{x^2}{\sqrt{x^2-4}} = \frac{2x^2-4}{\sqrt{x^2-4}}$$

$$f''(x) = \frac{4x\sqrt{x^2-4} - (2x^2-4)\frac{1}{2}(x^2-4)^{-1/2}(2x)}{x^2-4} = \frac{4x(x^2-4)x - (2x^2-4)}{(x^2-4)^{3/2}} = \frac{2x^3-12x}{(x^2-4)^{3/2}} = \frac{2x(x^2-6)}{(x^2-4)^{3/2}}$$

$f'(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$
$f''(x) > 0$ on $\left(-\sqrt{6}, 2\right) \cup \left(\sqrt{6}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{6}\right) \cup \left(2, \sqrt{6}\right)$
f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, concave up on $\left(-\sqrt{6}, -2\right) \cup \left(\sqrt{6}, \infty\right)$, concave down on $\left(-\infty, -\sqrt{6}\right) \cup \left(2, \sqrt{6}\right)$, $x = \pm\sqrt{6}$ are inflection points.



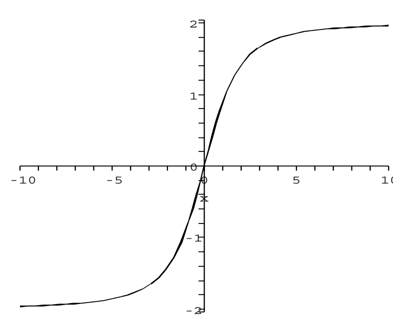
34. $f(x) = \dfrac{2x}{\sqrt{x^2+4}}$

$$f'(x) = \frac{2\sqrt{x^2+4} - 2x(\frac{1}{2})(x^2+4)^{-1/2}2x}{(x^2+4)} = \frac{8}{(x^2+4)^{3/2}}$$

$f'(x)$ is always positive, so there are no critical points and $f(x)$ is always increasing.

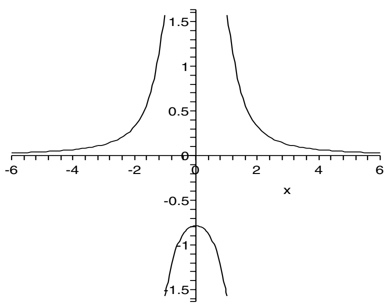
$$f''(x) = 8(-\tfrac{3}{2})(x^2+4)^{-5/2}(2x) = \frac{-24x}{(x^2+4)^{5/2}}$$

The only critical point is $x = 0$. Since $f''(-1) > 0$, $f(x)$ is concave up on $(-\infty, 0)$. Also $f''(1) < 0$, so $f(x)$ is concave down on $(0, \infty)$ and $x = 0$ is an inflection point for f.

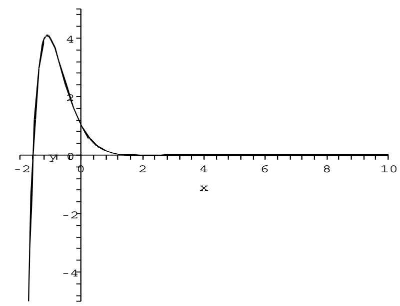


35. The function has horizontal asymp-

tote $y = 0$, and is undefined at $x = \pm 1$.
$y' = \dfrac{-2x}{x^4 - 2x^2 + 2} = 0$
only when $x = 0$.
$y'' = \dfrac{2(3x^4 - 2x^2 - 2)}{(x^4 - 2x^2 + 2)^2} = 0$
at approximately $x = \pm 1.1024$ and changes sign there, so these are inflection points (very easy to miss by looking at the graph). The Second Derivative Test shows that $x = 0$ is a local maximum.



36. $f(x) = e^{-2x} \cos x$
$f'(x) = -2e^{-2x} \cos x - e^{-2x} \sin x$
$= e^{-x}(-2 \cos x - \sin x)$
$f''(x) = -2e^{-2x}(-2 \cos x - \sin x)$
$+ e^{-2x}(2 \sin x - \cos x)$
$= e^{-2x}(4 \sin x + 3 \cos x)$
$f'(x) = 0$ when $\sin x = -2 \cos x$ so when $x = k\pi + \tan^{-1}(-2)$ for any integer k.
$f''(2k\pi + \tan^{-1}(-2)) < 0$ so there are local maxima at all such points, while $f''((2k+1)\pi + \tan^{-1}(-2)) > 0$, so there are local minima at all such points. $f''(x) = 0$ when $4 \sin x = -3 \cos x$ or $x = k\pi + \tan^{-1}(-3/4)$ for any integer k. All such points x are inflection points.



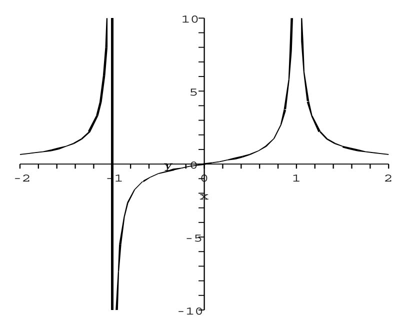
37. $f(x)$ is concave up on $(-\infty, -0.5)$ and $(0.5, \infty)$;
$f(x)$ is concave down on $(-0.5, 0.5)$.

38. $f(x)$ is concave up on $(-\infty, 0)$;
$f(x)$ is concave down on $(0, \infty)$.

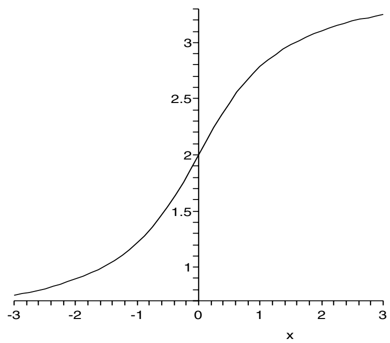
39. $f(x)$ is concave up on $(1, \infty)$;
$f(x)$ is concave down on $(-\infty, 1)$.

40. $f(x)$ is concave up on $(-1, 0)$ and $(1, \infty)$;
$f(x)$ is concave down on $(-\infty, -1)$ and $(0, 1)$.

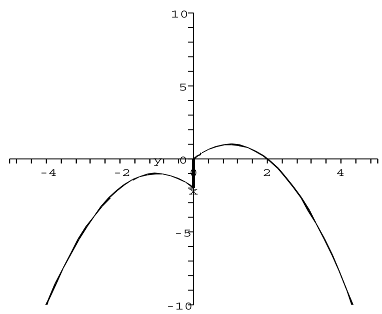
41. One possible graph:



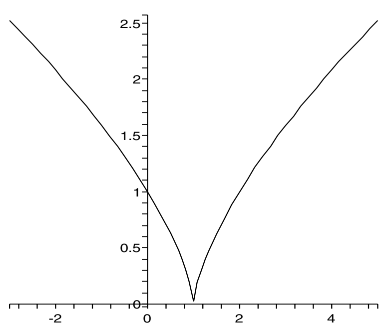
42. One possible graph:



43. One possible graph:



44. One possible graph:



45. $f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = 3ax^2 + 2bx + c$
$f''(x) = 6ax + 2b$
Thus, $f''(x) = 0$ for $x = -b/3a$. Since f'' changes sign at this point, f has an inflection point at $x = -b/3a$. Note that $a \neq 0$.

For the quartic function (where again $a \neq 0$),
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$
$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$
$$\begin{aligned} f''(x) &= 12ax^2 + 6bx + 2c \\ &= 2(6ax^2 + 3bx + c) \end{aligned}$$
The second derivative is zero when
$$\begin{aligned} x &= \frac{-3b \pm \sqrt{9b^2 - 24ac}}{12a} \\ &= \frac{-3b \pm \sqrt{3(3b^2 - 8ac)}}{12a} \end{aligned}$$
There are two distinct solutions to the previous equation (and therefore two inflection points) if and only if $3b^2 - 8ac > 0$.

46. Since $f'(0) = 0$ and $f''(0) > 0$, $f(x)$ must have a local minimum at $x = 0$. Since we also know that $f(0) = 0$, this means that there is some neighborhood (possibly very small) of 0 such that for all x in this neighborhood (exluding $x = 0$), $f(x) > 0$.

Similarly, $g'(0) = 0$ and $g''(0) < 0$ implies that $g(x)$ must have a local maximum at $x = 0$. Again we know that $g(0) = 0$, so there is some neighborhood of 0 such that for all x in this neighborhood (exluding $x = 0$), $g(x) < 0$.

On the smaller of these two neighborhoods, we know that $g(x) < 0 < f(x)$.

47. The function has the following properties:
increasing on $(0, \infty)$;
decreasing on $(-\infty, 0)$;
local minimum at $x = 0$;
concave up on $(-\infty, \infty)$;
no inflection points.

48. The function has the following properties:
increasing on $(-\infty, 1)$ and $(2, \infty)$;
decreasing on $(1, 2)$;
local maximum at $x = 1$;
local minimum at $x = 2$;
concave up on $(1.5, \infty)$;

concave down on $(-\infty, 1.5)$;
inflection point at $x = 1.5$.

49. For #47:
increasing on $(-\infty, -1)$ and $(1, \infty)$;
decreasing on $(-1, 1)$;
local maximum at $x = -1$;
local minimum at $x = 1$;
concave up on $(0, \infty)$;
concave down on $(-\infty, 0)$;
inflection point at $x = 0$.

For # 48:
increasing on $(0, 2)$ and $(2, \infty)$;
decreasing on $(-\infty, 0)$;
local minimum at $x = 0$;
concave up on $(-\infty, 1)$ and $(2, \infty)$;
concave down on $(1, 2)$;
inflection points at $x = 1$ and $x = 2$.

50. For #47:
concave up on $(-\infty, -1)$ and $(1, \infty)$;
concave down on $(-1, 1)$;
inflection points at $x = -1$ and $x = 1$.

For # 48:
concave up on $(0, 2)$ and $(2, \infty)$;
concave down on $(-\infty, 0)$;
inflection point at $x = 0$.

51. We need to know $w'(0)$ to know if the depth is increasing.

52. We assume the sick person's temperature is too high, and not too low. We do need to know $T'(0)$ in order to tell which is better.

If $T''(0) = 2$ and $T' > 0$, the person's temperature is rising alarmingly.

If $T''(0) = -2$ and $T' > 0$, the person's temperature is increasing, but leveling off.
Negative T'' is better if $T' > 0$.

If $T''(0) = 2$ and $T' < 0$, the person's temperature is decreasing and leveling off.
If $T''(0) = -2$ and $T' < 0$, the person's temperature is dropping too steeply to be safe.
Positive T'' is probably better if $T' < 0$.

53. $s(x) = -3x^3 + 270x^2 - 3600x + 18000$
$s'(x) = -9x^2 + 540x - 3600$
$s''(x) = -18x + 540 = 0$
$x = 30$. This is a max because the graph of $s'(x)$ is a parabola opening down. So spend \$30,000 on advertising to maximize the rate of change of sales. This is also the inflection point of $s(x)$.

54. $Q'(t)$ measures the number of units produced per hour. If this number is larger, the worker is more efficient.
$Q'(t) = -3t^2 + 12t + 12$ will be maximized where
$Q'' = -6t + 12 = 0$, or $t = 2$ hours. (This is a maximum by the First Derivative Test.) It is reasonable to call this inflection point the point of diminishing returns, because after this point, the efficiency of the worker decreases.

55. $C(x) = .01x^2 + 40x + 3600$
$\overline{C}(x) = \dfrac{C(x)}{x} = .01x + 40 + 3600x^{-1}$
$\overline{C}'(x) = .01 - 3600x^{-2} = 0$
$x = 600$. This is a min because $\bar{C}''(x) = 7200x^{-3} > 0$ for $x > 0$, so the graph is concave up. So manufacture 600 units to minimize average cost.

56. Solving $c' = 0$ yields $t = 19.8616$. The Second Derivative Test shows this is a maximum. Solving $c'' = 0$ yields $t = 41.8362$. Suppose a second drug produced a similar plasma concentration graph, with the same maximum, but a later inflection point.

Then the plasma concentration decays faster for the second drug, since it takes longer for the rate of decay to level off.

57. Both functions are increasing for $x > 0$ and have the same asymptote, $y = 1$, so that is no help. However, if
$f(x) = \dfrac{x}{27+x}$,
then
$f'(x) = \dfrac{27}{(27+x)^2}$.
Hence $f'(x)$ is decreasing for $x > 0$ and so f has no inflection points on this interval. On the other hand, if
$f(x) = \dfrac{x^3}{c^3+x^3}$,
then
$f'(x) = \dfrac{3cx^2}{(c^3+x^3)^2}$
and
$f''(x) = \dfrac{6c^3x(c^3-2x^3)}{(c^3+x^3)^3}$
and so there is an inflection point at $x = c/\sqrt[3]{2}$. When $c = 27$, $27/\sqrt[3]{2} \approx 21.4$, in excellent agreement with the given graph.

58. $\tan^{-1} x$ is not a good fit to the data (even after appropriately scaling) because it is not concave up near $x = 0$.
$g_1(x) = \dfrac{2}{\pi}\tan^{-1}\dfrac{x}{30}$
fits the data weakly. $g_1(27)$ is close to 0.5, but then $g_1(40)$ is too small.

$g_2(x) = \dfrac{1}{1+99e^{-t/2}}$
has the desired change in concavity, but it grows too rapidly. For example, if the function is scaled so that $g_2(27) \approx 0.5$, then $g_2(40)$ is very nearly 1 instead of 0.75.

59. Let $f(x) = -1 - x^2$. Then
$f'(x) = -2x$
$f''(x) = -2$
so f is concave down for all x, but $-1 - x^2 = 0$ has no solution.

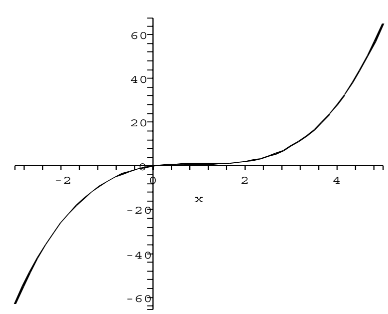
60. The statement is true.

61. Since the tangent line points above the sun, the sun appears higher in the sky than it really is.

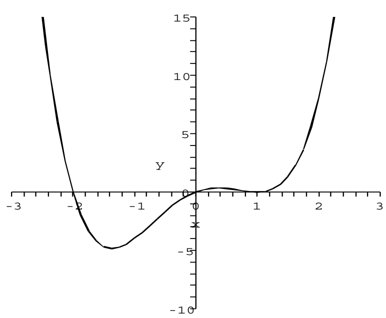
62. If $f''(c) < 0$, then f' is decreasing at c. Because $f'(c) = 0$, this means that $f' > 0$ to the left of c and $f' < 0$ to the right of c. Therefore, by the First Derivative Test, $f(c)$ is a local maximum. The proof of the second claim is similar.

3.6 Overview of Curve Sketching

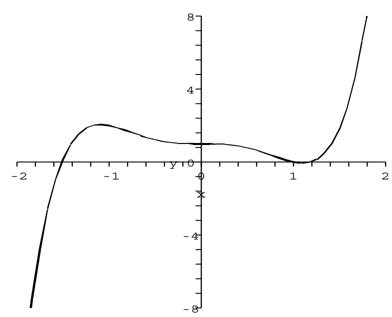
1. $f(x) = x^3 - 3x^2 + 3x$
$= x(x^2 - 3x + 3)$
The only x-intercept is $x = 0$; the y-intercept is $(0, 0)$.
$f'(x) = 3x^2 - 6x + 3$
$= 3(x^2 - 2x + 1) = 3(x-1)^2$
$f'(x) > 0$ for all x, so $f(x)$ is increasing for all x and has no local extrema.
$f''(x) = 6x - 6 = 6(x - 1)$
There is an inflection point at $x = 1$: $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.
Finally, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.



2. $f(x) = x^4 - 3x^2 + 2x$
$= x(x^3 - 3x + 2)$
The x-intercepts are $x = -2$, $x = 1$ and $x = 0$; the y-intercept is $(0, 0)$.
$f'(x) = 4x^3 - 6x + 2$
$= 2(x^3 - 3x + 1)$
The critical numbers are $x = -1.366$, 0.366 and 1.
$f'(x) > 0$ on $(-1.366, 0.366)$ and $(1, \infty)$, so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-\infty, -1.366)$ and $(0.366, 1)$, so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has local minima at $x = -1.366$ and $x = 1$ and a local maximum at $x = 0.366$.
$f''(x) = 12x^2 - 6 = 6(2x^2 - 1)$
The critical numbers here are $x = \pm 1/\sqrt{2}$. $f''(x) > 0$ on $(-\infty, -1/\sqrt{2})$ and $(1/\sqrt{2}, \infty)$ so $f(x)$ is concave up on these intervals. $f''(x) < 0$ on $(-1/\sqrt{2}, 1/\sqrt{2})$ so $f(x)$ is concave down on this interval. Thus $f(x)$ has inflection points at $x = \pm 1/\sqrt{2}$.
Finally, $f(x) \to \infty$ as $x \to \pm\infty$.

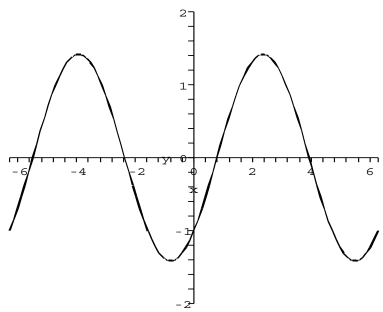


3. $f(x) = x^5 - 2x^3 + 1$
The x-intercepts are $x = 1$ and $x \approx -1.5129$; the y-intercept is $(0, 1)$.
$f'(x) = 5x^4 - 6x^2 = x^2(5x^2 - 6)$
The critical numbers are $x = 0$ and $x = \pm\sqrt{6/5}$. Plugging values from each of the intervals into $f'(x)$, we find that $f'(x) > 0$ on $(-\infty, -\sqrt{6/5})$ and $(\sqrt{6/5}, \infty)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-\sqrt{6/5}, 0)$ and $(0, \sqrt{6/5})$ so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has a local maximum at $-\sqrt{6/5}$ and a local minimum at $\sqrt{6/5}$.
$f''(x) = 20x^3 - 12x = 4x(5x^2 - 3)$
The critical numbers are $x = 0$ and $x = \pm\sqrt{3/5}$. Plugging values from each of the intervals into $f''(x)$, we find that $f''(x) > 0$ on $(-\sqrt{3/5}, 0)$ and $(\sqrt{3/5}, \infty)$ so $f(x)$ is concave up on these intervals. $f''(x) < 0$ on $(-\infty, -\sqrt{3/5})$ and $(0, \sqrt{3/5})$ so $f(x)$ is concave down on these intervals. Thus $f(x)$ has inflection points at all three of these critical numbers.
Finally, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.



4. $f(x) = \sin x - \cos x$
This has y-intercept $(0, -1)$ and x-intercepts everywhere that $\sin x = \cos x$, i.e., at $x = \pi/4 + k\pi$ for any integer k.
$f'(x) = \cos x + \sin x$
The critical numbers here are $x = 3\pi/4 + k\pi$ for any integer k. We find that $f(x)$ is decreasing on intervals of the form $(3\pi/4 + 2k\pi, 3\pi/4 + (2k+1)\pi)$ and increasing on intervals of the form $(3\pi/4 + (2k-1)\pi, 3\pi/4 + 2k\pi)$.
$f''(x) = -\sin x + \cos x$
The critical numbers are $x = \pi/4 + k\pi$ for any integer k. $f(x)$ is concave down on the intervals of the form $(\pi/4 + 2k\pi, \pi/4 + (2k+1)\pi)$ and

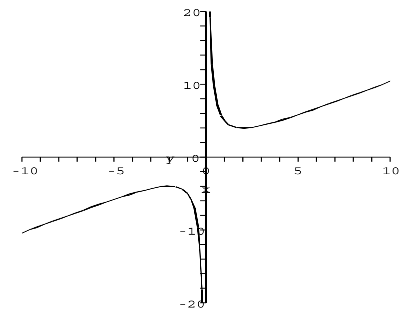
concave up on intervals of the form $(\pi/4+(2k-1)\pi, \pi/4+2k\pi)$.
$f(x)$ has no vertical or horizontal asymptotes.



5. $f(x) = x + \dfrac{4}{x} = \dfrac{x^2+4}{x}$
This function has no x- or y-intercepts. The domain is $\{x|x \neq 0\}$.
$f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to -\infty$ as $x \to 0^-$ and $f(x) \to \infty$ as $x \to 0^+$.

$$f'(x) = 1 - 4x^{-2} = \frac{x^2-4}{x^2}$$

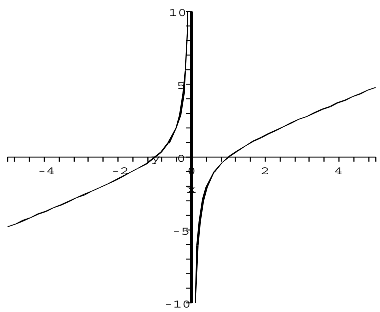
The critical numbers are $x = \pm 2$. We find that $f'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-2, 0)$ and $(0, 2)$, so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.
$f''(x) = 8x^{-3}$
$f''(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is concave down on this interval and $f''(x) > 0$ on $(0, \infty)$ so $f(x)$ is concave up on this interval, but $f(x)$ has an asymptote (not an inflection point) at $x = 0$.
Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



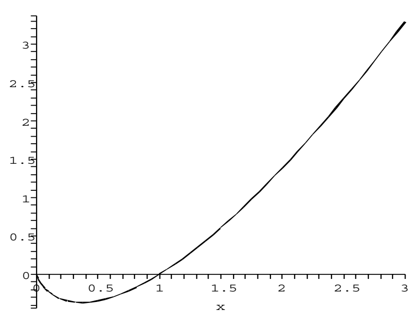
6. $f(x) = \dfrac{x^2-1}{x} = x - \dfrac{1}{x}$
There are x-intercepts at $x - \pm 1$, but no y-intercepts. The domain is $\{x|x \neq 0\}$.
$f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as $x \to 0^-$ and $f(x) \to -\infty$ as $x \to 0^+$.

$$f'(x) = 1 - x^{-2} = \frac{x^2-1}{x^2}$$

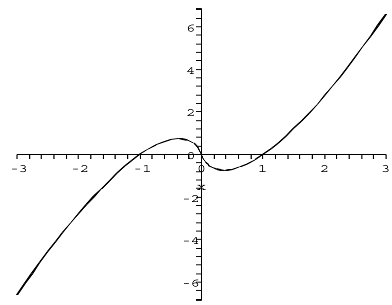
The critical numbers are $x = \pm 1$. $f(x)$ is increasing $(-\infty, -1)$ and $(1, \infty)$ and $f(x)$ is decreasing on $(-1, 0)$ and $(0, 1)$. Thus $f(x)$ has a local maximum at $x = -1$ and a local minimum at $x = 1$.
$f''(x) = 2x^{-3}$
$f''(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is concave down on this interval and $f''(x) > 0$ on $(0, \infty)$ so $f(x)$ is concave up on this interval, but $f(x)$ has an asymptote (not an inflection point) at $x = 0$.
Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



7. $f(x) = x \ln x$
The domain is $\{x|x > 0\}$. There is an x-intercept at $x = 1$ and no y-intercept.
$f'(x) = \ln x + 1$
The only critical number is $x = e^{-1}$. $f'(x) < 0$ on $(0, e^{-1})$ and $f'(x) > 0$ on (e^{-1}, ∞) so $f(x)$ is decreasing on $(0, e^{-1})$ and increasing on (e^{-1}, ∞). Thus $f(x)$ has a local minimum at $x = e^{-1}$.
$f''(x) = 1/x$, which is positive for all x in the domain of f, so $f(x)$ is always concave up.
$f(x) \to \infty$ as $x \to \infty$.



