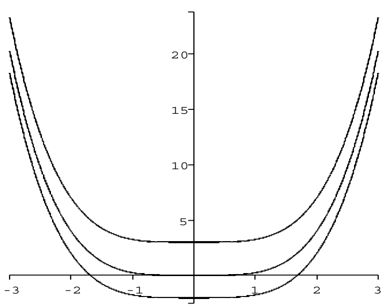
Chapter 4

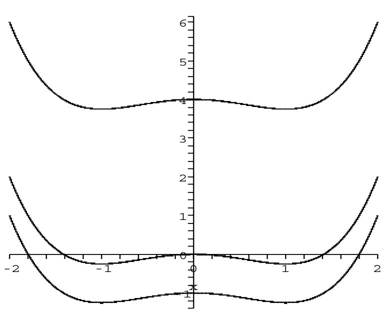
Integration

4.1 Antiderivatives

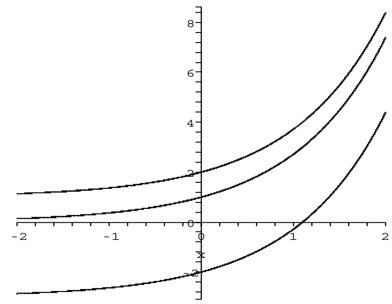
1. $\dfrac{x^4}{4}, \dfrac{x^4}{4}+3, \dfrac{x^4}{4}-2$



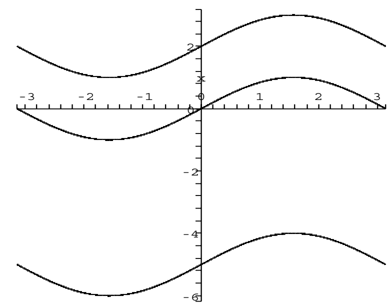
2. $\dfrac{x^4}{4}-\dfrac{x^2}{2}, \dfrac{x^4}{4}-\dfrac{x^2}{2}-1, \dfrac{x^4}{4}-\dfrac{x^2}{2}+4$



3. e^x, e^x+1, e^x-3



4. $\sin x, \sin x+2, \sin x-5$



5. $\displaystyle\int (3x^4-3x)\,dx = \frac{3}{5}x^5 - \frac{3}{2}x^2 + c$

6. $\displaystyle\int (x^3-2)\,dx = \frac{1}{4}x^4 - 2x + c$

7. $\displaystyle\int \left(3\sqrt{x} - \frac{1}{x^4}\right)\,dx = 2x^{3/2} + \frac{x^{-3}}{3} + c$

8. $\displaystyle\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right)\,dx$
$= -2x^{-1} + 2x^{1/2} + c$

9. $\displaystyle\int \frac{x^{1/3}-3}{x^{2/3}}\,dx$
$\displaystyle = \int (x^{-1/3} - 3x^{-2/3})\,dx$
$\displaystyle = \frac{3}{2}x^{2/3} - 9x^{1/3} + c$

10. $\displaystyle\int \frac{x+2x^{3/4}}{x^{5/4}}\,dx$
$\displaystyle = \int (x^{-1/4} + 2x^{-1/2})\,dx$
$\displaystyle = \frac{4}{3}x^{3/4} + 4x^{1/2} + c$

11. $\int (2\sin x + \cos x)\, dx$
$= -2\cos x + \sin x + c$

12. $\int (3\cos x - \sin x)\, dx$
$= 3\sin x + \cos x + c$

13. $\int 2\sec x \tan x\, dx = 2\sec x + c$

14. $\int \frac{4}{\sqrt{1-x^2}}\, dx = 4\arcsin x + c$

15. $\int 5\sec^2 x\, dx = 5\tan x + c$

16. $\int \frac{4\cos x}{\sin^2 x}\, dx = -4\csc x + c$

17. $\int (3e^x - 2)\, dx = 3e^x - 2x + c$

18. $\int (4x - 2e^x)\, dx = 2x^2 - 2e^x + c$

19. $\int (3\cos x - 1/x)\, dx$
$= 3\sin x - \ln|x| + c$

20. $\int (2x^{-1} + \sin x)\, dx = 2\ln|x| - \cos x + c$

21. $\int \frac{4x}{x^2+4}\, dx = 2\ln|x^2+4| + c$

22. $\int \frac{3}{4x^2+4}\, dx = \frac{3}{4}\tan^{-1} x + c$

23. $\int \left(5x - \frac{3}{e^x}\right) dx = \frac{5}{2}x^2 + \frac{3}{e^x} + c$

24. $\int (2\cos x - e^{2x})\, dx = 2\sin x - \frac{e^{2x}}{2} + c$

25. $\int \frac{e^x}{e^x+3}\, dx = \ln|e^x+3| + c$

26. $\int \frac{\cos x}{\sin x}\, dx = \ln|\sin x| + c$

27. $\int \frac{e^x+3}{e^x}\, dx = \int (1+3e^{-x})\, dx$
$= x - 3e^{-x} + c$

28. $\int \frac{(e^x)^2-2}{e^x}\, dx = \int (e^x - 2e^{-x})\, dx$
$= e^x + 2e^{-x} + c$

29. $\int x^{1/4}(x^{5/4} - 4)\, dx$
$= \int (x^{3/2} - 4x^{1/4})\, dx$
$= \frac{2}{5}x^{5/2} - \frac{16}{5}x^{5/4} + c$

30. $\int x^{2/3}(x^{-4/3} - 3)\, dx$
$= \int (x^{-2/3} - 3x^{2/3})\, dx$
$= 3x^{1/3} - \frac{9}{5}x^{5/3} + c$

31. a) N/A
b) $\int (\sqrt{x^3} + 4)\, dx = \frac{2}{5}x^{5/2} + 4x + c$

32. a) $\int \frac{3x^2-4}{x^2}\, dx = \int (3 - 4x^{-2})\, dx$
$= 3x + 4x^{-1} + c$
b) N/A

33. a) N/A
b) $\int \sec^2 x\, dx = \tan x + c$

34. a) $\int \left(\frac{1}{x^2} - 1\right) dx = -\frac{1}{x} - x + c$
b) N/A

35. Use a CAS to find antiderivatives and verify by computing the derivatives:

1.11(b) $\int \sec x\, dx = \ln|\sec x + \tan x| + c$
Verify:
$\frac{d}{dx}\ln|\sec x + \tan x|$
$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

1.11(f) $\int x \sin 2x \, dx$
$= \frac{\sin 2x}{4} - \frac{x \cos 2x}{2} + c$
Verify:
$\frac{d}{dx}\left(\frac{\sin 2x}{4} - \frac{x \cos 2x}{2}\right)$
$= \frac{2\cos 2x}{4} - \frac{\cos 2x - 2x \sin 2x}{2}$
$= x \sin 2x$

36. Use a CAS to find antiderivatives and verify by computing the derivatives:

31(a) The answer is too complicated to be presented here.

32(b) $\frac{1}{9}\left(3x + \sqrt{3}\ln\left|\frac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right|\right) + c$
Verify:
$\frac{d}{dx}\left[\frac{1}{9}\left(3x + \sqrt{3}\ln\left|\frac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right|\right)\right]$
$= \frac{1}{9}\left(3 + \frac{2\sqrt{3} + 3x}{2\sqrt{3} - 3x} \cdot \frac{-3(2\sqrt{3} + 3x) - 3(2\sqrt{3} - 3x)}{(2\sqrt{3} + 3x)^2}\right)$
$= \frac{1}{9}\left(3 - \frac{36}{12 - 9x^2}\right)$
$= \frac{x^2}{3x^2 - 4}$

33(a) Almost the same as in Exercise 35, example 1.11 (b).

34(b) $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$
Verify:
$\frac{d}{dx}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)$
$= \frac{1}{2} \cdot \frac{x+1}{x-1} \cdot \frac{(x+1) - (x-1)}{(x+1)^2}$
$= \frac{1}{x^2 - 1}$

37. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\int x^2 e^{-x^3}\, dx = -\frac{1}{3}e^{-x^3} + c$
Verify:
$\frac{d}{dx}\left(-\frac{1}{3}e^{-x^3}\right)$
$= -\frac{1}{3}e^{-x^3} \cdot (-3x^2)$
$= x^2 e^{-x^3}$

(b) $\int \frac{1}{x^2 - x}\, dx = \ln|x-1| - \ln|x| + c$
Verify:
$\frac{d}{dx}(\ln|x-1| - \ln|x|)$
$= \frac{1}{x-1} - \frac{1}{x} = \frac{x - (x-1)}{x(x-1)}$
$= \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$