8. $f(x) = x \ln x^2$
The domain is $\{x|x \neq 0\}$. There are x-intercepts at $x = \pm 1$ but no y-intercept.
$f'(x) = \ln x^2 + 2$
The critical numbers are $x = \pm e^{-1}$. $f''(x) = 2/x$, so $x = -e^{-1}$ is a local maximum and $x = e^{-1}$ is a local minimum. $f(x)$ is increasing on $(-\infty, -e^{-1})$ and (e^{-1}, ∞); $f(x)$ is decreasing on $(-e^{-1}, 0)$ and $(0, e^{-1})$. $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
$f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to \infty$.



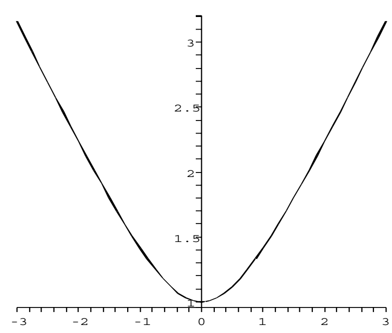
9. $f(x) = \sqrt{x^2 + 1}$
The y-intercept is $(0, 1)$. There are no x-intercepts.

$$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + 1}}$$

The only critical number is $x = 0$. $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$ so $f(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. Thus $f(x)$ has a local minimum at $x = 0$.

$$f''(x) = \frac{\sqrt{x^2 + 1} - x\frac{1}{2}(x^2 + 1)^{-1/2} 2x}{x^2 + 1} = \frac{1}{(x^2 + 1)^{3/2}}$$

Since $f''(x) > 0$ for all x, we see that $f(x)$ is concave up for all x.
$f(x) \to \infty$ as $x \to \pm\infty$.



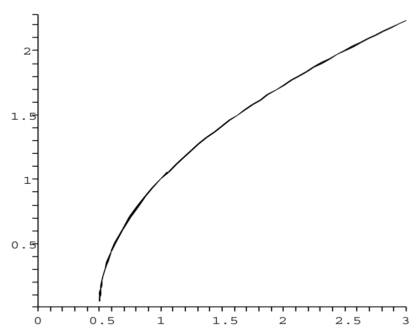
10. $f(x) = \sqrt{2x-1}$
The domain is $\{x|x \geq 1/2\}$. There is an x-intercept at $x = 1/2$.

$$f'(x) = \frac{1}{2}(2x-1)^{-1/2}2 = \frac{1}{\sqrt{2x-1}}$$

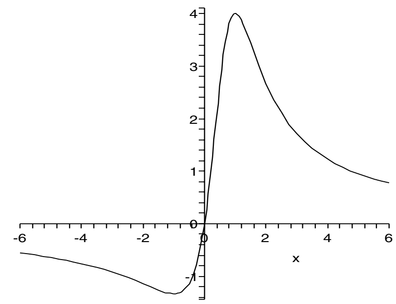
$f'(x)$ is undefined at $x = 1/2$, but this is an endpoint of $f(x)$ and there are no other critical points. Since $f'(x)$ is positive for all x in the domain of f, we see that $f(x)$ is increasing for all x in the domain.

$$f''(x) = -\frac{1}{2}(2x-1)^{-3/2}2 = \frac{-1}{(2x-1)^{3/2}}$$

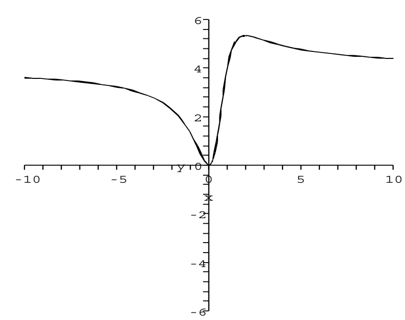
$f''(x) < 0$ for all x in the domain of f, so f is concave down for all x for which it is defined.
$f(x) \to \infty$ as $x \to \infty$.



11. $f(x) = \dfrac{4x}{x^2-x+1}$
The function has horizontal asymptote at $y = 0$.
$f'(x) = \dfrac{4(1-x^2)}{(x^2-x+1)^2}$
There are critical numbers at $x = \pm 1$.
$f''(x) = \dfrac{8(x^3-3x+1)}{(x^2-x+1)^3}$
with critical numbers at approximately $x = -1.8793$, 0.3473, and 1.5321. $f''(x)$ changes sign at these values, so these are inflection points. The Second Derivative test shows that $x = -1$ is a minimum, and $x = 1$ is a maximum.

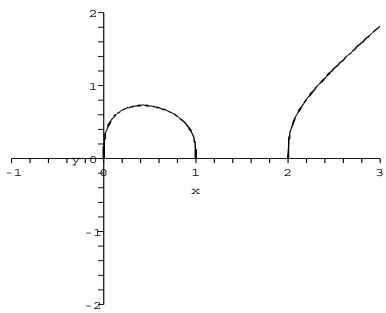


12. $f(x) = \dfrac{4x^2}{x^2-x+1}$
The function has horizontal asymptote at $y = 4$.
$f'(x) = \dfrac{-4x(x-2)}{(x^2-x+1)^2}$
There are critical numbers at $x = 0$ and $x = 2$.
$f''(x) = \dfrac{8(x^3-3x^2+1)}{(x^2-x+1)^3}$
with critical numbers at approximately $x = -0.5321$, 0.6527, and 2.8794. $f''(x)$ changes sign at these values, so these are inflection points. The Second Derivative test shows that $x = 0$ is a minimum, and $x = 2$ is a maximum.

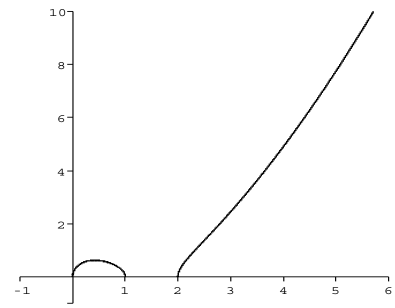


13. $f(x) = (x^3 - 3x^2 + 2x)^{1/3}$
$f'(x) = \dfrac{3x^2-6x+2}{3(x^3-3x^2+2x)^{2/3}}$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$f''(x) = \dfrac{-6x^2+12x-8}{9(x^3-3x^2+2x)^{5/3}}$

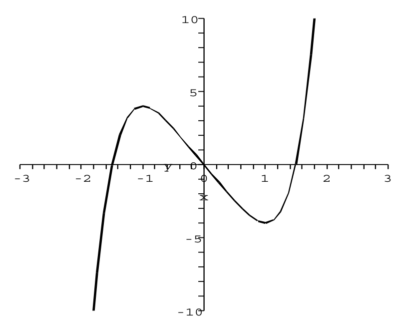
with critical numbers $x = 0$, 1 and 2. $f''(x)$ changes sign at these values, so these are inflection points. The Second Derivative test shows that $x = \dfrac{3+\sqrt{3}}{3}$ is a minimum, and $x = \dfrac{3-\sqrt{3}}{3}$ is a maximum.
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



14. $f(x) = (x^3 - 3x^2 + 2x)^{1/2}$
$f(x)$ is defined for $0 \le x \le 1$ and $x \ge 2$. $f(x) \to \infty$ as $x \to \infty$.
$f'(x) = \dfrac{3x^2 - 6x + 2}{2(x^3 - 3x^2 + 2x)^{1/2}}$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$f''(x) = \dfrac{3x^4 - 12x^3 + 12x^2 - 4}{4(x^3 - 3x^2 + 2x)^{3/2}}$
with critical numbers $x = 0$, 1 and 2 and $x \approx -0.4679$ and 2.4679. $f(x)$ is undefined at $x = -0.4679$, so we do not consider this point. $f''(x)$ changes sign at $x = 2.4679$, so this is an inflection point. The Second Derivative test shows that $x = \dfrac{3-\sqrt{3}}{3}$ is a maximum.

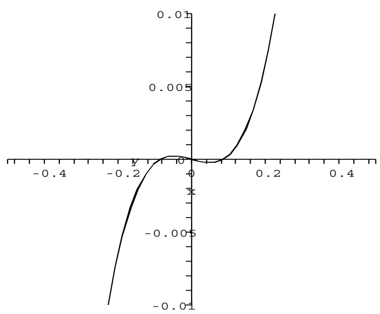


15. $f(x) = x^5 - 5x = x(x^4 - 5)$
x-intercepts are $x = 0$ and $x = \pm\sqrt[4]{5}$.
The y-intercept is $(0, 0)$.
$f'(x) = 5x^4 - 5 = 5(x^4 - 1)$
The critical numbers are $x = \pm 1$.
$f''(x) = 20x^3$ so $x = -1$ is a local maximum and $x = 1$ is a local minimum. $f(x)$ is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$. It is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$, with an inflection point at $x = 0$.
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



16. $f(x) = x^3 - \dfrac{3}{400}x = x(x^2 - \dfrac{3}{400})$
The y-intercept (also an x-intercept) is $(0, 0)$ and there are also x-intercepts at $x = \pm\sqrt{3}/20$.
$f'(x) = 3x^2 - \dfrac{3}{400}$
The critical numbers are $x = \pm 1/20$.
$f''(x) = 6x$, so $x = -1/20$ is a local maximum and $x = 1/20$ is a local minimum. $f(x)$ is increasing on

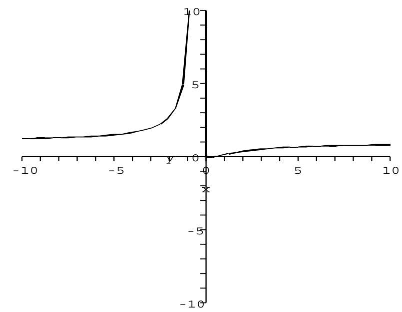
$(-\infty, -1/20)$ and $(1/20, \infty)$ and decreasing on $(-1/20, 1/20)$. It is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$, with an inflection point at $x = 0$.
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



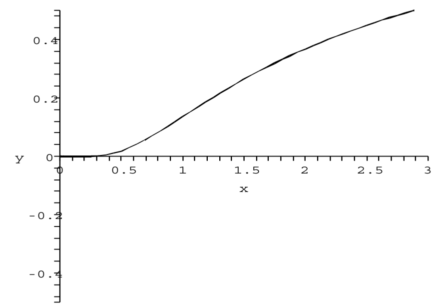
17.

$$f(x) = e^{-2/x}$$
$$f'(x) = e^{-2/x}\left(\frac{2}{x^2}\right) = \frac{2}{x^2}e^{-2/x}$$
$$f''(x) = \frac{-4}{x^3}e^{-2/x} + \frac{2}{x^2}e^{-2/x}\left(\frac{2}{x^2}\right)$$
$$= \frac{4}{x^4}e^{-2/x} - \frac{4}{x^3}e^{-2/x}$$

$f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$
$f''(x) > 0$ on $(-\infty, 0) \cup (0, 1)$
$f''(x) < 0$ on $(1, \infty)$
f increasing on $(-\infty, 0)$ and on $(0, \infty)$, concave up on $(-\infty, 0) \cup (0, 1)$, concave down on $(1, \infty)$, inflection point at $x = 1$. f is undefined at $x = 0$.
$\lim_{x\to 0^+} e^{-2/x} = \lim_{x\to 0^+} \frac{1}{e^{2/x}} = 0$ and
$\lim_{x\to 0^-} e^{-2/x} = \infty$
So f has a vertical asymptote at $x = 0$. $\lim_{x\to\infty} e^{-2/x} = \lim_{x\to-\infty} e^{-2/x} = 1$
So f has a horizontal asymptote at $y = 1$.
Global graph of $f(x)$:



Local graph of $f(x)$:



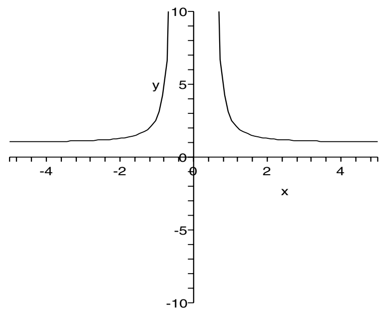
18. $f(x) = e^{1/x^2}$
The function has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as x approaches 0 from the right or left. There is a horizontal asymptote of $y = 1$ as $x \to \pm\infty$.

$$f'(x) = \frac{-2}{x^3} \cdot e^{1/x^2}$$

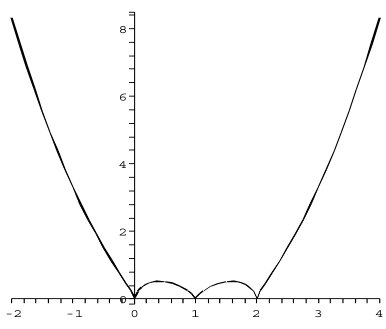
$f'(x) > 0$ for $x < 0$, so $f(x)$ is increasing on $(-\infty, 0)$ and $f'(x) < 0$ for $x > 0$, so $f(x)$ is decreasing on $(-\infty, 0)$.

$$f''(x) = \frac{2e^{1/x^2}(3x^2 + 2)}{x^6}$$

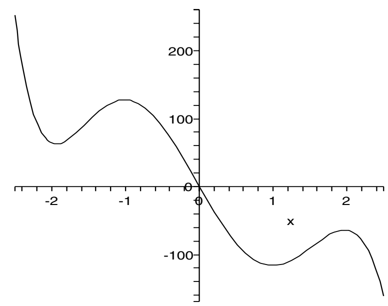
is positive for all $x \neq 0$, so $f(x)$ is concave up for all $x \neq 0$.



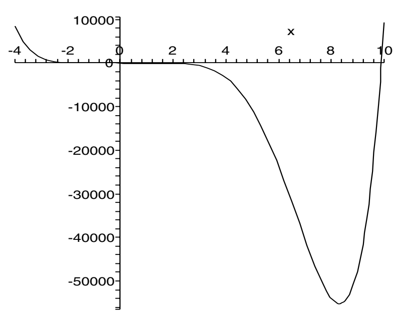
19. $f(x) = (x^3 - 3x^2 + 2x)^{2/3}$
$f'(x) = \dfrac{2(3x^2 - 6x + 2)}{3(x^3 - 3x^2 + 2x)^{1/3}}$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$f''(x) = \dfrac{18x^4 - 72x^3 + 84x^2 - 24x - 8}{9(x^3 - 3x^2 + 2x)^{4/3}}$
with critical numbers $x = 0$, 1 and 2 and $x \approx -0.1883$ and 2.1883. $f''(x)$ changes sign at these last two values, so these are inflection points. The Second Derivative test shows that $x = \dfrac{3 \pm \sqrt{3}}{3}$ are both maxima. Local minima occur at $x = 0$, 1 and 2.
$f(x) \to \infty$ as $x \to \pm\infty$.



20. $f(x) = x^6 - 10x^5 - 7x^4 + 80x^3 + 12x^2 - 192x$
$f(x) \to \infty$ as $x \to \pm\infty$.
$f'(x) = 6x^5 - 50x^4 - 28x^3 + 240x^2 + 24x - 192$
Critical numbers at approximately $x = -1.9339$, -1.0129, 1, 1.9644, and 8.3158.
$f''(x) = 30x^4 - 200x^3 - 84x^2 + 480x + 24$
Critical numbers at approximately $x = -1.5534$, -0.0496, 1.5430, and 6.7267, and changes sign at each of these values, so these are inflection points. The Second Derivative Test shows that $x = -1.9339$, 1, and 8.3158 are local minima, and $x = -1.0129$ and 1.9644 are local maxima. The extrema near $x = 0$ look like this:



The inflection points, and the global behavior of the function can be seen on the following graph.



21. $f(x) = \dfrac{x^2 + 1}{3x^2 - 1}$
Note that $x = \pm\sqrt{1/3}$ are not in the domain of the function, but yield vertical asymptotes.
$$f'(x) = \frac{2x(3x^2 - 1) - (x^2 + 1)(6x)}{(3x^2 - 1)^2}$$
$$= \frac{(6x^3 - 2x) - (6x^3 + 6x)}{(3x^2 - 1)^2}$$

$= \dfrac{-8x}{(3x^2-1)^2}$

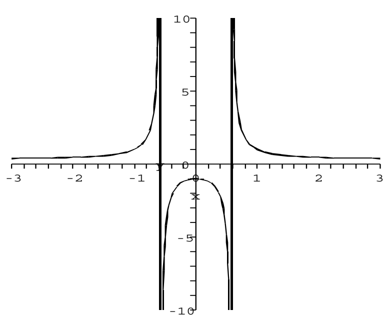
So the only critical point is $x = 0$.

$f'(x) > 0$ for $x < 0$
$f'(x) < 0$ for $x > 0$
so f is increasing on $(-\infty, -\sqrt{1/3})$ and on $(-\sqrt{1/3}, 0)$; decreasing on $(0, \sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$. Thus there is a local max at $x = 0$.

$f''(x) = 8 \cdot \dfrac{9x^2+1}{(3x^2-1)^3}$

$f''(x) > 0$ on $(-\infty, -\sqrt{1/3}) \cup (\sqrt{1/3}, \infty)$
$f''(x) < 0$ on $(-\sqrt{1/3}, \sqrt{1/3})$
Hence f is concave up on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$; concave down on $(-\sqrt{1/3}, \sqrt{1/3})$.

Finally, when $|x|$ is large, the function approached $1/3$, so $y = 1/3$ is a horizontal asymptote.



22. $f(x) = \dfrac{2x^2}{x^3+1}$

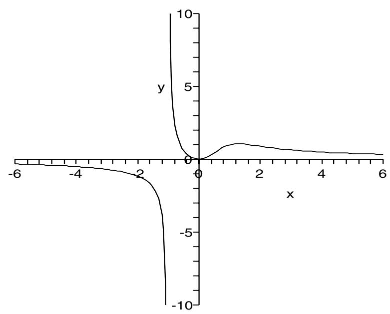
The function has a vertical asymptote at $x = -1$, and horizontal asymptote $y = 0$.

$$f'(x) = \frac{2x(2-x^3)}{(x^3+1)^2}.$$

The critical numbers are $x = 0$ and $x = \sqrt[3]{2}$.

$$f''(x) = \frac{4(x^6-7x^3+1)}{(x^3+1)^3}$$

Critical numbers at approximately $x = 0.5264$ and $x = 1.8995$, and changes sign at these values, so these are inflection points. The Second Derivative Test shows that $x = 0$ is a local minimum, and $x = \sqrt[3]{2}$ is a local maximum.



23. $f(x) = \dfrac{5x}{x^3-x+1}$

Looking at the graph of $x^3 - x + 1$, we see that there is one real root, at approximately -1.325; so the domain of the function is all x except for this one point, and $x = -1.325$ will be a vertical asymptote. There is a horizontal asymptote of $y = 0$.

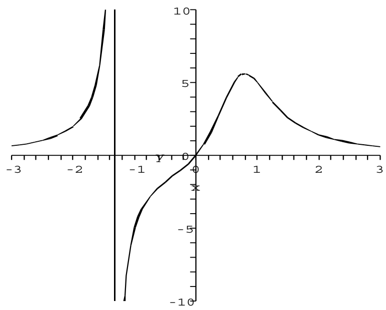
$$f'(x) = 5\frac{1-2x^3}{(x^3-x-1)^2}$$

The only critical point is $x = \sqrt[3]{1/2}$. By the first derivative test, this is a local max.

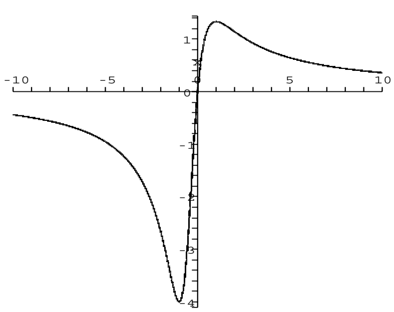
$$f''(x) = 10\frac{3x^5+x^3-6x^2+1}{(x^3-x+1)^3}$$

The numerator of f'' has three real roots, which are approximately $x = -.39018$, $x = .43347$, and $x = 1.1077$.
$f''(x) > 0$ on $(-\infty, -1.325) \cup (-.390, .433) \cup (1.108, \infty)$
$f''(x) < 0$ on $(-1.325, -.390) \cup (.433, 1.108)$
So f is concave up on $(-\infty, -1.325) \cup (-.390, .433) \cup (1.108, \infty)$ and concave down on $(-1.325, -.390) \cup$

$(.433, 1.108)$. Hence $x = -.39018$, $x = .43347$, and $x = 1.1077$ are inflection points.



24. $f(x) = \dfrac{4x}{x^2+x+1}$
The function has horizontal asymptote at $y = 0$.
$f'(x) = \dfrac{-4(x^2-1)}{(x^2+x+1)^2}$
There are critical numbers at $x = \pm 1$.
$f''(x) = \dfrac{8(x^3-3x-1)}{(x^2+x+1)^3}$
with critical numbers at approximately $x = -1.5321$, -0.3473, and 1.8794. $f''(x)$ changes sign at these values, so these are inflection points. The Second Derivative test shows that $x = -1$ is a minimum, and $x = 1$ is a maximum.



25. $f(x) = x^2\sqrt{x^2-9}$
f is undefined on $(-3, 3)$.
$f'(x) =$
$2x\sqrt{x^2-9} + x^2\left(\dfrac{1}{2}(x^2-9)^{-1/2}\cdot 2x\right)$

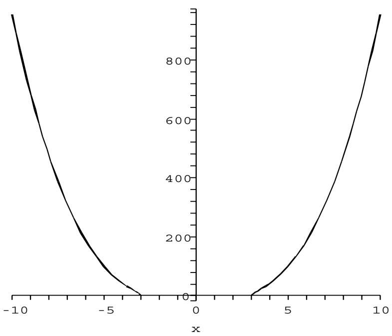
$$= 2x\sqrt{x^2-9} + \frac{x^3}{\sqrt{x^2-9}}$$
$$= \frac{2x(x^2-9)+x^3}{\sqrt{x^2-9}}$$
$$= \frac{3x^3-18x}{\sqrt{x^2-9}} = \frac{3x(x^2-6)}{\sqrt{x^2-9}}$$
$$= \frac{3x(x+\sqrt{6})(x-\sqrt{6})}{\sqrt{x^2-9}}$$

Critical points ± 3. (Note that f is undefined at $x = 0$, $\pm\sqrt{6}$.)