(c) $\int \sec x dx = \ln|\sec x + \tan x| + c$
Verify:
$\frac{d}{dx}[\ln|\sec x + \tan x|]$
$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$
$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

38. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\int \frac{x}{x^4 + 1}\, dx = \frac{1}{2}\arctan x^2 + c$
Verify:
$\frac{d}{dx}\left(\frac{1}{2}\arctan x^2\right)$
$= \frac{1}{2} \cdot \frac{1}{x^4 + 1} \cdot 2x = \frac{x}{x^4 + 1}$

(b) $\int 3x \sin 2x \, dx$
$= \frac{3}{4}\sin 2x - \frac{3x}{2}\cos 2x + c$
Verify:
$\frac{d}{dx}\left(\frac{3}{4}\sin 2x - \frac{3x}{2}\cos 2x\right)$
$= \frac{3}{2}\cos 2x - \frac{3}{2}\cos 2x + 3x \sin 2x$
$= 3x \sin 2x$

(c) $\int \ln x\, dx = x\ln x - x + c$

Verify:

$\frac{d}{dx}(x\ln x - x) = \ln x + 1 - 1$
$= \ln x$

39. Finding the antiderivative,
$f(x) = 3e^x + \frac{x^2}{2} + c.$

Since $f(0) = 4$, we have
$4 = f(0) = 3 + c$. Therefore,
$f(x) = 4\sin x + 1.$

40. Finding the antiderivative,
$f(x) = 4\sin x + c.$

Since $f(0) = 3$, we have $3 = f(0) = c$.
Therefore,
$f(x) = 4\sin x + 3.$

41. Finding the antiderivative of f'' gives
$f'(x) = 12x + c_1.$

Since $f'(0) = 2$, we have $2 = f'(0) = c_1$ and therefore
$f'(x) = 12x + 2.$

Finding the antiderivative of $f'(x)$ gives
$f(x) = 6x^2 + 2x + c_2.$

Since $f(0) = 3$, we have $3 = f(0) = c_2$ and
$f(x) = 6x^2 + 2x + 3.$

42. Finding the antiderivative of f'' gives
$f'(x) = x^2 + c_1.$

Since $f'(0) = -3$, we have
$-3 = f'(0) = c_1$ and therefore
$f'(x) = x^2 - 3.$

Finding the antiderivative of $f'(x)$ gives
$f(x) = \frac{1}{3}x^3 - 3x + c_2.$

Since $f(0) = 2$, we have $2 = f(0) = c_2$ and $f(x) = \frac{1}{3}x^3 - 3x + 2.$

43. Taking antiderivatives,
$f''(x) = 3\sin x + 4x^2$
$f'(x) = -3\cos x + \frac{4}{3}x^3 + c_1$
$f(x) = -3\sin x + \frac{1}{3}x^4 + c_1x + c_2.$

44. Taking antiderivatives,
$f''(x) = x^{1/2} - 2\cos x$
$f'(x) = \frac{2}{3}x^{3/2} - 2\sin x + c_1$
$f(x) = \frac{4}{15}x^{5/2} + 2\cos x + c_1x + c_2.$

45. Taking antiderivatives,
$f'''(x) = 4 - 2/x^3$
$f''(x) = 4x + x^{-2} + c_1$
$f'(x) = 2x^2 - x^{-1} + c_1x + c_2$
$f(x) = \frac{2}{3}x^3 - \ln|x| + \frac{c_1}{2}x^2 + c_2x + c_3$

46. Taking antiderivatives,
$f'''(x) = \sin x - e^x$
$f''(x) = -\cos x - e^x + c_1$
$f'(x) = -\sin x - e^x + c_1x + c_2$
$f(x) = \cos x - e^x + \frac{c_1}{2}x^2 + c_2x + c_3$

47. Position is the antiderivative of velocity,
$s(t) = 3t - 6t^2 + c.$
Since $s(0) = 3$, we have $c = 3$. Thus,
$s(t) = 3t - 6t^2 + 3.$

48. Position is the antiderivative of velocity,
$s(t) = -3e^{-t} - 2t + c.$
Since $s(0) = 0$, we have $-3 + c = 0$ and therefore $c = 3$. Thus,
$s(t) = -3e^{-t} - 2t + 3.$

49. First we find velocity, which is the antiderivative of acceleration,
$v(t) = -3\cos t + c_1.$
Since $v(0) = 0$ we have
$-3 + c_1 = 0$, $c_1 = 3$ and

$v(t) = -3\cos t + 3$.

Position is the antiderivative of velocity,
$s(t) = -3\sin t + 3t + c_2$.
Since $s(0) = 4$, we have $c_2 = 4$. Thus, $s(t) = -3\sin t + 3t + 4$.

50. First we find velocity, which is the antiderivative of acceleration,
$v(t) = \frac{1}{3}t^3 + t + c_1$.
Since $v(0) = 4$ we have $c_1 = 4$ and
$v(t) = \frac{1}{3}t^3 + t + 4$.

Position is the antiderivative of velocity,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t + c_2$.
Since $s(0) = 0$, we have $c_2 = 0$. Thus,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t$.

51. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 30 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (30mph = 44ft/s). $v(t) = at + 44$.

We know that the car accelerates to 50 mph (50mph = 73ft/s) in 4 seconds, so $v(4) = 73$. Therefore, $a \cdot 4 + 44 = 73$ and $a = \frac{29}{4}$ ft/s

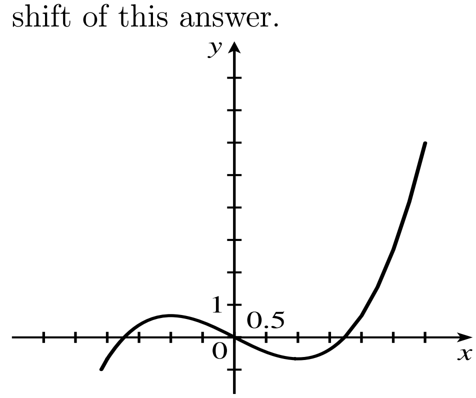
So,
$v(t) = \frac{29}{4}t + 44$ and
$s(t) = \frac{29}{8}t^2 + 44t + s_0$
where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.

Then, $s(t) = \frac{29}{8}t^2 + 44t$ and the distance traveled by the car during the 4 seconds is $s(4) = 234$ feet.

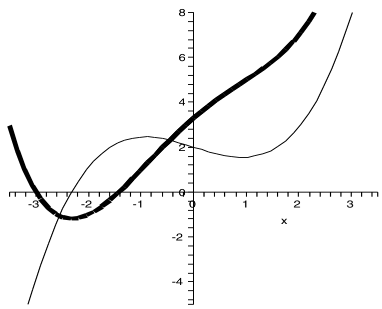
52. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 60 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (60mph = 88ft/s). $v(t) = at + 88$.

We know that the car comes to rest in 3 seconds, so $v(3) = 0$. Therefore, $a(3) + 88 = 0$ and $a = -88/3$ft/s (the acceleration should be negative since the car is actually decelerating.
So,
$v(t) = -\frac{88}{3}t + 88$ and
$s(t) = -\frac{44}{3}t^2 + 88t + s_0$ where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.
Then, $s(t) = -\frac{44}{3}t^2 + 88t$ and the stopping distance is $s(3) = 132$ feet.

53. There are many correct answers, but any correct answer will be a vertical shift of this answer.

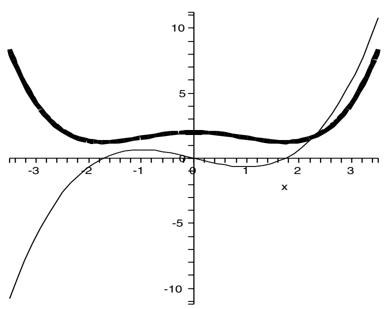


54. There are many correct answers, but any correct answer will be a vertical shift of this answer. $f(x)$ and $f'(x)$ are both shown, with $f(x)$ shown in bold.



55. All functions that have the derivative shown in Exercise 53 are vertical translations of the graph given as the answer for Exercise 53.

56. There is not one correct answer here. The different answers can be more than just vertical translations of each other–it depends on what the graph of $f'(x)$ is used. In any case, one possibility is shown here (with a possible graph of $f'(x)$ shown also, $f(x)$ is shown in bold).



57. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 0.5]$:
$a \approx \dfrac{v(0.5) - v(0)}{0.5 - 0} = -31.6 \text{ m/s}^2$.
Notice, acceleration should be negative since the object is falling.