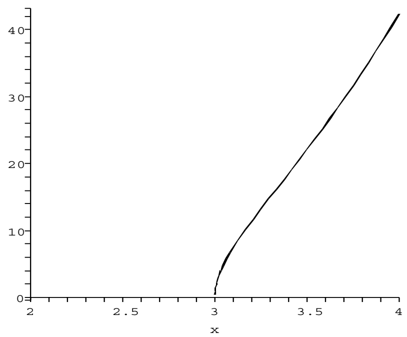
$$f''(x) = \frac{(9x^2-18)\sqrt{x^2-9}}{x^2-9} - \frac{(3x^3-18x)\cdot\frac{1}{2}(x^2-9)^{-1/2}\cdot 2x}{x^2-9}$$
$$= \frac{(9x^2-18)(x^2-9) - x(3x^3-18x)}{(x^2-9)^{3/2}}$$
$$= \frac{(6x^4-81x^2+162)}{(x^2-9)^{3/2}}$$

$f''(x) = 0$ when
$$x^2 = \frac{81 \pm \sqrt{81^2 - 4(6)(162)}}{2(6)}$$
$$= \frac{81 \pm \sqrt{2673}}{12} = \frac{1}{4}(27 \pm \sqrt{297})$$

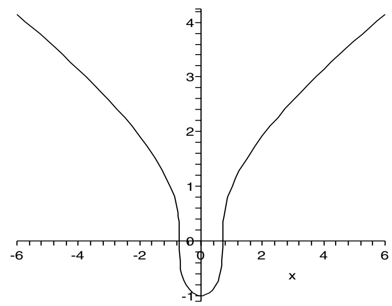
So $x \approx \pm 3.325$ or $x \approx \pm 1.562$, but these latter values are not in the same domain. So only ± 3.325 are potential inflection points.
$f'(x) > 0$ on $(3, \infty)$
$f'(x) < 0$ on $(-\infty, -3)$
$f''(x) > 0$ on $(-\infty, -3.3) \cup (3.3, \infty)$
$f''(x) < 0$ on $(-3.3, 3) \cup (3, 3.3)$
f is increasing on $(3, \infty)$, decreasing on $(-\infty, -3)$, concave up on $(-\infty, -3.3) \cup (3.3, \infty)$, concave down on $(-3.3, -3) \cup (3, 3.3)$. $x = \pm 3.3$ are inflection points.
Global graph of $f(x)$:



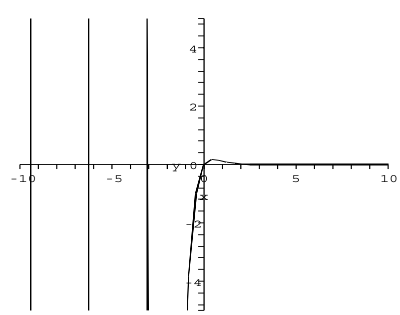
Local graph of $f(x)$:



26. $f(x) = \sqrt[3]{2x^2 - 1}$
$f'(x) = \dfrac{4x}{3(2x^2-1)^{2/3}}$
$f'(x) = 0$ at $x = 0$ and is undefined at $x = \pm\sqrt{1/2}$.
$f''(x) = \dfrac{-4(2x^2+3)}{9(2x^2-1)^{5/3}}$
$f''(x)$ is never 0, and is undefined where f' is. The function changes concavity at $x = \pm\sqrt{1/2}$, so these are inflection points. The slope does not change at these values, so they are not extrema. The Second Derivative Test shows that $x = 0$ is a minimum.
$f(x) \to \infty$ as $x \to \pm\infty$.

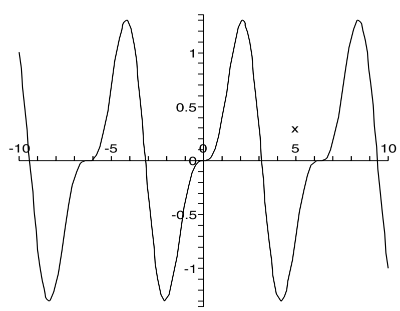


27. $f(x) = e^{-2x} \sin x$
$f'(x) = e^{-2x}(\cos x - 2\sin x)$
$f''(x) = e^{-2x}(3\sin x - 4\cos x)$
$f'(x) = 0$ when $\cos x = 2\sin x$; that is, when $\tan x = 1/2$; that is, when $x = k\pi + \tan^{-1}(1/2)$, where k is any integer.
$f'(x) < 0$, and f is decreasing, on intervals of the form $(2k\pi + \tan^{-1}(\frac{1}{2}), (2k+1)\pi + \tan^{-1}(\frac{1}{2}))$
$f'(x) > 0$ and f is increasing, on intervals of the form $((2k-1)\pi + \tan^{-1}(\frac{1}{2}), 2k\pi + \tan^{-1}(\frac{1}{2}))$
Hence f has a local max at $x = 2k\pi + \tan^{-1}(1/2)$ and a local min at $x = (2k+1)\pi + \tan^{-1}(1/2)$.
$f''(x) = 0$ when $3\sin x = 4\cos x$; that is, when $\tan x = 4/3$; that is, when $x = k\pi + \tan^{-1}(4/3)$. The sign of f'' changes at each of these points, so all of them are inflection points.

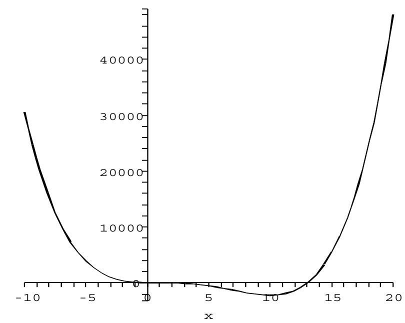


28. $f(x) = \sin x - \frac{1}{2}\sin 2x$
$f'(x) = \cos x - \cos 2x$
$f'(x) = 0$ when $x = 2k\pi$, $2\pi/3 + 2k\pi$,

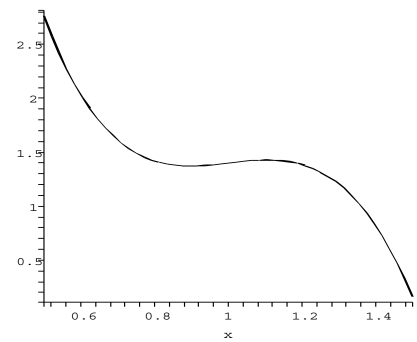
or $4\pi/3 + 2k\pi$.
$f''(x) = -\sin x + 2\sin 2x$
$f''(x) = 0$ when $x = 0$, π and approximately ± 1.3181, and the pattern repeats with period 2π. f'' changes sign at each of these values, so these are inflection points. The First Derivative Test shows that $x = 2k\pi$ is neither a minimum nor a maximum. The Second Derivative Test shows that the other critical numbers are extrema that alternate between minima and maxima.



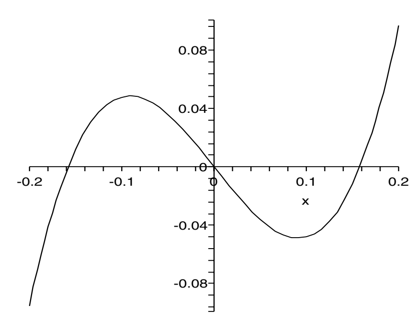
29. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
$f'(x) = 4x^3 - 48x^2 + 84x - 39.6$
$f''(x) = 12x^2 - 96x + 84$
$= 12(x^2 - 8x + 7)$
$= 12(x - 7)(x - 1)$
$f'(x) > 0$ on $(.8952, 1.106) \cup (9.9987, \infty)$
$f'(x) < 0$ on $(-\infty, .8952) \cup (1.106, 9.9987)$
$f''(x) > 0$ on $(-\infty, 1) \cup (7, \infty)$
$f''(x) < 0$ on $(1, 7)$
f is increasing on $(.8952, 1.106)$ and on $(9.9987, \infty)$, decreasing on $(-\infty, .8952)$ and on $(1.106, 9.9987)$, concave up on $(-\infty, 1) \cup (7, \infty)$, concave down on $(1, 7)$, $x = .8952$, 9.9987 are local min, $x = 1.106$ is local max, $x = 1, 7$ are inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.
Global graph of $f(x)$:



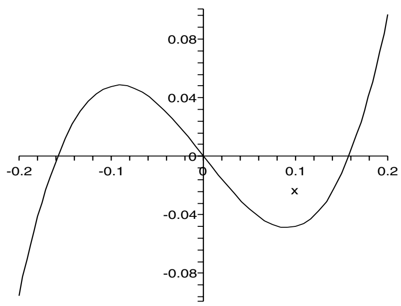
Local graph of $f(x)$:



30. $f(x) = x^4 + 32x^3 - 0.02x^2 - 0.8x$
$f'(x) = 4x^3 + 96x^2 - 0.04x - 0.8$
$f'(x) = 0$ at approximately $x = -24$, -0.09125, and 0.09132.
$f''(x) = 12x^2 + 192x - 0.04$
$f''(x) = 0$ at approximately $x = 16.0002$ and 0.0002, and changes sign at these values, so these are inflection points. The Second Derivative Test shows that $x = -24$ and 0.09132 are minima, and that $x = -0.09125$ is a maxima. The extrema near $x = 0$ look like this:



The global behavior looks like this:



31. $f(x) = \dfrac{25 - 50\sqrt{x^2 + 0.25}}{x}$

$$= 25\left(\frac{1 - 2\sqrt{x^2 + 0.25}}{x}\right)$$

$$= 25\left(\frac{1 - \sqrt{4x^2 + 1}}{x}\right)$$

Note that $x = 0$ is not in the domain of the function.

$$f'(x) = 25\left(\frac{1 - \sqrt{4x^2 + 1}}{x^2\sqrt{4x^2 + 1}}\right)$$

We see that there are no critical points. Indeed, $f' < 0$ wherever f is defined. One can verify that

$f''(x) > 0$ on $(0, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$
Hence the function is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

$$\lim_{x\to\infty} \frac{25 - 50\sqrt{x^2 + 0.25}}{x}$$

$$= \lim_{x\to\infty} \frac{25}{x} - \frac{50\sqrt{x^2 + 0.25}}{x}$$

$$= \lim_{x\to\infty} 0 - 50\frac{x\sqrt{1 + \frac{0.25}{x^2}}}{x}$$

$$= \lim_{x\to\infty} -50\sqrt{1 + \frac{0.25}{x^2}} = -50$$

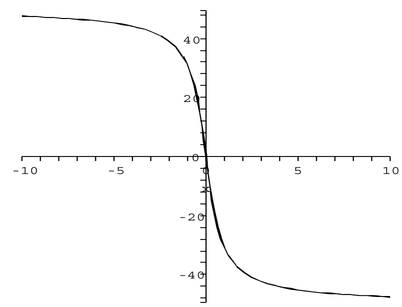
$$\lim_{x\to-\infty} \frac{25 - 50\sqrt{x^2 + 0.25}}{x}$$

$$= \lim_{x\to\infty} \frac{25}{x} - \frac{50\sqrt{x^2 + 0.25}}{x}$$

$$= \lim_{x\to-\infty} 0 - 50\frac{(-x)\sqrt{1 + \frac{0.25}{x^2}}}{x}$$

$$= \lim_{x\to\infty} 50\sqrt{1 + \frac{0.25}{x^2}} = 50$$

So f has horizontal asymptotes at $y = 50$ and $y = -50$.



32. $f(x) = \tan^{-1}\left(\dfrac{1}{x^2 - 1}\right)$

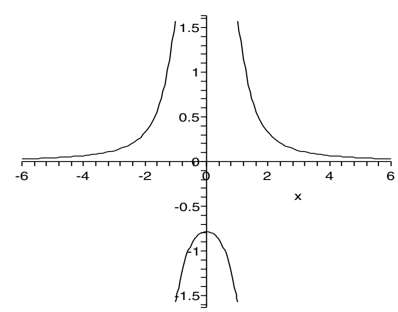
The function has horizontal asymptote $y = 0$, and is undefined at $x = \pm 1$.

$$f'(x) = \frac{-2x}{x^4 - 2x^2 + 2}$$

$f'(x) = 0$ only when $x = 0$.

$$f''(x) = \frac{2(3x^4 - 2x^2 - 2)}{(x^4 - 2x^2 + 2)^2}$$

$f''(x) = 0$ at approximately $x = \pm 1.1024$ and changes sign there, so these are inflection points (very easy to miss by looking at the graph). The Second Derivative Test shows that $x = 0$ is a local maximum.



33. $f(x) = x^4 + cx^2$
$f'(x) = 4x^3 + 2cx$
$f''(x) = 12x^2 + 2c$

$c = 0$: 1 extremum, 0 inflection points
$c < 0$: 3 extrema, 2 inflection points
$c > 0$: 1 extremum, 0 inflection points
$c \to -\infty$: the graph widens and lowers
$c \to +\infty$: the graph narrows

34. $f(x) = x^4 + cx^2 + x$
$f'(x) = 4x^3 + 2cx + 1$
$f''(x) = 12x^2 + 2c$

If c is negative, there will be two solutions to $f'' = 0$, and these will be inflection points. For $c > 0$ there will be no solutions to $f'' = 0$, and no inflection points. For $c = 0$, $f'' = 0$ when $x = 0$, but does not change sign there, so this is not an inflection point. $f' = 0$ has one solution, corresponding to a minimum, for all $c > -1.5$. For $c = -1.5$, there is a second critical point which is neither a minimum nor a maximum. For $c < -1.5$ there are three critical points, two minima and a maximum.

As $c \to \infty$ the curve has one minimum, and narrows.

As $c \to -\infty$, the two minima get farther apart and drop lower. The local maximum approaches $(0, 0)$.

35. $f(x) = \dfrac{x^2}{x^2 + c^2}$
$f'(x) = \dfrac{2c^2 x}{(x^2 + c^2)^2}$
$f''(x) = \dfrac{2c^4 - 6c^2x^2}{(x^2 + c^2)^3}$

If $c = 0$: $f(x) = 1$, except that f is undefined at $x = 0$.

$c < 0$, $c > 0$: horizontal asymptote at $y = 1$, local min at $x = 0$, since the derivative changes sign from negative to positive at $x = 0$; also there are inflection points at $x = \pm c/\sqrt{3}$.

As $c \to -\infty$, $c \to +\infty$: the graph widens.

36. $f(x) = e^{-x^2/c}$
$f'(x) = \dfrac{-2x}{c} \cdot e^{-x^2/c}$
$f''(x) = \dfrac{-2c + 4x^2}{c^2} \cdot e^{-x^2/c}$
For $c > 0$ the graph is a bell curve centered at its maximum point $(0, 1)$, and the inflection points are at $x = \pm\sqrt{c/2}$. As $c \to \infty$, the curve widens.

The function is not defined for $c = 0$.

For $c < 0$, there are no inflection points, and $x = 0$ is a minimum. The graph is cup shaped and widens as $c \to -\infty$.

37. When $c = 0$, $f(x) = \sin(0) = 0$.

Since $\sin x$ is an odd function, $\sin(-cx) = -\sin(cx)$. Thus negative values of c give the reflection through the x-axis of their positive counterparts. For large values of c, the graph looks just like $\sin x$, but with a very small period.

38. When $c = 0$, we have $f(x) = x^2\sqrt{-x^2}$, which is undefined.

Since $x^2\sqrt{c^2 - x^2} = x^2\sqrt{(-c)^2 - x^2}$, the function is the same regardless of whether c is negative or positive. The function is always 0 at $x = 0$ and undefined for $|x| > |c|$. Where it is defined, $f(x) \geq 0$, attaining its minimum at $x = 0$. It reaches its maximum value at $x = \pm\sqrt{2c^2/3}$. At these points, f attains the value $2\sqrt{3}|c|^3/9$. The function looks generally the same as $|c|$ gets large, with the domain and range increasing as $|c|$ does.

39. $f(x) = xe^{-bx}$
$f(0) = 0$
$f(x) > 0$ for $x > 0$
$\lim_{x\to\infty} xe^{-bx} = \lim_{x\to\infty} \frac{x}{e^{bx}} = \lim_{x\to\infty} \frac{1}{be^{bx}} = 0$
(by L'Hôspital's rule)
$f'(x) = e^{-bx}(1 - bx)$, so there is a unique critical point at $x = 1/b$, which must be the maximum. The bigger b is, the closer the max is to the origin. For time since conception, $1/b$ represents the most common gestation time. For survival time, $1/b$ represents the most common life span.

40. From the graph we can count 15 maxima and 16 minima in the range $0 \le x \le 10$. Using a CAS to solve
$f'(x) =$
$-\sin(10x + 2\cos x)(10 - 2\sin x)$
$= 0,$
we find the following values of x at the extrema

Minima	Maxima
0.11549	0.44806
0.80366	1.18055
1.57080	1.96104
2.33793	2.69353
3.02610	3.33776
3.63216	3.91326
4.18477	4.45009
4.71239	7.97469
5.24001	5.51152
5.79261	6.08702
6.39868	6.73125
7.08685	7.46374
7.85398	8.24422
8.62112	8.97672
9.30929	9.62094
9.91535	

41. No: Let $f(x) = \dfrac{x+1}{x^2+1}$. The roots of the denominator are complex, so there are no vertical asymptotes.

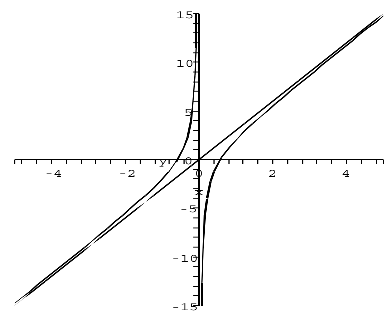
No: Let $f(x) = \dfrac{x^4 - 2x + 3}{x^2 + 1}$. This function goes to ∞ as $x \to \pm\infty$.

42.

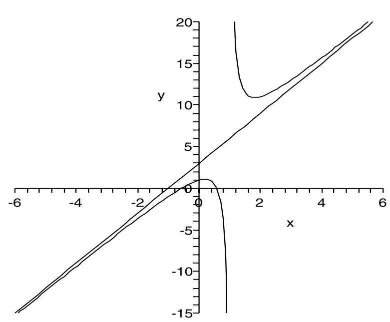
$$\lim_{x\to\infty}\left[\frac{x^4 - x^2 + 1}{x^2 - 1} - x^2\right]$$
$$= \lim_{x\to\infty}\left[\frac{x^4 - x^2 + 1 - x^2(x^2 - 1)}{x^2 - 1}\right]$$
$$= \lim_{x\to\infty}\left[\frac{1}{x^2 - 1}\right] = 0$$

Thus $f(x) = \dfrac{x^4 - x^2 + 1}{x^2 - 1}$ has x^2 as an asymptote.

43. $f(x) = \dfrac{3x^2 - 1}{x} = 3x - \dfrac{1}{x}$
$y = 3x$ is a slant asymptote.

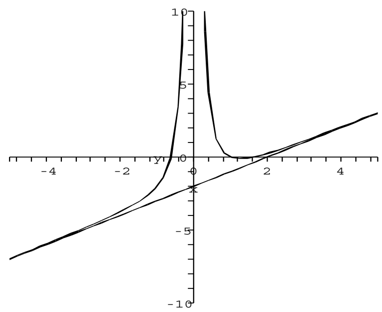


44. $f(x) = \dfrac{3x^2 - 1}{x - 1} = 3x + 3 + \dfrac{2}{x - 1}$,
so the slant asymptote is $y = 3x + 3$.



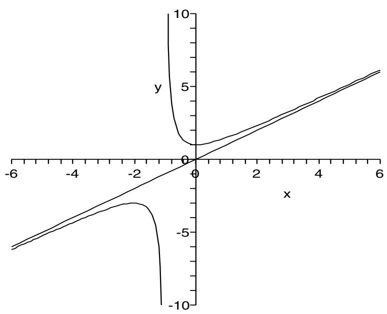
45. $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2} = x - 2 + \dfrac{1}{x^2}$

$y = x - 2$ is a slant asymptote.



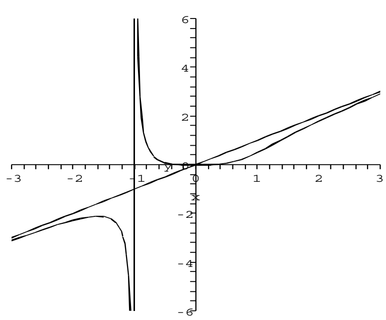
46. $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = x + \dfrac{x - 1}{x^2 - 1}$,

so the slant asymptote is $y = x$.



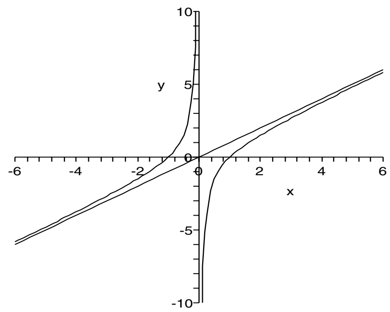
47. $f(x) = \dfrac{x^4}{x^3 + 1} = x - \dfrac{x}{x^3 + 1}$

$y = x$ is a slant asymptote.



48. $f(x) = \dfrac{x^4 - 1}{x^3 + x} = x + \dfrac{-x^2 - 1}{x^3 + x}$,

so the slant asymptote is $y = x$.



49. One possibility:

$$f(x) = \frac{3x^2}{(x-1)(x-2)}$$

50. One possibility:

$$f(x) = \frac{x}{x^2 - 1}$$

51. One possibility:

$$f(x) = \frac{2x}{\sqrt{(x-1)(x+1)}}$$

52. One possibility:

$$f(x) = \frac{2x^2}{(x-1)(x-3)}$$

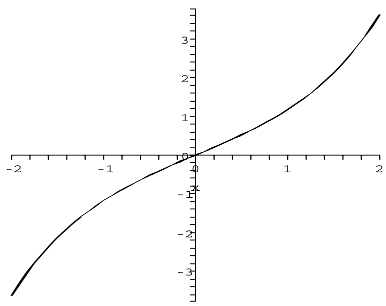
53. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$f'(x) = \dfrac{e^x + e^{-x}}{2}$

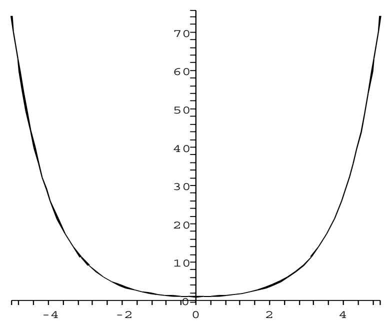
$f'(x) > 0$ for all x so $f(x)$ is always increasing and has no extrema.

$f''(x) = \dfrac{e^x - e^{-x}}{2}$

$f''(x) = 0$ only when $x = 0$ and changes sign here, so $f(x)$ has an inflection point at $x = 0$.

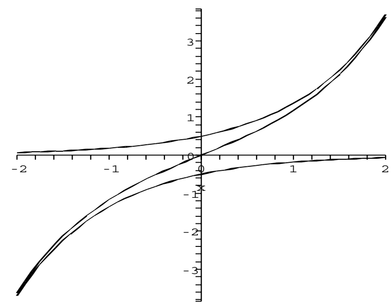


$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$
$$f'(x) = \frac{e^x - e^{-x}}{2}$$
$f'(x) = 0$ only when $x = 0$.
$$f''(x) = \frac{e^x + e^{-x}}{2}$$
$f''(x) > 0$ for all x, so $f(x)$ has no inflection points, but $x = 0$ is a minimum.



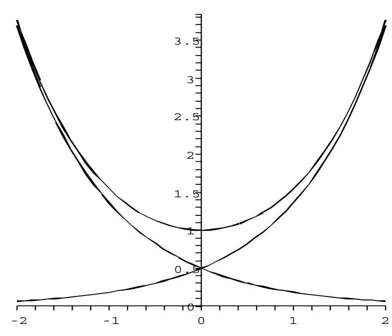
54. For $y = \sinh x$ we need to use $-\frac{1}{2}e^{-x}$ instead of $\frac{1}{2}e^{-x}$. To explain the enveloping behavior, note that

$$\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to-\infty} -\frac{e^{-x}}{2}$$
$$\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to\infty} \frac{e^x}{2}$$



To explain the enveloping behavior for $y = \cosh x$, note that

$$\lim_{x\to-\infty} \cosh x = \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} = \lim_{x\to-\infty} \frac{e^{-x}}{2}$$
$$\lim_{x\to\infty} \cosh x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{2} = \lim_{x\to\infty} \frac{e^x}{2}$$



3.7 Optimization

1. $f(x) = x^2 + 1$ has a minimum at $x = 0$, while $\sin(x^2 + 1)$ has minima where $x^2 + 1 = 3\pi/2 + 2np\pi$.

2. True, since e^x is an increasing function, the x values which make $f(x)$ the smallest will also make $e^{f(x)}$ the smallest.

3.

$$
\begin{aligned}
A &= xy = 1800 \\
y &= \frac{1800}{x} \\
P &= 2x + y = 2x + \frac{1800}{x} \\
P' &= 2 - \frac{1800}{x^2} = 0 \\
2x^2 &= 1800 \\
x &= 30
\end{aligned}
$$

$P'(x) > 0$ for $x > 30$
$P'(x) < 0$ for $0 < x < 30$
So $x = 30$ is min.

$$y = \frac{1800}{x} = \frac{1800}{30} = 60$$

So the dimensions are $30' \times 60'$ and the minimum perimeter is 120 ft.

4. If y is the length of fence opposite the river, and x is the length of the other two sides, then we have the constraint $2x + y = 96$. We wish to maximize $A = xy = x(96 - 2x)$.
$A' = 96 - 4x = 0$ when $x = 24$.
$A'' = -4 < 0$ so this gives a maximum. Reasonable possible values of x range from 0 to 48, and the area is 0 at these extremes. The maximum area is $A = 1152$, and the dimensions are $x = 24$, $y = 48$.

5.

$$
\begin{aligned}
P &= 2x + 3y = 120 \\
3y &= 120 - 2x \\
y &= 40 - \frac{2}{3}x \\
A &= xy \\
A(x) &= x\left(40 - \frac{2}{3}x\right) \\
A'(x) &= 1\left(40 - \frac{2}{3}x\right) + x\left(-\frac{2}{3}\right) \\
&= 40 - \frac{4}{3}x = 0 \\
40 &= \frac{4}{3}x \\
x &= 30
\end{aligned}
$$

$A'(x) > 0$ for $0 < x < 30$
$A'(x) < 0$ for $x > 30$

So $x = 30$ is max, $y = 40 - \frac{2}{3} \cdot 30 = 20$

So the dimensions are $20' \times 30'$.

6. Let x be the length of the sides facing each other and y be the length of the third side. We have the constraint that $xy = 800$, or $y = 800/x$. We also know that $x > 6$ and $y > 10$. The function we wish to minimize is the length of walls needed, or the side length minus the width of the doors. $L = (y - 10) + 2(x - 6) = 800/x + 2x - 22$.
$L' = -800/x^2 + 2 = 0$ when $x = 20$.
$L'' = 1600/x^3 > 0$ when $x = 20$ so this is a minimum. Possible values of x range from 6 to 80. $L(6) = 123.3$, $L(80) = 148$, and $L(20) = 58$. To minimize the length of wall, the facing sides should be 20 feet, and the third side should be 40 feet.

7.

$$\begin{aligned} A &= xy \\ P &= 2x + 2y \\ 2y &= P - 2x \\ y &= \frac{P}{2} - x \\ A(x) &= x\left(\frac{P}{2} - x\right) \\ A'(x) &= 1 \cdot \left(\frac{P}{2} - x\right) + x(-1) \\ &= \frac{P}{2} - 2x = 0 \\ P &= 4x \\ x &= \frac{P}{4} \end{aligned}$$

$A'(x) > 0$ for $0 < x < P/4$
$A'(x) < 0$ for $x > P/4$

So $x = P/4$ is max,

$$y = \frac{P}{2} - x = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

So the dimensions are $\frac{P}{4} \times \frac{P}{4}$. Thus we have a square.

8. We have a rectangle with sides x and y and area $A = xy$, and that we wish to minimize the perimeter,
$P = 2x + 2y = 2x + 2 \cdot \dfrac{A}{x}$.
$P' = 2 - \dfrac{2A}{x^2} = 0$ when $x = \sqrt{A}$.
$P'' = 4A/x^3 > 0$ here, so this is a minimum. Possible values of x range from 0 to ∞. As x approaches these values the perimeter grows without bound. For fixed area, the rectangle with minimum perimeter has dimensions $x = y = \sqrt{A}$, a square.

9.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-1)^2} \\ y &= x^2 \\ d &= \sqrt{x^2 + (x^2-1)^2} \\ &= (x^4 - x^2 + 1)^{1/2} \\ d'(x) &= \frac{1}{2}(x^4 - x^2 + 1)^{-1/2}(4x^3 - 2x) \\ &= \frac{2x(2x^2-1)}{2\sqrt{x^4 - x^2 + 1}} = 0 \end{aligned}$$

$x = 0, \pm\sqrt{1/2}$;
$f(0) = 1$, $f(\sqrt{1/2}) = 3/4$, $f(-\sqrt{1/2}) = \frac{3}{4}$;
Thus $x = \pm\sqrt{1/2}$ are min, and the points on $y = x^2$ closest to $(0, 1)$ are $(\sqrt{1/2}, 1/2)$ and $(-\sqrt{1/2}, 1/2)$.

10. Points on the curve $y = x^2$ can be written (x, x^2). The distance from such a point to $(3, 4)$ is

$$\begin{aligned} D &= \sqrt{(x-3)^2 + (x^2-4)^2} \\ &= \sqrt{x^4 - 7x^2 - 6x + 25}. \end{aligned}$$

We numerically approximate the solution of

$$D' = \frac{2x^3 - 7x - 3}{\sqrt{x^4 - 7x^2 - 6x + 25}} = 0$$

to be $x \approx 2.05655$, and two negative solutions. The negative critical numbers clearly do not minimize the distance. The closest point is approximately $(2.05655, 4.22940)$.

11.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-0)^2} \\ y &= \cos x \\ d &= \sqrt{x^2 + \cos^2 x} \\ d'(x) &= \frac{2x - 2\cos x \sin x}{2\sqrt{x^2 + \cos^2 x}} = 0 \\ x &= \cos x \sin x \\ x &= 0 \end{aligned}$$

So $x = 0$ is min and the point on $y = \cos x$ closest to $(0, 0)$ is $(0, 1)$.

12. Points on the curve $y = \cos x$ can be written $(x, \cos x)$. The distance from such a point to $(1, 1)$ is

$$\begin{aligned} D &= \sqrt{(x-1)^2 + (\cos x - 1)^2} \\ &= \sqrt{x^2 - 2x + \cos^2 x - 2\cos x + 2} \end{aligned}$$

We numerically approximate the solution of

$$\begin{aligned} D' &= \frac{x - 1 - \cos x \sin x + \sin x}{\sqrt{x^2 - 2x + \cos^2 x - 2\cos x + 2}} \\ &= 0 \end{aligned}$$

to be $x \approx 0.789781$. The First or Second Derivative Test shows that this is a minimum distance. The closest point is approximately $(0.789781, 0.704001)$.

13. For $(0, 1)$, $(\sqrt{1/2}, 1/2)$ on $y = x^2$, we have
$y' = 2x$, $y'(\sqrt{1/2}) = 2 \cdot \sqrt{1/2} = \sqrt{2}$ and

$$m = \frac{\frac{1}{2} - 1}{-\sqrt{\frac{1}{2}} - 0} = \frac{1}{\sqrt{2}}.$$

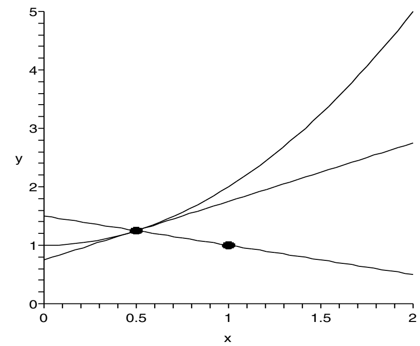
For $(0, 1)$, $(-\sqrt{1/2}, 1/2)$ on $y = x^2$, we have
$y'(-\sqrt{1/2}) = 2(-\sqrt{1/2}) = -\sqrt{2}$ and

$$m = \frac{\frac{1}{2} - 1}{-\sqrt{\frac{1}{2}} - 0} = \frac{1}{\sqrt{2}}.$$

For $(3, 4)$, $(2.06, 4.2436)$ on $y = x^2$, we have $y'(2.06) = 2(2.06) = 4.12$ and

$$m = \frac{4.2436 - 4}{2.06 - 3} = -0.2591 \approx -\frac{1}{4.12}.$$

14. Consider a curve, a point not on the curve, and a point on the curve.



If the line connecting the points is not perpendicular to the tangent line to the curve, then the point on the curve can move closer to the point off the curve by moving in the direction of the acute angle on the same side as the point off the curve. When the lines are perpendicular, the points on the curve nearby are not closer to the point off the curve.

15.

$$\begin{aligned} V &= l \cdot w \cdot h \\ V(x) &= (10 - 2x)(6 - 2x) \cdot x,\ 0 \le x \le 3 \\ V'(x) &= -2(6 - 2x) \cdot x + (10 - 2x)(-2) \cdot x \\ &\quad + (10 - 2x)(6 - 2x) \\ &= 60 - 64x + 12x^2 \\ &= 4(3x^2 - 16x + 15) \\ &= 0 \\ x &= \frac{16 \pm \sqrt{(-16)^2 - 4 \cdot 3 \cdot 15}}{6} \\ &= \frac{8}{3} \pm \frac{\sqrt{19}}{3} \\ x &= \frac{8}{3} + \frac{\sqrt{19}}{3} > 3. \end{aligned}$$

$V'(x) > 0$ for $x < 8/3 - \sqrt{19}/3$
$V'(x) < 0$ for $x > 8/3 - \sqrt{19}/3$

So $x = \dfrac{8}{3} - \dfrac{\sqrt{19}}{3}$ is a max.

16. If we cut squares out of the corners of a 12” by 16” sheet and fold it into a box, the volume of the resulting box will be

$$V = x(12-2x)(16-2x) \\ = 4x^3 - 56x^2 + 192x,$$

where the value of x must be between 0 and 6.

$$V' = 12x^2 - 112x + 192 = 0$$

when $x = \frac{14\pm2\sqrt{13}}{3} \approx 7.07$ and 2.26. The critical value $x = \frac{14+2\sqrt{13}}{3}$ is outside of the reasonable range. The volume is 0 when x is 0 or 6. The First Derivative Test shows that $x = \frac{14-2\sqrt{13}}{3}$ gives the maximum volume.

17. Let x be the distance from the connection point to the easternmost development. Then $0 \le x \le 5$.

$$\begin{aligned} f(x) &= \sqrt{3^2+(5-x)^2} + \sqrt{4^2+x^2}, \\ &\quad 0 \le x \le 5 \\ f'(x) &= -(9+(5-x)^2)^{-1/2}(5-x) \\ &\quad + \frac{1}{2}(16+x^2)^{-1/2}(2x) \\ &= \frac{x-5}{\sqrt{9+(5-x)^2}} + \frac{x}{\sqrt{16+x^2}} \\ &= 0 \\ x &= \frac{20}{7} \approx 2.857 \\ f(0) &= 4+\sqrt{34} \approx 9.831 \\ f\left(\frac{20}{7}\right) &= \sqrt{74} \approx 8.602 \\ f(5) &= 3+\sqrt{41} \approx 9.403 \end{aligned}$$

So $x = 20/7$ is minimum. The length of new line at this point is approximately 8.6 miles. Since $f(0) \approx 9.8$ and $f(5) \approx 9.4$, the water line should be 20/7 miles west of the second development.

18. Say the pipeline intersects the shore at a distance x from the closest point on the shore to the oil rig. Then x will be between 0 and 8. The length of underwater pipe is then $W = \sqrt{x^2+25^2}$, and the length of pipe constructed on land will be $L = \sqrt{(8-x)^2+5^2}$. The total cost will be $C = 50W + 20L$.

We numerically solve

$$C' = \frac{50x}{\sqrt{625+x^2}} + \frac{10(2x-16)}{\sqrt{x^2-16x+89}} = 0$$

to find $x \approx 5.108987$. The first derivative test shows that this gives a minimum. The cost at this value is \$1391 thousand. The cost when $x = 0$ is \$1439 thousand, and the cost when $x = 8$ is \$1412 thousand, so $x = 5.108987$ gives the absolute minimum cost.

19.

$$\begin{aligned} C(x) &= 5\sqrt{16+x^2} + 2\sqrt{36+(8-x)^2} \\ &\quad 0 \le x \le 8 \\ C(x) &= 5\sqrt{16+x^2} + 2\sqrt{100-16x+x^2} \\ C'(x) &= 5\left(\frac{1}{2}\right)(16+x^2)^{-1/2}\cdot 2x \\ &\quad + 2\left(\frac{1}{2}\right)(100-16x+x^2)^{-1/2}(2x-16) \\ &= \frac{5x}{\sqrt{16+x^2}} + \frac{2x-16}{\sqrt{100-16x+x^2}} \\ &= 0 \\ x &\approx 1.2529 \\ C(0) &= 40 \\ C(1.2529) &\approx 39.0162 \\ C(8) &\approx 56.7214 \end{aligned}$$

The highway should emerge from the marsh 1.2529 miles east of the bridge. If we build a straight line to the interchange, we have $x = (3.2)$.

Since $C(3.2) - C(1.2529) \approx 1.963$, we save \$1.963 million.

20. Say the road intersects the edge of the marsh at a distance x from the closest point on the edge to the bridge. Then x will be between 0 and 8. The length of road over marsh is now $M = \sqrt{x^2+4^2}$, and the length of road constructed on dry land will be $L = \sqrt{(8-x)^2+6^2}$. The total cost will be $C = 6M + 2L$.

We numerically solve

$$C' = \frac{6x}{\sqrt{16+x^2}} + \frac{2x-16}{\sqrt{x^2-16x+100}} = 0$$

to find $x \approx 1.04345$. The first derivative test shows that this gives a minimum. The cost at this value is \$43.1763 million. The cost when we use the solution $x = 1.2529$ from exercise 19 is \$43.2078 million, so the increase is \$31,500.

21.

$$C(x) = 5\sqrt{16+x^2} + 3\sqrt{36+(8-x)^2}$$
$$0 \le x \le 8$$
$$C'(x) = \frac{5x}{\sqrt{16+x^2}} + \frac{3x-24}{\sqrt{100-16x+x^2}}$$

Setting $C'(x) = 0$ yields
$x \approx 1.8941$
$C(0) = 50$
$C(1.8941) \approx 47.8104$
$C(8) \approx 62.7214$

The highway should emerge from the marsh 1.8941 miles east of the bridge. So if we must use the path from exercise 21, the extra cost is
$C(1.2529) - C(1.8941)$
$= 48.0452 - 47.8104 = 0.2348$
or about \$234.8 thousand.

22. Say the contestant swims to a point on shore distance x from the closest point on shore. Then x will be between 0 and 3. The distance travelled in water will be $W = \sqrt{2^2+x^2}$ and the distance travelled on land will be $L = \sqrt{(3-x)^2+2^2}$. The total time will be $T = W/4 + L/10$.

We numerically solve

$$T' = \frac{x}{4\sqrt{4+x^2}} + \frac{2x-6}{20\sqrt{x^2-6x+13}} = 0$$

to find $x \approx .6407871171$. The total time in the water will be 0.5256 hours. The total time on land will be 0.3087 hours.

23.

$$\begin{aligned} T(x) &= \frac{\sqrt{1+x^2}}{v_1} + \frac{\sqrt{1+(2-x)^2}}{v_2} \\ T'(x) &= \frac{1}{v_1}\cdot\frac{1}{2}(1+x^2)^{-1/2}\cdot 2x \\ &\quad + \frac{1}{v_2}(1+(2-x)^2)^{-1/2}\cdot(2-x)(-1) \\ &= \frac{x}{v_1\sqrt{1+x^2}} + \frac{x-2}{v_2\sqrt{1+(2-x)^2}} \end{aligned}$$

Note that

$$\begin{aligned} T'(x) &= \frac{1}{v_1}\cdot\frac{x}{\sqrt{1+x^2}} \\ &\quad - \frac{1}{v_2}\cdot\frac{(2-x)}{\sqrt{1+(2-x)^2}} \\ &= \frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2 \end{aligned}$$

When $T'(x) = 0$, we have

$$\begin{aligned} \frac{1}{v_1}\sin\theta_1 &= \frac{1}{v_2}\sin\theta_2 \\ \frac{\sin\theta_1}{\sin\theta_2} &= \frac{v_1}{v_2} \end{aligned}$$

24. The distance light travels is

$$D = \sqrt{2^2+x^2} + \sqrt{1^2+(4-x)^2}.$$

We maximize this by solving

$$D' = \frac{x}{\sqrt{4+x^2}} + \frac{2x-8}{2\sqrt{x^2-8x+17}} = 0$$

to find $x = 8/3$. For this value of x, $\theta_1 = \theta_2 = \tan^{-1}(3/4)$. (Or simply note similar triangles.)

25. Cost: $C = 2(2\pi r^2) + 2\pi rh$
Convert from fluid ounces to cubic inches:
$12 \text{ fl oz} = 12 \text{ fl oz} \cdot 1.80469 \text{ in}^3/\text{fl oz}$
$= 21.65628 \text{ in}^3$
Volume: $V = \pi r^2 h$ so

$$h = \frac{V}{\pi r^2} = \frac{21.65628}{\pi r^2}$$

$$C = 4\pi r^2 + 2\pi r\left(\frac{21.65628}{\pi r^2}\right)$$
$$C(r) = 4\pi r^2 + 43.31256r^{-1}$$
$$C'(r) = 8\pi r - 43.31256r^{-2}$$
$$= \frac{8\pi r^3 - 43.31256}{r^2}$$

$$r = \sqrt[3]{\frac{43.31256}{8\pi}} = 1.1989''$$

when $C'(r) = 0$.
$C'(r) < 0$ on $(0, 1.1989)$
$C'(r) > 0$ on $(1.1989, \infty)$
Thus $r = 1.1989$ minimizes the cost.

$$h = \frac{21.65628}{\pi(1.1989)^2} = 4.7957''$$

26. If the top and bottom of the cans are 2.23 times as thick as the sides, then the new cost function will be

$$C(r) = 2\pi\left(2.23r^2 + \frac{21.65628}{\pi r}\right).$$

Then $C'(r) = 2\pi(4.46r - \frac{21.65628}{\pi r^2}) = 0$ when $r = \sqrt[3]{\frac{21.65628}{4.46\pi}} \approx 1.156$. The First Derivative Test shows this is a minimum, and we can verify that the cost increases without bound as $r \to 0$ and $r \to \infty$.

27.

$$V(r) = cr^2(r_0 - r)$$
$$V'(r) = 2cr(r_0 - r) + cr^2(-1)$$
$$= 2crr_0 - 3cr^2$$
$$= cr(2r_0 - 3r)$$

$V'(r) = 0$ when $r = 2r_0/3$
$V'(r) > 0$ on $(0, 2r_0/3)$
$V'(r) < 0$ on $(2r_0/3, \infty)$
Thus $r = 2r_0/3$ maximizes the velocity.
$r = 2r_0/3 < r_0$, so the windpipe contracts.

28. We wish to minimize

$$E(\theta) = \frac{\csc\theta}{r^4} + \frac{1-\cot\theta}{R^4}.$$

We find

$$E'(\theta) = -\frac{\csc\theta\cot\theta}{r^4} + \frac{1+\cot^2\theta}{R^4}$$
$$= \frac{-\cos\theta R^4 + r^4}{r^4R^4\sin^2\theta}.$$

This is zero when $\cos\theta = r^4/R^4$, so $\theta = \cos^{-1}(r^4/R^4)$. The derivative changes from negative to positive here, so this gives a minimum as desired.

29. $p(x) = \dfrac{V^2x}{(R+x)^2}$

$$p'(x) = \frac{V^2(R+x)^2 - V^2x \cdot 2(R+x)}{(R+x)^4}$$
$$= \frac{V^2R^2 - V^2x^2}{(R+x)^4}$$

$p'(x) = 0$ when $x = R$
$p'(x) > 0$ on $(0, R)$
$p'(x) < 0$ on (R, ∞)
Thus $x = R$ maximizes the power absorbed.

30. If the meter registers 115 volts, then $v = 115\sqrt{2}$. The function $V(t) = v\sin(2\pi ft)$ has amplitude v, so the maximum value of the voltage is $115\sqrt{2}$.

31. $\pi r + 4r + 2w = 8 + \pi$

$w = \dfrac{8 + \pi - r(\pi + 4)}{2}$

$A(r) = \dfrac{\pi r^2}{2} + 2rw$

$= \dfrac{\pi r^2}{2} + r(8 + \pi - r(\pi + 4))$

$= r^2 \left(-4 - \dfrac{\pi}{2}\right) + r(8 + \pi)$

$A'(r) = -2r \left(4 + \dfrac{\pi}{2}\right) + (8 + \pi) = 0$

$A'(r) = 0$ when $r = 1$

$A'(r) > 0$ on $(0, 1)$

$A'(r) < 0$ on $(1, \infty)$

Thus $r = 1$ maximizes the area so $w = \dfrac{8 + \pi - (\pi + 4)}{2} = 2$. The dimensions of the rectangle are 2×2.

32. Let x be the distance from the end at which the wire is cut. Due to symmetry, we may consider $0 \le x \le 1$. We wish to minimize the area of the squares formed by the two pieces. The total area is

$A(x) = \left(\dfrac{x}{4}\right)^2 + \left(\dfrac{2 - x}{4}\right)^2$

$= \dfrac{2x^2 - 4x + 4}{16}.$

We compute

$A'(x) = \dfrac{x}{4} - \dfrac{1}{4} = 0$ when $x = 1$.

$A'' = \dfrac{1}{4} > 0$, so this is a minimum.

We check $A(0) = 1/4$ and $A(1) = 1/8$ and see that cutting the wire in half minimizes the area of the two squares.

33. $l \times w = 92$, $w = 92/l$

$A(l) = (l + 4)(w + 2)$

$= (l + 4)(92/l + 2)$

$= 92 + 368/l + 2l + 8$

$= 100 + 368l^{-1} + 2l$

$A'(l) = -368l^{-2} + 2$

$= \dfrac{2l^2 - 368}{l^2}$

$A'(l) = 0$ when $l = \sqrt{184} = 2\sqrt{46}$

$A'(l) < 0$ on $(0, 2\sqrt{46})$

$A'(l) > 0$ on $(2\sqrt{46}, \infty)$

So $l = 2\sqrt{46}$ minimizes the total area. When $l = 2\sqrt{46}$, $w = \frac{92}{2\sqrt{46}} = \sqrt{46}$.

For the minimum total area, the printed area has width $\sqrt{46}$ in. and length $2\sqrt{46}$ in., and the advertisement has overall width $\sqrt{46} + 2$ in. and overall length $2\sqrt{46} + 4$ in.

34. Let x and y be the width and height of the advertisement. Then $xy = 120$ and $y = 120/x$. We wish to maximize the printed area

$A = (x-2)(y-3) = (x-2)(\dfrac{120}{x} - 3)$

$= 126 - 3x - \dfrac{240}{x}.$

We find $A' = -3 + \dfrac{240}{x^2} = 0$ when $x = 4\sqrt{5}$. The first Derivative Test shows that this is a maximum. The smallest x could be is 2, and this gives $A(2) = 0$. The largest x could be is 40, and this also gives $A(40) = 0$. Thus, we see that the dimensions which maximize the printed area are $x = 4\sqrt{5}$ and $y = 6\sqrt{5}$.

35. Let L represent the length of the ladder. Then from the diagram, it follows that

$L = a \sec\theta + b \csc\theta.$

Therefore,

$$\frac{dL}{d\theta} = a \sec\theta \tan\theta - b \csc\theta \cot\theta$$

$$0 = a \sec\theta \tan\theta - b \csc\theta \cot\theta$$

$$a \sec\theta \tan\theta = b \csc\theta \cot\theta$$

$$\frac{b}{a} = \frac{\sec\theta \tan\theta}{\csc\theta \cot\theta}$$

$$= \frac{1}{\cos\theta} \frac{\sin\theta}{\cos\theta} \frac{\sin\theta}{1} \frac{\sin\theta}{\cos\theta}$$

$$= \tan^3\theta$$

Thus,

$\tan\theta = \sqrt[3]{b/a}$

$\theta = \tan^{-1}\left(\sqrt[3]{b/a}\right)$
$= \tan^{-1}\left(\sqrt[3]{4/5}\right)$
≈ 0.748 rad or 42.87 degrees
Thus, the length of the longest ladder that can fit around the corner is approximately
$L = a\sec\theta + b\csc\theta$
$= 5\sec(0.748) + 4\csc(0.748)$
≈ 12.7 ft

36. From exercise 35, we have that $\theta = \tan^{-1}(\sqrt[3]{b/a})$ is the critical number limiting the length of the ladder. Thus $\tan\theta = b^{1/3}/a^{1/3}$. We can then draw a right triangle with θ as one angle and the length of the side opposite θ equal to $b^{1/3}$ and the length of the side adjacent to θ equal to $a^{1/3}$. By the Pythagorean Theorem, the hypotenuse of this triangle is $(a^{2/3} + b^{2/3})^{1/2}$. From this triangle, we find

$$\sin\theta = \frac{b^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}} \text{ and}$$
$$\cos\theta = \frac{a^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}}$$

so

$$\csc\theta = \frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}} \text{ and}$$
$$\sec\theta = \frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}}.$$

Thus

$$\begin{aligned} L &= a\sec\theta + b\csc\theta \\ &= a\cdot\frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} \\ &\quad + b\cdot\frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}} \\ &= a^{2/3}(a^{2/3}+b^{2/3})^{1/2} \\ &\quad + b^{2/3}(a^{2/3}+b^{2/3})^{1/2} \\ &= (a^{2/3}+b^{2/3})(a^{2/3}+b^{2/3})^{1/2} \\ &= (a^{2/3}+b^{2/3})^{3/2}. \end{aligned}$$

37. Using the result of exercise 36 and solving for b:

$$\begin{aligned} L &= (a^{2/3}+b^{2/3})^{3/2} \\ L^{2/3} &= a^{2/3}+b^{2/3} \\ b^{2/3} &= L^{2/3}-a^{2/3} \\ b &= (L^{2/3}-a^{2/3})^{3/2} \\ &= (8^{2/3}-5^{2/3})^{3/2} \\ &\approx 1.16 \text{ ft} \end{aligned}$$

38. This was already done in exercise 37 while solving for b:

$$b = (L^{2/3}-a^{2/3})^{3/2}.$$

39.

$$\begin{aligned} R(x) &= \frac{35x-x^2}{x^2+35} \\ R'(x) &= -35\frac{x^2+2x-35}{(x^2+35)^2} \\ &= -35\frac{(x-5)(x+7)}{(x^2+35)^2} \end{aligned}$$

Hence the only critical number for $x \geq 0$ is $x = 5$ (that is, 5000 items). This must correspond to the absolute maximum, since $R(0) = 0$ and $R(x)$ is negative for large x. So maximum revenue is $R(5) = 2.5$ (that is, \$2500).

40. To maximize

$$R(x) = \frac{cx-x^2}{x^2+c},$$

we compute

$$R'x = \frac{c(c-2x-x^2)}{(x^2+c)^2}.$$

This is zero when $x^2+2x-c=0$, so

$$x = \frac{-2\pm\sqrt{4+4c}}{2}.$$

The First Derivative Test shows that

$$x = \frac{-2+\sqrt{4+4c}}{2}$$

is a maximum.

41. $Q'(t)$ is efficiency because it represents the number of additional items produced per unit time.
$Q(t) = -t^3 + 12t^2 + 60t$
$Q'(t) = -3t^2 + 24t + 60$
$= 3(-t^2 + 8t + 20)$
This is the quantity we want to maximize.

$Q''(t) = 3(-2t+8)$ so the only critical number is $t = 4$ hours. This must be the maximum since the function $Q'(t)$ is a parabola opening down.

42. The worker's efficiency, Q' is maximized at the point of diminishing returns because at this point Q'' changes from positive to negative. The First Derivative Test applied to Q' shows that Q' has a local maximum at this point. (This assumes that the graph of Q changes from concave up to concave down at the inflection point. If this was reversed, the inflection point would not be a point of diminishing returns, and the efficiency would be minimized at such a point.)

43. Let $C(t)$ be the total cost of the tickets. Then
$C(t) =$(price per ticket)(# of tickets)
$C(t) = (40 - (t - 20))(t)$
$= (60 - t)(t) = 60t - t^2$
for $20 < t < 50$. Then $C'(t) = 60 - 2t$, so $t = 30$ is the only critical number. This must correspond to the maximum since $C(t)$ is a parabola opening down.

44. If each additional ticket over 20 reduces the cost-per-ticket by c dollars, then the total cost for ordering x tickets (with x between 20 and 50) is
$C(x) = (40 - c(x - 20))x$
$= (40 + 20c)x - cx^2.$
This is a downward facing parabola with one maximum at $x = \dfrac{20 + 10c}{c}$.
If we want the maximum cost to be at $x = 50$, we must choose c so that the peak of the parabola is at or to the right of 50. The value of $x = \dfrac{20 + 10c}{c}$ increases as c decreases, and equals 50 when $c = \frac{1}{2}$. Any discount of 50 cents or less will cause the maximum cost to occur when the group orders 50 tickets.

45.

$$\begin{aligned}
R &= \frac{2v^2\cos^2\theta}{g}(\tan\theta - \tan\beta) \\
R'(\theta) &= \frac{2v^2}{g}\left[2\cos\theta(-\sin\theta)(\tan\theta - \tan\beta)\right. \\
&\quad \left. + \cos^2\theta \cdot \sec^2\theta\right] \\
&= \frac{2v^2}{g}\left[-2\cos\theta\sin\theta \cdot \frac{\sin\theta}{\cos\theta}\right. \\
&\quad +2\cos\theta\sin\theta\tan\beta \\
&\quad \left. + \cos^2\theta \cdot \frac{1}{\cos^2\theta}\right] \\
&= \frac{2v^2}{g}\left[-2\sin^2\theta + \sin(2\theta)\tan\beta + 1\right] \\
&= \frac{2v^2}{g}\left[-2\sin^2\theta + \sin(2\theta)\tan\beta\right. \\
&\quad \left. +(\sin^2\theta + \cos^2\theta)\right] \\
&= \frac{2v^2}{g}\left[\sin(2\theta)\tan\beta\right. \\
&\quad \left. +(\cos^2\theta - \sin^2\theta)\right] \\
&= \frac{2v^2}{g}\left[\sin(2\theta)\tan\beta + \cos(2\theta)\right]
\end{aligned}$$

$R'(\theta) = 0$ when

$$\begin{aligned}
\tan\beta &= \frac{-\cos(2\theta)}{\sin(2\theta)} = -\cot(2\theta) \\
&= -\tan\left(\frac{\pi}{2} - 2\theta\right) \\
&= \tan\left(2\theta - \frac{\pi}{2}\right)
\end{aligned}$$

Hence $\beta = 2\theta - \pi/2$, so

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right) = \frac{\beta}{2} + \frac{\pi}{4} = \frac{\beta^\circ}{2} + 45^\circ$$

(a) $\beta = 10^\circ$, $\theta = 50^\circ$

(b) $\beta = 0^\circ$, $\theta = 45^\circ$

(c) $\beta = -10^\circ$, $\theta = 40^\circ$

46. One example: For golf, the results of exercise 45 mean that if we want to maximize distance on a level shot, the ball should be hit so that the initial angle is 45°. If the target is β° above or below the ball, we maximize the distance by raising or lowering the initial angle by $\beta^\circ/2$.

47. $T = \frac{-1}{c}\ln\left(1 - c\cdot\frac{b-a}{v_0}\right)$
$b = 300$, $a = 0$, $v_0 = 125$, $c = 0.1$
$T = \frac{-1}{0.1}\ln\left(1 - 0.1\cdot\frac{300-0}{125}\right)$
$= 2.744$ sec
$T(x) = -10\ln(1 - 0.0008(300-x)) - 10\ln(1 - 0.0008x) + 0.1$
$T'(x) = -10\left(\frac{0.0008}{0.76 + 0.0008x} - \frac{0.0008}{1 - 0.0008x}\right) = 0$
$0.0008(1 - 0.0008x) = 0.0008(0.76 + 0.0008x)$
$T'(x) = 0$ when $x = \frac{1 - 0.76}{0.0016} = 150$ ft. $T'(x) < 0$ on $(0, 150)$
$T'(x) > 0$ on $(150, 300)$
Hence $x = 150$ minimizes the total time.
$T(150) = -10\ln(1 - 0.0008(300-150)) - 10\ln(1 - 0.0008(150)) + 0.1$
$= 2.656$ sec.
So the relay is faster.

If the delay is 0.2 sec, the relay takes longer.

48. If it takes 0.1877 seconds of delay to catch and relay the ball, then the relay and the direct throw take the same amount of time. It is difficult to catch and throw again that fast. Relays are important in baseball for the increased accuracy of shorter throws and the ability of fielders to change targets to react to their opponent's actions.

49. $T(x) = -10\ln\left(1 - 0.1\frac{300-x}{125}\right) - 10\ln\left(1 - 0.1\frac{x}{100}\right) + 0.1$
$= -10(\ln(1 - 0.0008(300-x)) - 10\ln(1 - .001x) + 0.01$
$T'(x) = -10\left(\frac{0.0008}{0.76 + 0.0008x} - \frac{0.001}{1 - 0.001x}\right) = 0$
$0.0008(1 - 0.001x) = 0.001(0.76 + 0.0008x)$
$T'(x) = 0$ when $x = 25$ ft.
$T'(x) < 0$ on $(0, 25)$
$T'(x) > 0$ on $(25, 300)$
Hence $x = 25$ minimizes the total time.
$T(25) = -10\ln(1 - 0.0008(300-25)) - 10\ln(1 - 0.001(25)) + 0.1$
$= 2.838$ sec.
So the relay takes longer. Without the delay, the relay would take 2.738 sec, so a delay of $2.744 - 2.738 = .006$ sec makes the two times equal.

50. For any delay, the best relay is halfway. For a delay of 0.1 s, the time required for a relay throw is $T_r = -20\ln(1 - \frac{15}{v_0}) + 0.1$ and the time required for a direct throw is $T_d = -10\ln(1 - \frac{30}{v_0})$. Solving $T_r = T_d$ for v_0 yields $v_0 = 165$ feet per second.

51. $A = 4xy$
$\frac{dA}{dx} = 4\left(xy' + y\right)$

To determine $y' = \dfrac{dy}{dx}$, use the equation for the ellipse:

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
$$0 = \frac{2x}{a^2} + \frac{2yy'}{b^2}$$
$$\frac{2yy'}{b^2} = -\frac{2x}{a^2}$$
$$y' = -\frac{b^2}{a^2}\frac{x}{y}$$

Substituting this expression for y' into the expression for $\dfrac{dA}{dx}$, we get

$$\begin{aligned}\frac{dA}{dx} &= xy' + y \\ &= x\left(-\frac{b^2}{a^2}\frac{x}{y}\right) + y \\ &= -\frac{b^2}{a^2}\frac{x^2}{y} + y\end{aligned}$$

The area is maximized when its derivative is zero:

$$0 = -\frac{b^2}{a^2}\frac{x^2}{y} + y$$
$$\frac{b^2}{a^2}\frac{x^2}{y} = y$$
$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

Substituting the previous relationship into the equation for the ellipse, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2}$$

and therefore,

$$x = \frac{a}{\sqrt{2}} \quad \text{and} \quad y = \frac{b}{\sqrt{2}}$$

Thus, the maximum area is

$$A = 4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}} = 2ab$$

Since the area of the circumscribed rectangle is $4ab$, the required ratio is

$$2ab : \pi ab : 4ab = 1 : \frac{\pi}{2} : 2$$

52. Let V_c be the volume of the cylinder, h be the height of the cylinder and r the radius of the cylinder so that

$$V_c = h\pi r^2.$$

Let V_s be the volume of the sphere and R be the radius of the sphere so that

$$V_s = \frac{4}{3}\pi R^3.$$

Draw the sphere on coordinate axes with center $(0, 0)$ and inscribe the cylinder. Then draw a right triangle as follows: draw a straight line from the origin to the side of the cylinder (this line has length r, the radius of the cylinder); draw a line from this point to the point where the cylinder meets the sphere (this line has length $h/2$, half the height of the cylinder); connect this point with the origin to create the hypotenuse of the triangle (this line has length R, the radius of the sphere). Thus we see that

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2.$$

Now we have

$$V_s = \frac{4}{3}\pi\left(r^2 + \frac{h^2}{4}\right)^{3/2}.$$

Taking the derivative of both sides with respect to h gives

$$0 = 2\pi\left(r^2 + \frac{h^2}{4}\right)^{1/2}\left(2rr' + \frac{h}{2}\right).$$

Solving for r', we find $r' = -h/4r$. Taking the derivative with respect to h of both sides of the formula for the volume for the cylinder yields

$$\frac{dV_c}{dh} = \pi r^2 + 2h\pi rr'.$$

Plugging in the formula we found for r' gives

$$\begin{aligned}\frac{dV_c}{dh} &= \pi r^2 + 2h\pi r\left(\frac{-h}{4r}\right)\\ &= \pi r^2 - \frac{h^2\pi}{2}.\end{aligned}$$

To maximize the volume of the cylinder, we set this equal to 0 and find that the volume of the cylinder is maximized when $h^2 = 2r^2$. In this case, the formula relating R, r and h above gives

$$h = \sqrt{\frac{4}{3}R^2} = \frac{2R}{\sqrt{3}}.$$

The maximum volume of the cylinder is then

$$\begin{aligned}V_c &= h\pi r^2\\ &= \frac{\pi h^3}{2} = \frac{\pi\left(\frac{2R}{\sqrt{3}}\right)^3}{2}\\ &= \frac{1}{\sqrt{3}}\left(\frac{4}{3}\pi R^3\right)\\ &= \frac{1}{\sqrt{3}}V_s.\end{aligned}$$

53. Suppose that $a = b$ in the isosceles triangle, so that

$$A^2 = s(s-a)(s-b)(s-c) = s(s-a)^2(s-c)$$

Since $s = \frac{1}{2}(a+b+c)$, it follows that $s = \frac{1}{2}(2a+c) = a + \frac{c}{2}$, so that $s - a = \frac{c}{2}$. Thus,

$$\begin{aligned}A^2 &= s\left(\frac{c^2}{4}\right)(s-c)\\ &= \frac{s}{4}\left(sc^2 - c^3\right)\end{aligned}$$

Since s is a constant (it's half of the perimeter), we can now differentiate to get

$$\begin{aligned}2A\frac{dA}{dc} &= \frac{s}{4}\left(2sc - 3c^2\right)\\ 0 &= c(2s-3c)\end{aligned}$$

Thus, the area is maximized when $2s - 3c = 0$, which means $c = \frac{2}{3}s$. Solving for a, we get

$$a = s - \frac{c}{2} = s - \frac{s}{3} = \frac{2}{3}s.$$

Thus, the area is maximized when $a = b = c$; in other words the area is maximized when the triangle is equilateral.

The maximum area is

$$\begin{aligned}A &= \sqrt{s(s-c)^3} = \sqrt{s\left(\frac{s}{3}\right)^3}\\ &= \frac{s^2}{9}\sqrt{3} = \frac{p^2}{36}\sqrt{3}\end{aligned}$$

3.8 Related Rates

1. $V(t) = (\text{depth})(\text{area}) = \frac{\pi}{48}\left[r(t)\right]^2$ (units in cubic feet per min)

$$V'(t) = \frac{\pi}{48}2r(t)r'(t) = \frac{\pi}{24}r(t)r'(t)$$

We are given $V'(t) = \frac{120}{7.5} = 16$.

Hence $16 = \dfrac{\pi}{24}r(t)r'(t)$ so

$$r'(t) = \frac{(16)(24)}{\pi r(t)}.$$

(a) When $r = 100$,
$$\begin{aligned}r'(t) &= \frac{(16)(24)}{100\pi} = \frac{96}{25\pi}\\ &\approx 1.2223 \text{ ft/min},\end{aligned}$$

(b) When $r = 200$,
$$\begin{aligned}r'(t) &= \frac{(16)(24)}{200\pi} = \frac{48}{25\pi}\\ &\approx 0.61115 \text{ ft/min}\end{aligned}$$

2. $V = (\text{depth})(\text{area})$. $\dfrac{1''}{8} = \dfrac{1'}{96}$, so

$$V(t) = \frac{1}{96}\pi r(t)^2.$$

Differentiating we find

$$\frac{dV}{dt} = \frac{2\pi}{96} r(t) \frac{dr}{dt}.$$

Using 1 ft^3 = 7.5 gal, the rate of change of volume is $\frac{90}{7.5} = 12$. So when $r(t) = 100$,

$12 = \frac{2\pi}{96} 100 \frac{dr}{dt}$, and

$\frac{dr}{dt} = \frac{144}{25\pi}$ feet per minute.

3. From #1,

$$V'(t) = \frac{\pi}{48} 2r(t)r'(t) = \frac{\pi}{24} r(t)r'(t),$$

so $\frac{g}{7.5} = \frac{\pi}{24}(100)(.6) = 2.5\pi$,

so $g = (7.5)(2.5)\pi$

$= 18.75\pi \approx 58.905$ gal/min.

4. If the thickness is doubled, then the rate of change of the radius is halved.

5. t = hours elapsed since injury
r = radius of the infected area
A = area of the infection
$A = \pi r^2$
$A'(t) = 2\pi r(t) \cdot r'(t)$
When $r = 3$ mm, $r' = 1$ mm/hr,
$A' = 2\pi(3)(1) = 6\pi$ mm^2/hr

6. We have $A'(t) = 2\pi r r'(t)$, and $r'(t) =$ 1 mm/hr, so when the radius is 6 mm we have

$$A'(t) = 2\pi \cdot 6 \cdot 1 = 12\pi \text{ mm}^2/\text{hr}$$

. This rate is larger when the radius is larger because the area is changing by the same amount along the entire circumference of the circle. When the radius is larger, there is more circumference, so the same change in radius causes a larger change in area.

7. $V(t) = \frac{4}{3}\pi[r(t)]^3$
$V'(t) = 4\pi[r(t)]^2 r'(t) = Ar'(t)$
If $V'(t) = kA(t)$, then
$r'(t) = \frac{V'(t)}{A(t)} = \frac{kA(t)}{A(t)} = k.$

8. We have $A'(t) = 2\pi r r'(t)$, and $r'(t) =$ 5 ft/min, so when the radius is 200 ft we have
$A'(t) = 2\pi \cdot 200 \cdot 5 = 2,000\pi$ ft^2/min.

9.

$$\begin{aligned} 10^2 &= x^2 + y^2 \\ 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \\ &= -\frac{6}{8}(3) \\ &= -2.25 \text{ ft/s} \end{aligned}$$

10. We have

$$\cos\theta(t) = \frac{x(t)}{10}.$$

Differentiating with respect to t gives

$$-\sin\theta(t) \cdot \theta'(t) = \frac{x'(t)}{10}.$$

When the bottom is 6 feet from the wall, the top of the ladder is 8 feet from the floor and this distance is the opposite side of the triangle from *theta*. Thus, at this point, $\sin\theta = 8/10$. So

$-\frac{8}{10}\theta'(t) = \frac{3}{10}$

$\theta'(t) = -\frac{3}{8}$ rad/s.

11.

$$\theta = \pi - \tan^{-1}\left(\frac{40}{60-x}\right) - \tan^{-1}\left(\frac{20}{x}\right)$$

$$\frac{d\theta}{dx} = -\frac{40\left(\frac{1}{60-x}\right)^2}{1+\left(\frac{40}{60-x}\right)^2} + \frac{\frac{20}{x^2}}{1+\left(\frac{20}{x}\right)^2}$$

When $x = 30$, this becomes

$$\begin{aligned}\frac{d\theta}{dx} &= -\frac{40\left(\frac{1}{30}\right)^2}{1+\left(\frac{40}{30}\right)^2} + \frac{\frac{20}{900}}{1+\left(\frac{20}{30}\right)^2} \\ &= -\frac{1}{1625} \text{ rad/ft} \\ \frac{d\theta}{dt} &= \frac{d\theta}{dx}\frac{dx}{dt} \\ &= \left(-\frac{1}{1625}\right)(4) \\ &\approx -0.00246 \text{ rad/s}\end{aligned}$$

12. As in the solution to #11, let x be the distance from the $20'$ building to the person. To find the maximum θ, we set $\frac{d\theta}{dx} = 0$ and solve for x:

$$0 = -\frac{40\left(\frac{1}{60-x}\right)^2}{1+\left(\frac{40}{60-x}\right)^2} + \frac{\frac{20}{x^2}}{1+\left(\frac{20}{x}\right)^2}$$

$$\frac{20}{x^2+40} = \frac{40}{(60-x)^2+1}$$

$$0 = 20x^2 + 2400x - 56000$$

$$0 = x^2 + 120x - 2800$$

Using the quadratic formula, we find two roots:

$$x = -60 \pm 80$$

We discard the x value obtained from the minus sign as it is negative and does not make sense for our problem. The other value is $x = 20$. We find $\theta'(10) > 0$ and $\theta'(30) < 0$, so $x = 20$ must be a maximum as desired.

13. We know $[x(t)]^2 + 4^2 = [s(t)]^2$. Hence $2x(t)x'(t) = 2s(t)s'(t)$, so

$$x'(t) = \frac{s(t)s'(t)}{x(t)} = \frac{-240s(t)}{x(t)}.$$

When $x = 40$, $s = \sqrt{40^2+4^2} = 4\sqrt{101}$, so at that moment

$$x'(t) = \frac{(-240)(4\sqrt{101})}{40} = -24\sqrt{101}.$$

So the speed is $24\sqrt{101} \approx 241.2$mph.

14. From #13, we have

$$x'(t) = \frac{s(t)s'(t)}{x(t)} = \frac{-240s(t)}{x(t)}.$$

This time the height is 6 miles, so $s = \sqrt{40^2+6^2} = 2\sqrt{409}$, so at that moment

$$x'(t) = \frac{(-240)(2\sqrt{409})}{40} = -12\sqrt{409}.$$

So the speed is $12\sqrt{409} \approx 242.7$mph. The difference in height does not make a large difference in the speed of the plane.

15. If the police car is not moving, then $x'(t) = 0$, but all the other data are unchanged. So

$$\begin{aligned}d'(t) &= \frac{x(t)x'(t)+y(t)y'(t)}{\sqrt{[x(t)]^2+[y(t)]^2}} \\ &= \frac{-(1/2)(50)}{\sqrt{1/4+1/16}} \\ &= \frac{-100}{\sqrt{5}} \approx -44.721.\end{aligned}$$

This is more accurate.

16. If the police car is at the intersection, then the rate of change the police car measures is

$$\frac{0\cdot(-40)+\frac{1}{2}\cdot(-50)}{\sqrt{\frac{1}{4}+0}} = -50,$$

the true speed of the car.

17. $$\begin{aligned}d'(t) &= \frac{x(t)x'(t)+y(t)y'(t)}{\sqrt{[x(t)]^2+[y(t)]^2}} \\ &= \frac{-(1/2)(\sqrt{2}-1)(50)-(1/2)(50)}{\sqrt{1/4+1/4}} \\ &= -50.\end{aligned}$$

18. The radar gun will read less than the actual speed if the police car is not at the intersection, and is travelling away from the intersection.

19. $\overline{C}(x) = 10 + \frac{100}{x}$
$\overline{C}'(x(t)) = \frac{-100}{x^2} \cdot x'(t)$
$\overline{C}'(10) = -1(2) = -2$ dollars per item, so average cost is decreasing at the rate of \$2 per year.

20. The year 2 rate of change for the average cost is given by

$$\overline{C}'(t) = \frac{-94}{x^2} \cdot x'(t).$$

From the table we see that in year two $x = 9.4$ and $x' = 0.6$, so

$$\overline{C}'(t) = \frac{-94}{9.4^2} \cdot 0.6 = -0.6383 \text{ per year.}$$

21. From the table, we see that the recent trend is for advertising to increase by \$2000 per year. A good estimate is then $x'(2) \approx 2$ (in units of thousands). Starting with the sales equation
$s(t) = 60 - 40e^{-0.05x(t)}$,
we use the chain rule to obtain
$s'(t) = -40e^{-0.05x(t)}[-0.05x'(t)]$
$= 2x'(t)e^{-0.05x(t)}$.
Using our estimate that $x'(2) \approx 2$ and since $x(2) = 20$, we get $s'(2) \approx 2(2)e^{-1} \approx 1.471$. Thus, sales are increasing at the rate of approximately \$1471 per year.

22. The rate of change of sales is
$s' = 0.8e^{-0.04x}x'(t)$.
We are given $x = 40$ and $x'(t) = 1.5$, so
$s' = 0.8e^{-0.04 \cdot 40} \cdot 1.5 = 0.242$ thousand dollars per year.

23. We have $\tan\theta = \frac{x}{2}$, so

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{2}\right)$$
$$\sec^2\theta \cdot \theta' = \frac{1}{2}x'$$
$$\theta' = \frac{1}{2\sec^2\theta} \cdot x' = \frac{x'\cos^2\theta}{2}$$

at $x = 0$, we have $\tan\theta = \frac{x}{2} = \frac{0}{2}$ so $\theta = 0$ and we have $x' = -130$ft/s so

$$\theta' = \frac{(-130)\cdot\cos^2 0}{2} = -65 \text{ rad/s.}$$

24. $x = 2\tan\theta$, so $\frac{dx}{dt} = 2\sec^2\theta\frac{d\theta}{dt}$. $\theta = 0$ (and $\sec\theta = 1$) as the ball crosses home plate, so $\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$. For this to be less than 3 radians per sec, the pitch must be less than 6 ft/sec.

25. t = number of seconds since launch
x = height of rocket in miles after t seconds
θ = camera angle in radians after t seconds

$$\tan\theta = \frac{x}{2}$$
$$\frac{d}{dx}(\tan\theta) = \frac{d}{dx}\left(\frac{x}{2}\right)$$
$$\sec^2\theta \cdot \theta' = \frac{1}{2}x'$$
$$\theta' = \frac{\cos^2\theta \cdot x'}{2}$$

When $x = 3$, $\tan\theta = 3/2$, so $\cos\theta = 2/\sqrt{13}$.

$$\theta' = \frac{\left(\frac{2}{\sqrt{13}}\right)^2(.2)}{2} \approx .03 \text{ rad/s}$$

26. If the height of the rocket is x, then $x = 2\tan\theta$, and

$$\frac{dx}{dt} = 2\sec^2\theta\frac{d\theta}{dt}.$$

When $x = 1$ and $\dfrac{dx}{dt} = 0.2$, we have $0.2 = 2 \cdot \dfrac{5}{4} \cdot \dfrac{d\theta}{dt}$ and $\dfrac{d\theta}{dt} = 0.08$ radians per sec. This is larger because the angle changes more quickly when the rocket is close to the ground. When the rocket is far away, large changes in height result in small changes in the angle, since the angle is approaching a limit of $\pi/2$.

27. Let θ be the angle between the end of the shadow and the top of the lamppost. Then $\tan\theta = \dfrac{6}{s}$ and $\tan\theta = \dfrac{18}{s+x}$, so

$$\begin{aligned}
\frac{x+s}{18} &= \frac{s}{6}\\
\frac{d}{dx}\left(\frac{x+s}{18}\right) &= \frac{d}{dx}\left(\frac{s}{6}\right)\\
\frac{x'+s'}{18} &= \frac{s'}{6}\\
x'+s' &= 3s'\\
s' &= \frac{x'}{2}
\end{aligned}$$

Since $x' = 2$, $s' = 2/2 = 1$ ft/s.

28. From #27, $s' = x'/2$. Since $x' = -3$, $s' = -3/2$ ft/s.

29. $P(t) \cdot V'(t) + P'(t)V(t) = 0$

$$\frac{P'(t)}{V'(t)} = -\frac{P(t)}{V(t)} = -\frac{c}{V(t)^2}$$

30. Solving Boyle's Law for P gives $P = \dfrac{c}{V}$. Then differentiating gives

$P'(V) = \dfrac{-c}{V^2}$, the same as $P'(t)/V'(t)$.

31. Let $r(t)$ be the length of the rope at time t and $x(t)$ be the distance (along the water) between the boat and the dock.

$$\begin{aligned}
r(t)^2 &= 36 + x(t)^2\\
2r(t)r'(t) &= 2x(t)x'(t)\\
x'(t) &= \frac{r(t)r'(t)}{x(t)} = \frac{-2r(t)}{x(t)}\\
&= \frac{-2\sqrt{36+x^2}}{x}
\end{aligned}$$

When $x = 20$, $x' = -2.088$; when $x = 10$, $x' = -2.332$.

32. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, and we know that this cone has $r = \frac{h}{2}$, so we have $V = \frac{\pi}{12}h^3$. Differentiating gives

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}.$$

We are given that $\dfrac{dV}{dt} = 5\,\mathrm{m}^3/\mathrm{s}$, so when $h = 2$ meters, we have

$5 = \dfrac{\pi 2^2}{4} \cdot \dfrac{dh}{dt}$,

so $\dfrac{dh}{dt} = \dfrac{5}{\pi}$ meters per second.

33. $f(t) = \dfrac{1}{2L(t)}\sqrt{\dfrac{T}{\rho}} = \dfrac{110}{L(t)}$.

$f'(t) = \dfrac{-110}{L(t)^2}L'(t)$.

When $L = 1/2$, $f(t) = 220$ cycles per second. If $L' = -4$ at this time, then $f'(t) = 1760$ cycles per second per second. It will only take 1/8 second at this rate for the frequency to go from 220 to 440, and raise the pitch one octave.

34. $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = \dfrac{4}{3}\pi(3r^2)\dfrac{dr}{dt} = 4\pi r^2\dfrac{dr}{dt}$

$1 = 4\pi r^2\dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}$

When $r = .01$, $\dfrac{dr}{dt} = \dfrac{2500}{\pi}$

When $r = .1$, $\dfrac{dr}{dt} = \dfrac{25}{\pi}$.

At first, the radius expands rapidly; later it expands more slowly.

35. Let R represent the radius of the circular surface of the water in the tank.

$$V(R) = \pi\left[60^2(60^2 - R^2)^{1/2} - \frac{1}{3}(60^2 - R^2)^{3/2} + \frac{2}{3}60^3\right]$$

$$\begin{aligned}\frac{dV}{dR} &= \pi\left[60^2\left(\frac{1}{2}\right)(60^2 - R^2)^{-1/2}(-2R) - \frac{1}{3}\left(\frac{3}{2}\right)(60^2 - R^2)^{1/2}(-2R)\right]\\ &= \pi\left[\frac{-60^2 R}{\sqrt{60^2 - R^2}} + R\sqrt{60^2 - R^2}\right]\\ &= \pi R\left[\frac{-60^2 + 60^2 - R^2}{\sqrt{60^2 - R^2}}\right]\\ &= \frac{-\pi R^3}{\sqrt{60^2 - R^2}}\\ \frac{dR}{dt} &= \frac{dV/dt}{dV/dR}\\ &= \frac{10}{dV/dR}\\ &= \frac{-10\sqrt{60^2 - R^2}}{\pi R^3}\end{aligned}$$

(a) Substituting $R = 60$ into the previous equation, we get $\frac{dR}{dt} = 0$.

(b) We need to determine the value of R when the tank is three-quarters full. The volume of the spherical tank is $\frac{4}{3}\pi 60^3$, so when the tank is three-quarters full, $V(R) = \pi 60^3$. Substituting this value into the formula for $V(R)$ and solving for R (using a CAS, for example) we get $R \approx 56.265$. Substituting this value into the formula for dR/dt, we get

$$\begin{aligned}\frac{dR}{dt} &= \frac{-10\sqrt{60^2 - R^2}}{\pi R^3}\\ &\approx \frac{-10\sqrt{60^2 - 56.265^2}}{\pi 56.265^3}\\ &\approx -0.00037 \text{ ft/s}\end{aligned}$$

36. Assuming the tank is at least half full, we can represent the height of the water in the tank by

$h(t) = \sqrt{60^2 - R^2} + 60$.

Differentiating gives

$$\begin{aligned}h'(t) &= \frac{1}{2}(60^2 - R^2)^{-1/2}(-2R)R'(t)\\ &= -(60^2 - R^2)^{-1/2}R \cdot R'(t)\\ &= \frac{-(60^2 - R^2)^{-1/2}R \cdot (-10\sqrt{60^2 - R^2})}{\pi R^3}.\end{aligned}$$

Here we have used the expression for $R'(t)$ found in exercise 35.

(a) Substituting $R = 60$ into the previous equation, we get $h'(t) = 0$.

(b) Substituting $R \approx 56.265$ into the formula for $h'(t)$ gives $h'(t) \approx 0.001006$ ft/s.

37. The volume of the conical pile is $V = \frac{1}{3}\pi r^2 h$. Since $h = 2r$, we can write the volume as

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

Thus,

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi h^2}{4} \cdot \frac{dh}{dt}\\ 20 &= \frac{\pi 6^2}{4} \cdot \frac{dh}{dt}\\ \frac{dh}{dt} &= \frac{20}{9\pi}\\ \frac{dr}{dt} &= \frac{10}{9\pi}\end{aligned}$$

38. In this case, we have $r = h$ so

$$V = \frac{1}{3}\pi h^2 h = \frac{\pi h^3}{3}$$

Thus $V'(t) = \pi h^2 h'(t)$ so when the height is 6 feet,
$h'(t) = r'(t) = \frac{20}{36\pi} = \frac{5}{9\pi}$.

39.

$$\theta = \tan^{-1}\left(\frac{2s}{vT}\right)$$

$$\frac{d\theta}{dt} = \frac{\left(-\frac{2s}{T}\right) v^{-2} v'(t)}{1 + \left(\frac{2s}{vT}\right)^2} = \frac{-2sv'(t)}{Tv^2\left[1 + \frac{4s^2}{v^2T^2}\right]} = \frac{-2sTv'(t)}{T^2v^2 + 4s^2}$$

For $T = 1$, $s = 0.6$ and $v'(t) = 1$,

$$\frac{d\theta}{dT} = \frac{-1.2}{v^2 + 1.44}$$

(a) $v = 1$ m/s $\Rightarrow$

$$\frac{d\theta}{dT} = \frac{-1.2}{2.44} \approx -0.4918 \text{ rad/s}$$

(b) $v = 2$ m/s $\Rightarrow$

$$\frac{d\theta}{dT} = \frac{-1.2}{5.44} \approx -0.2206 \text{ rad/s}$$

3.9 Rates of Change in Economics and the Sciences

1. The marginal cost function is
$C'(x) = 3x^2 + 40x + 90$.
The marginal cost at $x = 50$ is $C'(50) = 9590$. The cost of producing the 50th item is $C(50) - C(49) = 9421$.

2. The marginal cost function is
$C'(x) = 4x^3 + 28x + 60$.
The marginal cost at $x = 50$ is $C'(50) = 501460$. The cost of producing the 50th item is $C(50) - C(49) = 486645$.

3. The marginal cost function is
$C'(x) = 3x^2 + 42x + 110$.
The marginal cost at $x = 100$ is $C'(100) = 34310$. The cost of producing the 100th item is $C(100) - C(99) = 33990$.

4. The marginal cost function is
$C'(x) = 3x^2 + 22x + 40$.
The marginal cost at $x = 100$ is $C'(100) = 32240$. The cost of producing the 100th item is $C(100) - C(99) = 31930$.

5. $C'(x) = 3x^2 - 60x + 300$
$C''(x) = 6x - 60 = 0$
$x = 10$ is the inflection point because $C''(x)$ changes from negative to positive at this value. After this point, cost rises more sharply.

6. A linear model doesn't reflect the capacity of the stadium, or the presence of a certain number of fans who would attend no matter what the price, but away from the extremes a linear model might serve adequately. For ticket price x, the revenue function is

$$R(x) = x(-3,000x + 57,000) = -3,000x^2 + 57,000x.$$

We solve
$R'(x) = -6,000x + 57,000 = 0$
and find that $x = 9.5$ dollars per ticket is the critical number. Since $R'' = -6,000 < 0$, this is a maximum.

7. $\overline{C}(x) = C(x)/x = 0.1x + 3 + \frac{2000}{x}$

$\overline{C}'(x) = 0.1 - \frac{2000}{x^2}$
Critical number is $x = 100\sqrt{2} \approx 141.4$.

$\overline{C}'(x)$ is negative to the left of the critical number and positive to the right, so this must be the minimum.

8. The average cost function is
$$\overline{C}(x) = \frac{0.2x^3 + 4x + 4000}{x} = 0.2x^2 + 4 + \frac{4000}{x}.$$
$$\overline{C}'(x) = 0.4x - \frac{4000}{x^2} = 0$$
when $x \approx 21.54$. This is a minimum because $\overline{C}'' = 0.4 + \frac{4000}{x^3} > 0$ at this x.

9. $\overline{C}(x) = C(x)/x = 10\frac{e^{0.02x}}{x}$
$\overline{C}'(x) = 10e^{.02x}\left(\frac{.02x - 1}{x^2}\right)$
Critical number is $x = 50$. $\overline{C}'(x)$ is negative to the left of the critical number and positive to the right, so this must be the minimum.

10. The average cost function is
$$\overline{C}(x) = \frac{\sqrt{x^3 + 800}}{x} \text{ and}$$
$$\overline{C}'(x) = \frac{x^3 - 1600}{2x^2\sqrt{x^3 + 800}}.$$
This is zero when $x = \sqrt[3]{1600}$. This is a minimum because
$$\overline{C}'' = \frac{5,120,000 + 12,800x^3 - x^6}{4x^3(x^3 + 800)^{3/2}} >$$
0 at this x.

11.

$C(x) = 0.01x^2 + 40x + 3600$
$C'(x) = 0.02x + 40$
$\overline{C}(x) = \frac{C(x)}{x} = 0.01x + 40 + \frac{3600}{x}$
$C'(100) = 42$
$\overline{C}(100) = 77$
so $C'(100) < \overline{C}(100)$
$\overline{C}(101) = 76.65 < \overline{C}(100)$

12.

$C'(x) = 0.02x + 40$
$C'(1000) = 60$
$\overline{C}(x) = \frac{0.01x^2 + 40x + 3600}{x}$
$\overline{C}(1000) = 53.6$
$\overline{C}(1001) = 53.6064$

13.

$\overline{C}'(x) = 0.01 - \frac{3600}{x^2} = 0$
so $x = 600$ is min and
$C'(600) = 52$
$\overline{C}(600) = 52$

14. If $C(x) = a + bx$, then $C'(x) = b$ and $\overline{C}(x) = b + a/x$.

$\overline{C}'(x) = -a/x^2 \neq 0$ for any x. The average cost function has no extrema, and is never equal to $C'(x)$ since $a/x \neq 0$ for any x.

15. $P(x) = R(x) - C(x)$
$P'(x) = R'(x) - C'(x) = 0$
$R'(x) = C'(x)$

16. $P(x) = (10x - 0.001x^2) - (2x + 5,000)$.
$P'(x) = 8 - 0.002x = 0$ if $x = 4,000$.
This is a maximum because $P''(x) = -0.002 < 0$.

17. $E = \dfrac{p}{f(p)} f'(p)$
$= \dfrac{p}{200(30-p)}(-200) = \dfrac{p}{p-30}$
To solve $\dfrac{p}{p-30} < -1$, multiply both sides by the negative quantity $p-30$, to get $p > (-1)(p-30)$ or $p > 30-p$, so $2p > 30$, so $15 < p < 30$.

18. $E = \dfrac{pf'(p)}{f(p)} = \dfrac{p(-200)}{200(20-p)} = \dfrac{p}{p-20}$
$\dfrac{p}{p-20} < -1$ when $p > 20-p$, so demand is elastic when $10 < p < 20$.

19. $f(p) = 100p(20-p) = 100(20p-p^2)$
$E = \dfrac{p}{f(p)} f'(p)$
$= \dfrac{p}{100p(20-p)}(100)(20-2p)$
$= \dfrac{20-2p}{20-p}$

To solve $\dfrac{20-2p}{20-p} < -1$, multiply both sides by the positive quantity $20-p$ to get $20-2p < (-1)(20-p)$, or $20-2p < p-20$, so $40 < 3p$, so $40/3 < p < 20$.

20. $E = \dfrac{pf'(p)}{f(p)}$
$= \dfrac{p(600-120p)}{60p(10-p)} = \dfrac{2p-10}{p-10}$
If $\dfrac{2p-10}{p-10} < -1$ for positive p, then $p-10$ must be negative. this means $\dfrac{2p-10}{p-10} < -1$ when $2p-10 > 10-p$, so demand is elastic when $\dfrac{20}{3} < p < 10$.

21. Elasticity of demand at price $p = 15$ is, by definition, the relative change in demand divided by the relative change in price, as price increases from 15 to an amount slightly larger than 15. So if (rel change in demand)/(rel change in price) is less than (-1), then rel change in demand is less than (-1)(rel change in price). This means that demand goes down more than price goes up, so revenue should decrease. (See problem 23.)

22. If the demand is inelastic for a given price, then raising the price will increase revenue.

23. $[pf(p)]' < 0$
if and only if $p'f(p) + pf'(p) < 0$
if and only if $f(p) + pf'(p) < 0$
if and only if $pf'(p) < -f(p)$
if and only if $\dfrac{pf'(p)}{f(p)} < -1$.

24. The percentage change in quantity purchased (using the chain rule) is
$$\frac{Q'(I)\cdot I'}{Q(I)}.$$
The percentage change in income is $\dfrac{I'}{I}$.
The income elasticity of demand is then
$\dfrac{Q'(I)\cdot I'}{Q(I)} \cdot \dfrac{I}{I'}$ or $\dfrac{Q'(I)\cdot I}{Q(I)}$.

25. $f(x) = 2x(4-x)$
$f'(x) = 2(4-x) + 2x(-1) = 8-4x$
$= 4(2-x) = 0$
$x = 2$ is a maximum since $f(x)$ is a downward opening parabola.

26. Re-write $x'(t)$ as $f(x) = 0.5x(5-x)$. $f'(x) = 2.5 - x = 0$ when $x = 2.5$. This is a maximum since $f''(x) = -1 < 0$.

27. $2x'(t) = 2x(t)[4-x(t)] = 0$
$x(t) = 0$, $x(t) = 4$ are critical numbers.
$x'(t) > 0$ for $0 < x(t) < 4$
$x'(t) < 0$ for $x(t) > 4$

So $x(t) = 4$ is the maximum concentration.

$x'(t) = 0.5x(t)[5 - x(t)]$
$x(t) = 0$, $x(t) = 5$ are critical numbers.
$x'(t) > 0$ for $0 < x(t) < 5$
$x'(t) < 0$ for $x(t) > 5$
So $x(t) = 5$ is the maximum concentration.

28. We need to find a function of the form $x'(t) = rx(t)(x_{\max} - x(t))$.
We are given $x_{\max} = 16$, and $x'(t) = 12$ when $x(t) = 8$. So $12 = r \cdot 8(16-8)$ and $r = 3/16$. The equation is
$x'(t) = \dfrac{3}{16}x(t)(16 - x(t))$.

29.

$$\begin{aligned} y'(t) &= c \cdot y(t)[K - y(t)] \\ y(t) &= Kx(t) \\ y'(t) &= Kx'(t) \\ Kx'(t) &= c \cdot Kx(t)[K - Kx(t)] \\ x'(t) &= c \cdot Kx(t)[1 - x(t)] \\ &= rx(t)[1 - x(t)] \\ r &= cK \end{aligned}$$

30. The given conditions translate into equations $3 = c \cdot 2(K - 2)$ and $4 = c \cdot 4(K - 4)$. Solving the first equation for c and substituting into the second equation gives
$4 = \dfrac{4 \cdot 3(K - 4)}{2(K - 2)} \Rightarrow K = 8$ and $c = 1/4$.

31. $x'(t) = [a - x(t)][b - x(t)]$
for $x(t) = a$,
$x'(t) = [a - a][b - a] = 0$
So the concentration of product is staying the same.

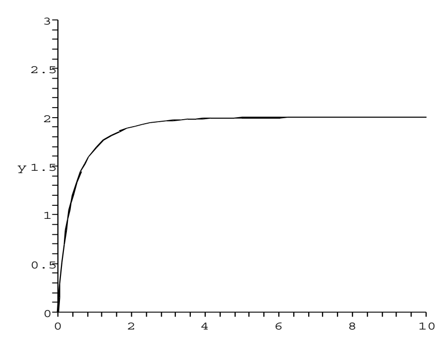
If $a < b$ and $x(0) = 0$ then $x'(t) > 0$ for $0 < x < a < b$
$x'(t) < 0$ for $a < x < b$
Thus $x(t) = a$ is a maximum.

32. The mathematical minimum of the function $(x - a)(x - b)$ is $\dfrac{a + b}{2}$. Since this is between a and b, the rate of reaction is negative. The concentration of the product can never get this large since it starts at 0 and cannot grow past a. When the concentration reaches a, the rate of reaction is 0, so no more product is produced. There are no critical values in the interval $[0, a]$. $x' = ab > 0$ when $x = 0$. $x' = 0$ at $x = a$. This makes $x = a$ the minimum reaction rate and $x = 0$ the maximum reaction rate.

33. $$x(0) = \frac{a[1 - e^{-(b-a)\cdot 0}]}{1 - \left(\frac{a}{b}\right) e^{-(b-a)\cdot 0}} = \frac{a[1 - 1]}{1 - \left(\frac{a}{b}\right)} = 0$$

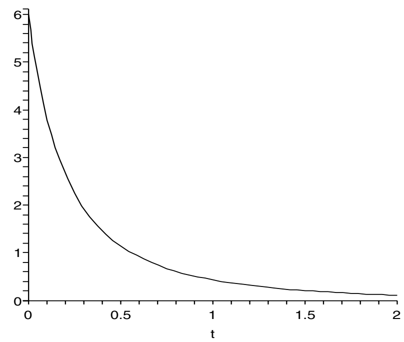
$$\lim_{t\to\infty} x(t) = \frac{a[1 - 0]}{1 - 0} = a$$

For $a = 2$ and $b = 3$ the graph looks like this:



34. $$x'(t) = \frac{ab(-b + a)^2 e^{-(b-a)t}}{(-b + ae^{-(b-a)t})^2}.$$

For $a = 2$ and $b = 3$ the graph looks like this:



The limit $\lim_{t\to\infty} x'(t) = 0$ because the numerator goes to 0 while the denominator approaches b^2. The reaction rate decreases and approaches 0 asymptotically.

35. The first inflection point occurs around $f = 1/3$, before the step up. The second occurs at the far right of the graph. The equivalence point is presumably more stable. The first inflection point would be hard to measure, since the pH takes drastic leap right after the inflection point occurs.

36. Recall that we are assuming $0 < f < 1$. As $f \to 1^-$,
$$p'(f) = \frac{1}{f(1-f)} \to \infty$$

37. $R(x) = \dfrac{rx}{k+x}$, $x \geq 0$
$$R'(x) = \frac{rk}{(k+x)^2}$$
There are no critical numbers. Any possible maximum would have to be at the endpoint $x = 0$, but in fact R is increasing on $[0, \infty)$, so there is no maximum (although as x goes to infinity, R approaches r).

38. To find $\dfrac{dV}{dP}$ in the equation
$$\left(P - \frac{n^2a}{V^2}\right)(V - nb) = nRT,$$
we implicitly differentiate to get
$$\left(1 + \frac{2n^2a}{V^3}\frac{dV}{dP}\right)(V - nb) + \left(P - \frac{n^2a}{V^2}\right)\frac{dV}{dP} = 0.$$
Solving for $\dfrac{dV}{dP}$ gives:
$$\frac{dV}{dP} = \frac{nb - V}{\frac{2n^2a}{V^3}(V - nb) + (P - \frac{n^2a}{V^2})}$$
This is the rate at which the volume changes with respect to changes in pressure.

39.
$$PV^{7/5} = c$$
$$\frac{d}{dP}\left(PV^{7/5}\right) = \frac{d}{dP}(c) = 0$$
$$V^{7/5} + \frac{7}{5}PV^{2/5}\frac{dV}{dP} = 0$$
$$V + \frac{7}{5}P\frac{dV}{dP} = 0$$
$$\frac{dV}{dP} = \frac{-5}{7}\frac{V}{P}.$$
But $V^{7/5} = c/P$, so $V = (c/P)^{5/7}$. Hence
$$\frac{dV}{dP} = \frac{-5}{7}\frac{V}{P} = \frac{-5}{7}\frac{(c/P)^{5/7}}{P} = \frac{-5c^{5/7}}{7P^{12/7}}.$$
As pressure increases, volume decreases.

40. If the equation $PV^{1.4} = c$ holds, and pressure decreases, then the volume must increase.

41. $m'(x) = 4 - \cos x$, so the rod is less dense at the ends.

42. $m'(x) = 3(x-1)^2 + 6$.
Density is maximum at the ends and at a minimum in the middle.

43. $m'(x) = 4$, so the rod is homogeneous.

44. $m'(x) = 8x$.
Density increases from 0 at the left end to a maximum at the right end.

45. $Q'(t) = e^{-2t} \cdot (-2)(\cos 3t - 2\sin 3t)$
$+ e^{-2t}((-\sin 3t \cdot 3) - 2\cos 3t \cdot 3)$
$= e^{-2t}(-8\cos 3t + \sin 3t)$ amps

46. $Q'(t) = e^{t}(3\cos 2t + \sin 2t)$
$+ e^{t}(-6\sin 2t + 2\cos 2t)$
$= 5e^{t}(\cos 2t - \sin 2t)$ amps

47. As $t \to \infty$, $Q(t) \to 4\sin 3t$, so $e^{-3t}\cos 2t$ is called the transient term and $4\sin 3t$ is called the steady-state value.
$Q'(t) = e^{-3t} \cdot (-3)\cos 2t$
$+ e^{-3t}(-\sin 2t \cdot 2) + 4\cos 3t \cdot 3$
$= e^{-3t}(-3\cos 2t - 2\sin 2t)$
$+ 12\cos 3t$
The transient term is $e^{-3t}(-3\cos 2t - 2\sin 2t)$ and the steady-state value is $12\cos 3t$.

48. $Q'(t) = -2e^{-2t}(\cos t - 2\sin t)$
$+ e^{-2t}(-\sin t - 2\cos t)$
$+ e^{-3t} - 3te^{-3t} - 8\sin 4t$
$Q'(t) = e^{-2t}(-4\cos t + 3\sin t)$
$+ e^{-3t}(1 - 3t) - 8\sin 4t$
The transient term is $e^{-2t}(-4\cos t + 3\sin t) + e^{-3t}(1 - 3t)$ and the steady-state value is $-8\sin 4t$.

49. The rate of population growth is given by
$f(p) = 4p(5 - p) = 4(5p - p^2)$
$f'(p) = 4(5 - 2p)$,
so the only critical number is $p = 2.5$. Since the graph of f is a parabola opening down, this must be a max.

50. The rate of growth $R = 2p(7 - 2p)$, so $R' = 14 - 8p = 0$ when $p = 7/4$. This is a maximum because $R'' = -8 < 0$.

51.

$$\begin{aligned} p'(t) &= \frac{-B(1 + Ae^{-kt})'}{(1 + Ae^{-kt})^2} \\ &= \frac{-B(-kAe^{-kt})}{(1 + Ae^{-kt})^2} \\ &= \frac{kABe^{-kt}}{(1 + Ae^{-kt})^2} \\ &= \frac{kABe^{-kt}}{1 + 2Ae^{-kt} + A^2e^{-2kt}} \\ &= \frac{kAB}{e^{kt} + 2A + A^2e^{-kt}} \end{aligned}$$

As t goes to infinity, the exponential term goes to 0, and so the limiting population is

$$\frac{B}{1 + A(0)} = B.$$

52. If the inflection point is $p = 120$, then the maximum population is $B = 240$. If the initial population is $p(0) = 40$, then

$$40 = \frac{240}{1 + A}.$$

We solve to get $A = 5$. If then $p(12) = 160$, we have the equation

$$160 = \frac{240}{1 + 5e^{-12k}}$$

which we can solve to get

$$k = \frac{\ln 10}{12}.$$

53. For $a = 70$, $b = 0.2$,

$$f(t) = \frac{70}{1 + 3e^{-0.2t}} = 70(1 + 3e^{-0.2t})^{-1}$$

$$f(2) = \frac{7 - 0}{1 + 3e^{-0.2 \cdot 2}} \approx 23$$

$$\begin{aligned} f'(t) &= -70(1+3e^{-0.2t})^{-2}(3e^{-0.2t})(-0.2) \\ &= \frac{42e^{-0.2t}}{(1 + 3e^{-0.2t})^2} \end{aligned}$$

$$f'(2) = \frac{42e^{0.2 \cdot 2}}{(1 + 3e^{-0.2 \cdot 2})^2} \approx 3.105$$

This says that at time $t = 2$ hours, the rate at which the spread of the rumor is increasing is about 3% of the population per hour.

$$\lim_{t\to\infty} f(t) = \frac{70}{1+0} = 70$$

so 70% of the population will eventually hear the rumor.

54. $f'(t) = -0.02e^{-0.02t} + 0.42e^{-0.42t}$
$f'(t) = 0$ when $0.42e^{-0.42t} = 0.02e^{-0.02t}$, or $e^{-0.4t} = 0.02/0.42$. So we see that

$$t = -\frac{\ln 0.047619}{0.4} \approx 7.6113$$

is the critical value. The Second Derivative Test shows that it is a maximum.

55. $$f'(x) = \frac{-64x^{-1.4}(4x^{-0.4}+15)}{(4x^{-0.4}+15)^2} - \frac{(160x^{-0.4}+90)(-1.6x^{-1.4})}{(4x^{-0.4}+15)^2} = \frac{-816x^{-1.4}}{(4x^{-0.4}+15)^2} < 0$$
So $f(x)$ is decreasing. This shows that pupils shrink as light increases.

56.

$$T(x) = 102 - \frac{1}{6}x^2 + \frac{1}{54}x^3.$$

To maximize $|T'(x)|$, we find all extrema of $T'(x)$ and compare their magnitudes.

$$T'(x) = \frac{-1}{3}x + \frac{1}{18}x^2.$$

$T''(x) = \frac{-1}{3} + \frac{1}{9}x = 0$ when $x = 3$. We test the critical numbers and the endpoints: $T'(0) = 0$, $T'(6) = 0$, and $T'(3) = \frac{-1}{2}$. The dosage that maximizes sensitivity is 3 mg.

57. If for some x marginal revenue equals marginal cost, then
$P'(x) = R'(x) - C'(x) = 0$,
so x is a critical number, but it may not be a maximum.

58. If $R'(x_0) = C'(x_0)$, then x_0 is a critical number of $P(x)$. If $R''(x_0) < C''(x_0)$, then $P'' < 0$, and the Second Derivative Test guarantees that x_0 gives a maximum.

59. If v is not greater than c, the fish will never make any headway.

$$E'(v) = \frac{v(v-2c)}{(v-c)^2}$$

so the only critical number is $v = 2c$. When v is large, $E(v)$ is large, and when v is just a little bigger than c, $E(v)$ is large, so we must have a minimum.

60. We wish to minimize $P = \frac{1}{v} + cv^3$.
$P' = \frac{-1}{v^2} + 3cv^2 = 0$ when $v = \sqrt[4]{\frac{1}{3c}}$.
$P'' = \frac{2}{v^3} + 6cv > 0$ at this velocity, so this gives the minimum power.

Ch. 3 Review Exercises

1. $f(x) = e^{3x}$, $x_0 = 0$,
$f'(x) = 3e^{3x}$
$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(0) + f'(0)(x-0) \\ &= e^{3\cdot 0} + 3e^{3\cdot 0}x \\ &= 1 + 3x \end{aligned}$$

2. $f'(x) = \dfrac{2x}{2\sqrt{x^2+3}}$.
$f(1) = 2$, and $f'(1) = 1/2$.
$L(x) = \frac{1}{2}(x-1) + 2$.

3. $f(x) = \sqrt[3]{x} = x^{1/3}$, $x_0 = 8$
$f'(x) = \frac{1}{3}x^{-2/3}$

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(8) + f'(8)(x-8) \\ &= \sqrt[3]{8} + \tfrac{1}{3}(8)^{-2/3}(x-8) \\ &= 2 + \tfrac{1}{12}(x-8) \\ L(7.96) &= 2 + \tfrac{1}{12}(7.96 - 8) \approx 1.99666 \end{aligned}$$

4. $\sin 3$ is close to $\sin \pi$. If $y = \sin x$, $y' = \cos x$. The point is $(\pi, 0)$ and the slope is -1. The linear approximation of $\sin x$ at $x = \pi$ is
$L(x) = -(x - \pi)$, so
$\sin 3 \approx -(3 - \pi) \approx 0.14159$.

5. From the graph of $f(x) = x^3 + 5x - 1$, there is one root.
$f'(x) = 3x^2 + 5$
Starting with $x_0 = 0$, Newton's method gives $x_1 = 0.2$, $x_2 = 0.198437$, and $x_3 = 0.198437$.

6. From the graph of $f(x) = x^3 - e^{-x}$, there is one root.
$f'(x) = 3x^2 + e^{-x}$
Starting with $x_0 = 1$, Newton's method gives $x_1 = 0.8123$, $x_2 = 0.7743$, and $x_3 = 0.7729$, which is accurate to 4 decimal places.

7. Near an inflection point, the rate of change of the rate of change of $f(x)$ is very small so there aren't any big dropoffs or sharp increases nearby to make the linear approximation inaccurate.

8. If $y = \dfrac{1}{1-x}$, then $y' = \dfrac{1}{(1-x)^2}$.
For "small" x, x is near 0. The point on the curve when $x = 0$ is $(0, 1)$, and the slope is 1, so the linear approximation is $L(x) = x + 1$, and this is valid for "small" x.

9. $\displaystyle\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$ is type $\frac{0}{0}$;
L'Hôpital's Rule gives

$$\lim_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}.$$

10. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2 + 3x}$ is type $\frac{0}{0}$;
L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{\cos x}{2x+3} = \frac{1}{3}.$$

11. $\displaystyle\lim_{x\to 0} \frac{e^{2x}}{x^4 + 2}$ is type $\frac{\infty}{\infty}$;
applying L'Hôpital's Rule twice gives:

$$\begin{aligned} &\lim_{x\to\infty} \frac{2e^{2x}}{4x^3} \\ &= \lim_{x\to\infty} \frac{4e^{2x}}{12x^2} = \lim_{x\to\infty} \frac{8e^{2x}}{24x} \\ &= \lim_{x\to\infty} \frac{16e^{2x}}{24} = \infty \end{aligned}$$

12. $\displaystyle\lim_{x\to\infty} (x^2 e^{-3x}) = \lim_{x\to\infty} \frac{x^2}{e^{3x}}$ is type $\frac{\infty}{\infty}$;
applying L'Hôpital's Rule twice gives:

$$\begin{aligned} &\lim_{x\to\infty} \frac{2x}{3e^{3x}} \\ &= \lim_{x\to\infty} \frac{2}{9e^{3x}} = 0 \end{aligned}$$

13.

$$\begin{aligned} L &= \lim_{x\to 2^+} \left|\frac{x+1}{x-2}\right|^{\sqrt{x^2-4}} \\ \ln L &= \lim_{x\to 2^+} \left(\sqrt{x^2-4}\,\ln\left|\frac{x+1}{x-2}\right|\right) \\ &= \lim_{x\to 2^+} \left(\frac{\ln\left|\frac{x+1}{x-2}\right|}{(x^2-4)^{-1/2}}\right) \\ &= \lim_{x\to 2^+} \left(\frac{\left|\frac{x-2}{x+1}\right| \frac{-3}{(x-2)^2}}{-x(x^2-4)^{-3/2}}\right) \\ &= \lim_{x\to 2^+} \left(\frac{3(x^2-4)^{3/2}}{x(x+1)(x-2)}\right) \\ &= \lim_{x\to 2^+} \left(\frac{3(x-2)^{1/2}(x+2)^{3/2}}{x(x+1)}\right) \\ \ln L &= 0 \\ L &= 1 \end{aligned}$$

14. $\lim\limits_{x\to\infty} x\ln\left(1+\dfrac{1}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$
is type $\frac{0}{0}$ so we can apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}(-x^{-2})}{-x^{-2}}$$
$$= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = 1$$

15.

$$\begin{aligned}\lim_{x\to 0^+} (\tan x \ln x) &= \lim_{x\to 0^+} \left(\frac{\ln x}{\cot x}\right)\\ &= \lim_{x\to 0^+} \left(\frac{1/x}{-\csc^2 x}\right)\\ &= \lim_{x\to 0^+} -\left(\frac{\sin^2 x}{x}\right)\\ &= -\lim_{x\to 0^+} \left(\frac{\sin x}{x}\sin x\right)\\ &= (-1)(0) = 0\end{aligned}$$

16. $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$ is type $\frac{0}{0}$;
we can apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 0} \frac{\sqrt{1-x^2}}{1+x^2} = 1$$

17. $f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3)$
$= 3(x+3)(x-1)$
So the critical numbers are $x = 1$ and $x = -3$.
$f'(x) > 0$ on $(-\infty, -3) \cup (1, \infty)$
$f'(x) < 0$ on $(-3, 1)$
Hence f is increasing on $(-\infty, -3)$ and on $(1, \infty)$ and f is decreasing on $(-3, 1)$. Thus there is a local max at $x = -3$ and a local min at $x = 1$.

$f''(x) = 3(2x + 2) = 6(x + 1)$
$f''(x) > 0$ on $(-1, \infty)$
$f''(x) < 0$ on $(-\infty, -1)$
Hence f is concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$, and there is an inflection point at $x = -1$.

18. $f'(x) = 4x^3 - 4$
$f'(x) = 0$ when $x = 1$, and this is the only critical number. The function is decreasing for $x < 1$ and increasing for $x > 1$.
$f'' = 12x^2 > 0$ when $x = 1$, so this is a local minimum. $f'' = 0$ when $x = 0$, but does not change sign there, so there are no inflection points. The function is concave up everywhere.

19. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$
$x = 0, 3$ are critical numbers.
$f'(x) > 0$ on $(3, \infty)$
$f'(x) < 0$ on $(-\infty, 0) \cup (0, 3)$
f increasing on $(3, \infty)$, decreasing on $(-\infty, 3)$ so $x = 3$ is a local min.
$f''(x) = 12x^2 - 24x = 12x(x - 2)$
$f''(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$
$f''(x) < 0$ on $(0, 2)$
f is concave up on $(-\infty, 0) \cup (2, \infty)$, concave down on $(0, 2)$ so $x = 0, 2$ are inflection points.

20. $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$
$f'(x) = 0$ when $x = 4$ and $x = -2$. The function is increasing for $x < -2$, then decreasing for $-2 < x < 4$, and increasing for $x > 4$. $x = -2$ represents a local maximum, and $x = 4$ represents a local minimum.
$f''(x) = 6x - 6$
$f''(x) = 0$ when $x = 1$, and changes sign there, so $x = 1$ is an inflection point. The function is concave down for $x < 1$ and concave up for $x > 1$.

21. $f'(x) = e^{-4x} + xe^{-4x}(-4) = e^{-4x}(1 - 4x)$ $x = 1/4$ is a critical number.
$f'(x) > 0$ on $\left(-\infty, \frac{1}{4}\right)$
$f'(x) < 0$ on $\left(\frac{1}{4}, \infty\right)$
f increasing on $\left(-\infty, \frac{1}{4}\right)$, decreasing on $\left(-\frac{1}{4}, \infty\right)$ so $x = 1/4$ is a local max.
$f''(x) = e^{-4x}(-4)(1 - 4x) + e^{-4x}(-4)$
$= -4e^{-4x}(2 - 4x)$
$f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$

$f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$
f is concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right)$ so $x = 1/2$ is inflection point.

22. $f'(x) = 2x\ln x + x = x(2\ln x + 1)$
$f'(x) = 0$ when $\ln x = -1/2$, so $x = e^{-1/2}$. ($x = 0$ is not a critical number because it is not in the domain of the function.) The function is decreasing for $0 < x < e^{-1/2}$, and increasing for $x > e^{-1/2}$. The critical number $x = e^{-1/2}$ represents a minimum.
$f''(x) = 2\ln x + 3$
$f''(x) = 0$ when $x = e^{-3/2}$ and the sign changes from negative to positive there, so this is an inflection point. The function is concave down for $0 < x < e^{-3/2}$ and concave up for $x > e^{-3/2}$.

23. $f'(x) = \dfrac{x^2 - (x-90)(2x)}{x^4}$
$= \dfrac{-(x-180)}{x^3}$
$x = 180$ is the only critical number.
$f'(x) < 0$ on $(-\infty, 0) \cup (180, \infty)$
$f'(x) > 0$ on $(0, 180)$
$f(x)$ is decreasing on $(-\infty, 0) \cup (180, \infty)$ and increasing on $(0, 180)$ so $f(x)$ has a local maximum at $x = 180$.
$f''(x) = -\dfrac{x^3 - (x-180)(3x^2)}{x^6}$
$= -\dfrac{-2x + 540}{x^4}$
$f''(x) < 0$ on $(-\infty, 0) \cup (0, 270)$
$f''(x) > 0$ on $(270, \infty)$ so $x = 90$ is an inflection point.

24. $f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$
$f'(x) = 0$ at $x = 0$ and is undefined at $x = \pm 1$. The function is decreasing for $x < -1$, increasing for $-1 < x < 0$, decreasing for $0 < x < 1$, and increasing for $1 < x$. Critical numbers $x = \pm 1$ are minima, and $x = 0$ is a maximum.
$f''(x) = \dfrac{4(x^2-3)}{9(x^2-1)^{4/3}}$
$f''(x) = 0$ when $x = \pm\sqrt{3}$, and undefined for $x = \pm 1$. The function is concave up for $x < -\sqrt{3}$, concave down for $-\sqrt{3} < x < -1$, concave down for $-1 < x < 1$, concave down for $1 < x < \sqrt{3}$, and concave up for $\sqrt{3} < x$. The inflection points are $x = \pm\sqrt{3}$.

25. $f'(x) = \dfrac{x^2 + 4 - x(2x)}{(x^2+4)^2}$
$= \dfrac{4 - x^2}{(x^2+4)^2}$
$x = \pm 2$ are critical numbers.
$f'(x) > 0$ on $(-2, 2)$
$f'(x) < 0$ on $(-\infty, -2) \cup (2, \infty)$
f increasing on $(-2, 2)$, decreasing on $(-\infty, -2)$ and on $(2, \infty)$ so f had a local min at $x = -2$ and a local max at $x = 2$.
$f''(x) =$
$\dfrac{-2x(x^2+4)^2 - (4-x^2)[2(x^2+4)\cdot 2x]}{(x^2+4)^4}$
$= \dfrac{2x^3 - 24x}{(x^2+4)^3}$
$f''(x) > 0$ on $\left(-\sqrt{12}, 0\right) \cup \left(\sqrt{12}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{12}\right) \cup \left(0, \sqrt{12}\right)$
f is concave up on $\left(-\sqrt{12}, 0\right) \cup \left(\sqrt{12}, \infty\right)$, concave down on $\left(-\infty, -\sqrt{12}\right) \cup \left(0, \sqrt{12}\right)$ so $x = \pm\sqrt{12}, 0$ are inflection points.

26. $f'(x) = \dfrac{2}{(x^2+4)^{3/2}}$
$f'(x)$ is never zero and is defined for all x, so there are no critical numbers. The function is increasing for all x.
$f''(x) = \dfrac{-6x}{(x^2+4)^{5/2}}$
$f''(x) = 0$ when $x = 0$. The function is concave up for $x < 0$, concave down

for $x > 0$, and the inflection point is $x = 0$.

27. $f'(x) = 3x^2 + 6x - 9$
$= 3(x+3)(x-1)$
$x = -3$, $x = 1$ are critical numbers, but $x = -3 \notin [0, 4]$.
$f(0) = 0^3 + 3 \cdot 0^2 - 9 \cdot 0 = 0$
$f(4) = 4^3 + 3 \cdot 4^2 - 9 \cdot 4 = 76$
$f(1) = 1^3 + 3 \cdot 1^2 - 9 \cdot 1 = -5$
So $f(4) = 76$ is absolute max on $[0, 4]$, $f(1) = -5$ is absolute min.

28. First note that $f(x) = \sqrt{x(x-1)(x-2)}$ is only defined on $[0, 1] \cup [2, \infty)$. So we are looking at the intervals $[0, 1] \cup [2, 3]$.

$$f'(x) = \frac{3x^2 - 6x + 2}{2\sqrt{x^3 - 3x^2 + 2x}}$$

The numerator has roots $x = \frac{3 \pm \sqrt{3}}{3}$, but $f(x)$ is only defined at $\frac{3-\sqrt{3}}{3}$. The denominator has zeros at $x = 0$, 1 and 2. Plus we have to check the values of f at the endpoint $x = 3$. We find:
$f(0) = 0$
$f(\frac{3-\sqrt{3}}{3}) \approx 0.6204$
$f(1) = 0$
$f(2) = 0$
$f(3) = \sqrt{6} \approx 2.4495$
Thus $f(x)$ has an absolute maximum on this interval at $x = 3$ and absolute minimums at $x = 0$, $x = 1$ and $x = 2$.

29. $f'(x) = \frac{4}{5}x^{-1/5}$
$x = 0$ is critical number.
$f(-2) = (-2)^{4/5} \approx 1.74$
$f(3) = (3)^{4/5} \approx 2.41$
$f(0) = (0)^{4/5} = 0$
$f(0) = 0$ is absolute min, $f(3) = 3^{4/5}$ is absolute max.

30. $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$
$f'(x) = 0$ when $x = 0$ and $x = 2$. We test $f(x)$ at the critical numbers in the interval $[-1, 4]$, and the endpoints.
$f(-1) = e \approx 2.718$
$f(0) = 0$
$f(2) = 4/e^2 \approx 0.541$
$f(4) = 16/e^4 \approx 0.293$
The absolute maximum is $f(-1) = e$, and the absolute minimum is $f(0) = 0$.

31. $f'(x) = 3x^2 + 8x + 2$

$f'(x) = 0$ when

$$x = \frac{-8 \pm \sqrt{64 - 24}}{6} = -\frac{4}{3} \pm \frac{\sqrt{10}}{3}$$

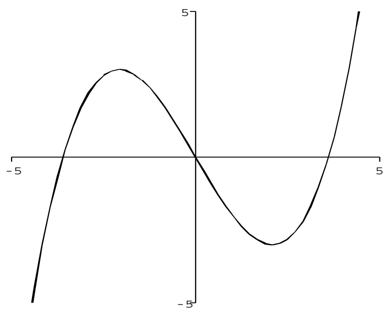
$x = -\frac{4}{3} - \frac{\sqrt{10}}{3}$ is local max, $x = -\frac{4}{3} + \frac{\sqrt{10}}{3}$ is local min.

32. $f'(x) = 4x^3 - 6x + 2$
$= 2(x-1)(2x^2 + 2x - 1)$
$f'(x) = 0$ when $x = 1$ and $x = \dfrac{-1 \pm \sqrt{3}}{2}$, and the derivative changes sign at these values, so these critical numbers are all extrema.

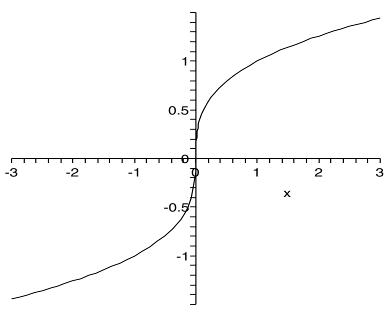
33. $f'(x) = 5x^4 - 4x + 1 = 0$
$x \approx 0.2553,\ 0.8227$
local min at $x \approx 0.8227$,
local max at $x \approx 0.2553$.

34. $f'(x) = 5x^4 + 8x - 4$
$f'(x) = 0$ at approximately $x = -1.3033$ and $x = 0.4696$ (found using Newton's method, or a CAS numerical solver). The derivative changes sign at these values so they correspond to extrema: $x = -1.3033$ is a local max and $x = 0.4696$ is a local min.

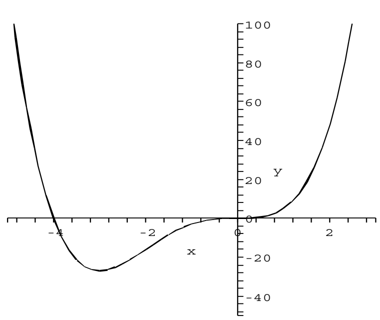
35. One possible graph:



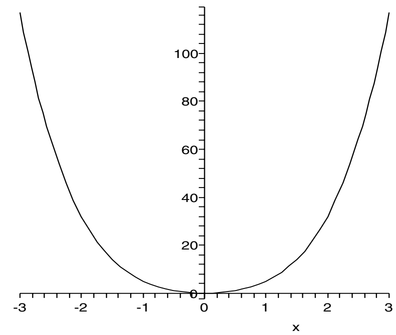
36. One possible graph:



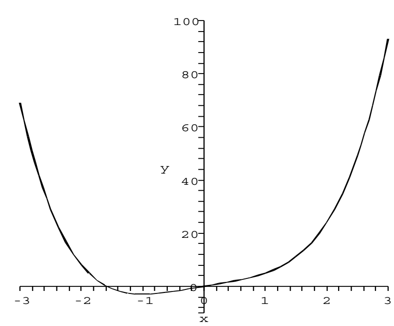
37. $f'(x) = 4x^3 + 12x^2 = 4x^2(4x + 3)$
$f''(x) = 12x^2 + 24x = 12x(x + 2)$
$f'(x) > 0$ on $(-3, 0) \cup (0, \infty)$
$f'(x) < 0$ on $(-\infty, -3)$
$f''(x) > 0$ on $(-\infty, -2) \cup (0, \infty)$
$f''(x) < 0$ on $(-2, 0)$
f increasing on $(-3, \infty)$, decreasing on $(-\infty, -3)$, concave up on $(-\infty, -2) \cup (0, \infty)$, concave down on $(-2, 0)$, local min at $x = -3$, inflection points at $x = -2$, 0.
$f(x) \to \infty$ as $x \to \pm\infty$.



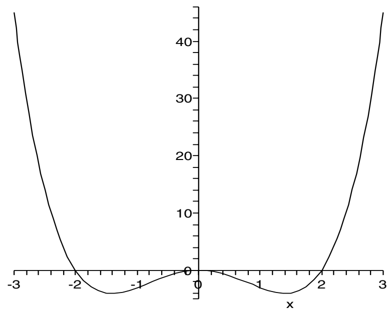
38. $f'(x) = 4x^3 + 8x$
$f'(x) = 0$ when $x = 0$.
$f'' = 12x^2 + 8 > 0$ at $x = 0$, so this is a minimum. $f''(x) > 0$ for all x so there are no inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.



39. $f'(x) = 4x^3 + 4 = 4(x^3 + 1)$
$f''(x) = 12x^2$
$f'(x) > 0$ on $(-1, \infty)$
$f'(x) < 0$ on $(-\infty, -1)$
$f''(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$
f increasing on $(-1, \infty)$, decreasing on $(-\infty, -1)$, concave up on $(-\infty, \infty)$, local min at $x = -1$.
$f(x) \to \infty$ as $x \to \pm\infty$.



40. $f'(x) = 4x^3 - 8x$
$f'(x) = 0$ when $x = 0$ and $x = \pm\sqrt{2}$.
$f'' = 12x^2 - 8 < 0$ at $x = 0$, so this is a maximum. $f''(x) > 0$ for $x = \pm\sqrt{2}$, so these are minima.
$f''(x) = 0$ when $x = \pm\sqrt{2/3}$, and changes sign there, so these are inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.



41. $f'(x) = \dfrac{x^2+1-x(2x)}{(x^2+1)^2}$

$= \dfrac{1-x^2}{(x^2+1)^2}$

$f''(x) =$

$\dfrac{-2x(x^2+1)^2-(1-x^2)2(x^2+1)2x}{(x^2+1)^4}$

$= \dfrac{2x(x^2-3)}{(x^2+1)^4}$

$f'(x) > 0$ on $(-1,1)$
$f'(x) < 0$ on $(-\infty,-1) \cup (1,\infty)$
$f''(x) > 0$ on $\left(-\sqrt{3},0\right) \cup \left(\sqrt{3},\infty\right)$
$f''(x) < 0$ on $\left(-\infty,-\sqrt{3}\right) \cup \left(0,\sqrt{3}\right)$
f increasing on $(-1,1)$, decreasing on $(-\infty,-1)$ and on $(1,\infty)$, concave up on

$$\left(-\sqrt{3},0\right) \cup \left(\sqrt{3},\infty\right),$$

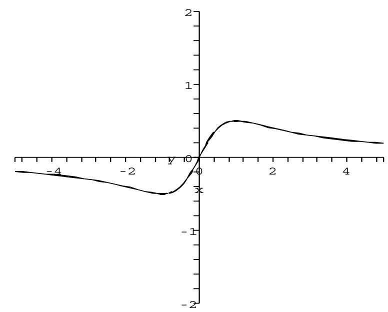
concave down on

$$\left(-\infty,-\sqrt{3}\right) \cup \left(0,\sqrt{3}\right),$$

local min at $x=-1$, local max at $x=1$, inflection points at 0, $\pm\sqrt{3}$.

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{x}{x^2+1} = 0$$

So f has a horizontal asymptote at $y=0$.

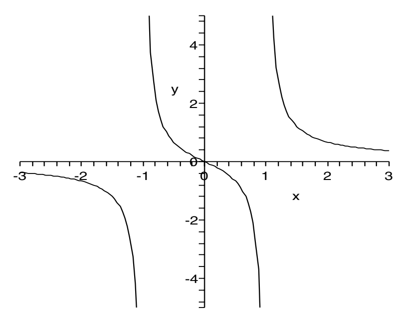


42. $f'(x) = -\dfrac{x^2+1}{(x^2-1)^2}$

is undefined when $f(x)$ is undefined, and is never zero. There are no extrema. There are vertical asymptotes at $x=\pm 1$, and horizontal asymptote $y=0$.

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0$ when $x=0$, and this is the inflection point: $f(x)$ is concave down on $(-\infty,-1)$ and $(0,1)$; $f(x)$ is concave up on $(-1,0)$ and $(1,\infty)$.



43. $f'(x) = \dfrac{(2x)(x^2+1)-x^2(2x)}{(x^2+1)^2}$

$= \dfrac{2x}{(x^2-1)^2}$

$f''(x) = \dfrac{2(x^2+1)^2-2x\cdot 2(x^2+1)2x}{(x^2+1)^4}$

$= \dfrac{2-6x^2}{(x^2+1)^3}$

$f'(x) > 0$ on $(0,\infty)$
$f'(x) < 0$ on $(-\infty,0)$

$f''(x) > 0$ on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$
f increasing on $(0, \infty)$ decreasing on $(-\infty, 0)$, concave up on

$$\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right),$$

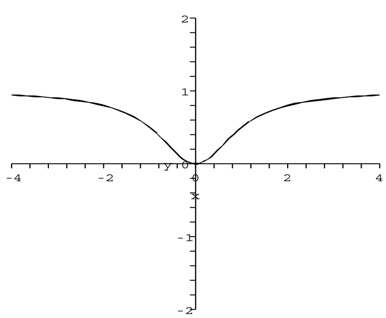
concave down on

$$\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right),$$

local min at $x = 0$, inflection points at $x = \pm\sqrt{1/3}$.

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} = 1$$

So f has a horizontal asymptote at $y = 1$.

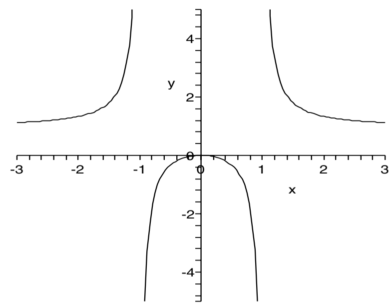


44. $f'(x) = -\dfrac{2x}{(x^2-1)^2}$

$f'(x) = 0$ when $x = 0$, and is undefined when $f(x)$ is undefined. There is a local maximum at $x = 0$. There are vertical asymptotes at $x = \pm 1$, and horizontal asymptote $y = 1$.

$$f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$$

$f''(x) \neq 0$ for any x, and there are no inflection points: $f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$.



45. $$f'(x) = \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2}$$

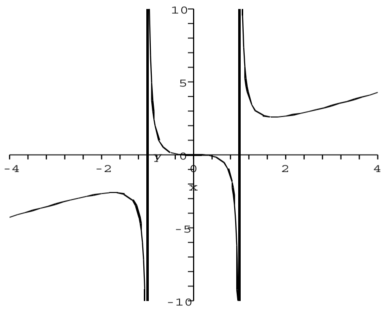
$$f''(x) = \frac{(4x^3-6x)(x^2-1)^2}{(x^2-1)^4} - \frac{(x^4-3x^2)2(x^2-1)2x}{(x^2-1)^4} = \frac{2x^3+6x}{(x^2-1)^4}$$

$f'(x) > 0$ on $\left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$
$f'(x) < 0$ on $\left(-\sqrt{3}, -1\right) \cup (-1, 0) \cup (0, 1) \cup \left(1, \sqrt{3}\right)$
$f''(x) > 0$ on $(-1, 0) \cup (1, \infty)$
$f''(x) < 0$ on $(-\infty, -1) \cup (0, 1)$
f increasing on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$; decreasing on $(-\sqrt{3}, -1)$ and on $(-1, 1)$ and on $(1, \sqrt{3})$; concave up on $(-1, 0) \cup (1, \infty)$, concave down on $(-\infty, -1) \cup (0, 1)$; $x = -\sqrt{3}$ local max; $x = \sqrt{3}$ local min; $x = 0$ inflection point. f is undefined at $x = -1$ and $x = 1$.

$\displaystyle\lim_{x\to 1^+} \frac{x^3}{x^2-1} = \infty$, and

$\displaystyle\lim_{x\to 1^-} \frac{x^3}{x^2-1} = -\infty$

So f has vertical asymptotes at $x = 1$ and $x = -1$.

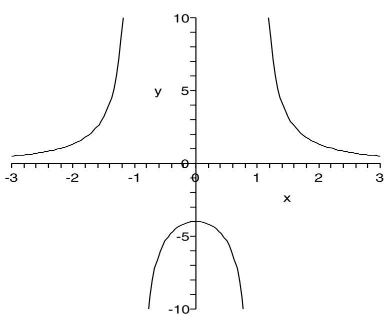


46. $f'(x) = -\dfrac{8x}{(x^2-1)^2}$

$f'(x) = 0$ when $x = 0$, and is undefined when $f(x)$ is undefined. $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0)$; $f(x)$ is decreasing on $(0, 1)$ and $(1, \infty)$. There is a local maximum at $x = 0$. There are vertical asymptotes at $x = \pm 1$, and horizontal asymptote $y = 0$.

$$f''(x) = \frac{8(3x^2+1)}{(x^2-1)^3}$$

$f''(x) \neq 0$ for any x, and there are no inflection points. $f(x)$ is concave up on $(-\infty, -1)$ and $(1, \infty)$; $f(x)$ is concave down on $(-1, 1)$.



47. $d = \sqrt{(x-2)^2 + (y-1)^2}$
$= \sqrt{(x-2)^2 + (2x^2-1)^2}$
$f(x) = (x-2)^2 + (2x^2-1)^2$
$f'(x) = 2(x-2) + 2(2x^2-1)4x$
$= 16x^3 - 6x - 4$
$f'(x) = 0$ when $x \approx 0.8237$
$f'(x) < 0$ on $(-\infty, 0.8237)$
$f'(x) > 0$ on $(0.8237, \infty)$
So $x \approx 0.8237$ corresponds to the closest point.
$y = 2x^2 = 2(0.8237)^2 = 1.3570$
$(0.8237, 1.3570)$ is closest to $(2, 1)$.

48. We compute the slope of the tangent line to $y = 2x^2$ at the closest point $(0.8237, 1.3570)$. When $x = 0.8237$, we get $y' = 3.2948$. The slope of the line between $(2, 1)$ and $(0.8237, 1.3570)$ is

$$\frac{1 - 1.3570}{2 - 0.8237} = -0.3035 = \frac{-1}{3.2948},$$

so the lines are perpendicular.

49. $C(x) = 6\sqrt{4^2 + (4-x)^2} + 2\sqrt{2^2 + x^2}$
$C'(x) =$
$6 \cdot \frac{1}{2}[16 + (4-x)^2]^{-1/2} \cdot 2(4-x)(-1)$
$+ 2\frac{1}{2}(4 + x^2)^{-1/2} \cdot 2x$
$= \dfrac{6(x-4)}{\sqrt{16 + (4-x)^2}} + \dfrac{2x}{\sqrt{4 + x^2}}$
$C'(x) = 0$ when $x \approx 2.864$
$C'(x) < 0$ on $(0, 2.864)$
$C'(x) > 0$ on $(2.864, 4)$
So $x \approx 2.864$ gives the minimum cost. Locate highway corner $4 - 2.864 = 1.136$ miles east of point A.

50. Let $F(v) = e^{-v/2}$. Then $F'(v) = -0.5e^{-v/2}$, so $F'(v) < 0$ for all v. Thus $F(v)$ is decreasing for all v. This says that as the speed of contraction increases, the force produced decreases.

Let $P(v) = ve^{-v/2}$. Then
$P'(v) = e^{-v/2}(1 - \frac{1}{2}v)$.
$P'(v) = 0$ when $v = 2$. We check that $P'(0) > 0$ and $P'(4) < 0$ so $v = 2$ is in fact a maximum.

51. Area: $A = 2\pi r^2 + 2\pi rh$
Convert to in^3:
16 fl oz $= 16$ fl oz $\cdot\ 1.80469$ in^3/fl oz
$= 28.87504\text{in}^3$

Volume: $V = \pi r^2 h$

$$h = \frac{\text{Vol}}{\pi r^2} = \frac{28.87504}{\pi r^2}$$

$$A(r) = 2\pi\left(r^2 + \frac{28.87504}{\pi r}\right)$$
$$A'(r) = 2\pi\left(2r - \frac{28.87504}{\pi r^2}\right)$$
$$2\pi r^3 = 28.87504$$

$$r = \sqrt[3]{\frac{28.87504}{2\pi}} \approx 1.663$$

$A'(r) < 0$ on $(0, 1.663)$
$A'(r) > 0$ on $(1.663, \infty)$

So $r \approx 1.663$ gives the minimum surface area.

$$h = \frac{28.87504}{\pi(1.663)^2} \approx 3.325$$

52. If $C(x) = 0.02x^2 + 4x + 1200$, then $C'(x) = 0.04x + 4 > 0$ for positive values of x (number of items manufactured). This must be positive because the cost function must be increasing. It must cost more to manufacture more items.
$C''(x) = 0.04 > 0$. This means that the cost per item is rising as the number of items produced increases. (For an efficient process, the cost per item should decrease as the number of items increases.)

53. Let θ_1 be the angle from the horizontal to the upper line segment defining θ and let θ_2 be the angle from the horizontal to the lower line segment defining θ. Then the length of the side opposite θ_2 is $\dfrac{H-P}{2}$ while the length of the side opposite θ_1 is $\dfrac{H+P}{2}$. Then

$$\begin{aligned}\theta(x) &= \theta_1 - \theta_2\\ &= \tan^{-1}\left(\frac{H+P}{2x}\right)\\ &\quad - \tan^{-1}\left(\frac{H-P}{2x}\right)\end{aligned}$$

and so

$$\begin{aligned}\theta'(x) = &\frac{1}{1+\left(\frac{H+P}{2x}\right)^2}\left(-\frac{H+P}{2x^2}\right)\\ &- \frac{1}{1+\left(\frac{H-P}{2x}\right)^2}\left(-\frac{H-P}{2x^2}\right).\end{aligned}$$

We set this equal to 0:

$$0 = \frac{-2(H+P)}{4x^2+(H+P)^2} + \frac{2(H-P)}{4x^2+(H-P)^2}$$

and solve for x:

$$\begin{aligned}&\frac{2(H+P)}{4x^2+(H+P)^2} = \frac{2(H-P)}{4x^2+(H-P)^2}\\ &8x^2(H+P) - 8x^2(H-P)\\ &\quad = 2(H-P)(H+P)^2\\ &\qquad - 2(H+P)(H-P)^2\\ &8x^2(2P) = 2(H-P)(H+P)(2P)\\ &x^2 = \frac{H^2-P^2}{4}\\ &x = \frac{\sqrt{H^2-P^2}}{2}.\end{aligned}$$

54. From exercise 53 we know that

$$\theta'(x) = \frac{-2(H+P)}{4x^2+(H+P)^2} + \frac{2(H-P)}{4x^2+(H-P)^2}$$

and that the function $\theta(x)$ is maximized at

$$x = \frac{\sqrt{H^2-P^2}}{2}.$$

Plugging in the appropriate H and P values for high school shows that $\theta(x)$

is maximized by $x \approx 23.9792$. This is not in the range specified. In order to find out whether $\theta(x)$ is increasing or decreasing in the interval specified we plug the H and P values into the expression for $\theta'(x)$ and then plug in a value in our interval, say 55. We find that $\theta'(55) \approx -0.00392$. Since this is negative, $\theta(x)$ is decreasing on this interval, so the announcers must be wrong.

Following the same procedure for college, we find that $\theta(x)$ is maximized by $x \approx 17.7324$ and $\theta'(55) \approx -0.00412$ so again the announcers would be wrong.

Finally, for pros we see that $\theta(x)$ is maximized at $x = 0$ and $\theta'(55) \approx -0.0055$ so the announcers would be wrong once again. In this situation there is no x value for which the announcers would be correct, but in the high school and college situations, if the field goal is taken from some x less than the x which maximized $\theta(x)$, the announcers would be correct.

55. $Q'(t) = -3e^{-3t}\sin 2t + e^{-3t}\cos 2t \cdot 2$
$= e^{-3t}(2\cos 2t - 3\sin 2t)$ amps

56. $f(x) = 0.3x(4 - x)$, $f'(x) = 1.2 - 0.6x = 0$ when $x = 2$, and changes from positive to negative there, so this represents a maximum.

57. $\rho(x) = m'(x) = 2x$
As you move along the rod to the right, its density increases.

58. With no studying, the person scores
$f(0) = \dfrac{90}{1+4} = 18.$
$f'(x) = \dfrac{144e^{-0.4t}}{(1+4e^{-0.4t})^2}.$
If the student were to study one hour, the score will increase by approximately
$f'(0) = \dfrac{144}{25} = 5.76$ points.

59. $C'(x) = 0.04x + 20$
$C'(20) = 0.04(20) + 20 = 20.8$
$C(20) - C(19) =$
$0.02(20)^2 + 20(20) + 1800$
$- [0.02(19)^2 + 20(19) + 1800]$
$= 20.78$

60. $\overline{C}(x) = \dfrac{0.02x^2 + 20x + 1800}{x}$
$= 0.02x + 20 + \dfrac{1800}{x},$
$\overline{C}'(x) = 0.02 - \dfrac{1800}{x^2}$
$\overline{C}'(x) = 0$ when $x = 300$, and the derivative changes from negative to positive here, so $x = 300$ gives the minimum average cost.