To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 0.5]$, the time interval is 0.5 and the velocity is -11.9. Therefore the position changed is $(-11.9)(0.5) = -5.95$ meters. The distance traveled will be 5.95 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 0.5]$	-31.6	5.95
$[0.5, 1.0]$	-24.2	12.925
$[1.0, 1.5]$	-11.6	17.4
$[1.5, 2.0]$	-3.6	19.3

58. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 1.0]$:

$$a \approx \frac{v(1.0) - v(0)}{1.0 - 0} = -9.8 \text{ m/s}^2.$$

Notice, acceleration should be negative since the object is falling.

To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 1.0]$, the time interval is 1.0 and the velocity is -4.9. Therefore the position changed is $(-4.9)(1.0) = -4.9$ meters. The distance traveled will be 4.9 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 1.0]$	-9.8	4.9
$[1.0, 2.0]$	-8.8	14.2
$[2.0, 3.0]$	-6.3	21.75
$[3.0, 4.0]$	-3.6	26.7

59. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. The slope of the tangent lines. For example, for the interval $[0, 0.5]$ the average acceleration is -0.9 and $v(0.5) = 70 + (-0.9)(0.5) = 69.55$.

And, the distance traveled is the speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{70 + 69.55}{2} \times 0.5 \approx 34.89$ meters.

Time	Speed	Dist
0	70	0
0.5	69.55	34.89
1.0	70.3	69.85
1.5	70.35	105.01
2.0	70.65	104.26

60. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. the slope of the tangent lines. For example, for the interval $[0.0, 0.5]$ the average acceleration is -0.8 and $v(0.5) = 20 + (-0.8)(.5) = 19.6$. Of course, speed is the absolute value of the velocity.

And, the distance traveled is the average speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{20 + 19.6}{2} \times 0.5 = 9.9$ meters.

Time	Speed	Dist
0	20	0
0.5	19.6	9.9
1.0	17.925	19.281
1.5	16.5	27.888
2.0	16.125	34.044

61. We start by taking antiderivatives:
$f'(x) = x^2/2 - x + c_1$
$f(x) = x^3/6 - x^2/2 + c_1 x + c_2$.

Now, we use the data that we are given. We know that $f(1) = 2$ and $f'(1) = 3$, which gives us
$3 = f'(1) = 1/2 - 1 + c_1$,
and
$1 = f(1) = 1/6 - 1/2 + c_1 + c_2$.

Therefore $c_1 = 7/2$ and $c_2 = -13/6$ and the function is
$f(x) = \dfrac{x^3}{6} - \dfrac{x^2}{2} + \dfrac{7x}{2} - \dfrac{13}{6}$.

62. We start by taking antiderivatives:
$f'(x) = 3x^2 + 4x + c_1$
$f(x) = x^3 + 2x^2 + c_1 x + c_2$.

Now, we use the data that we are given. We know that $f(-1) = 1$ and $f'(-1) = 2$, which gives us
$2 = f'(-1) = -1 + c_1$,
and
$1 = f(-1) = 1 - c_1 + c_2$.

Therefore $c_1 = 3$ and $c_2 = 3$ and the function is
$f(x) = x^3 + 2x^2 + 3x - 3$.

63. Let $u = x^2$; then $du = 2x dx$.

$$\int 2x \cos x^2\, dx = \int \cos u\, du$$
$= \sin u + c$
$= \sin x^2 + c$

64. $\dfrac{d}{dx}\left[(x^3+2)^{3/2}\right] = \dfrac{9}{2}x^2(x^3+2)^{1/2}$

Therefore,
$$\int x^2\sqrt{x^3+2}\,dx = \frac{2}{9}(x^3+2)^{3/2}+c$$

65. $\frac{d}{dx}\left[2x\sin 2x + x^2 2\cos 2x\right]$
$= 2(x\sin 2x + x^2\cos 2x)$

Therefore,
$$\int \left(x\sin 2x + x^2\cos 2x\right)dx$$
$$= \frac{1}{2}x^2\sin 2x + c$$

66. $\frac{d}{dx}\frac{x^2}{e^{3x}} = \frac{2xe^{3x}-3x^2e^{3x}}{e^{6x}}$

Therefore,
$$\int \frac{2xe^{3x}-3x^2e^{3x}}{e^{6x}}\,dx = \frac{x^2}{e^{3x}}+c$$

67. $\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}dx = \sqrt{\sin(x^2)}+c$

68. $\frac{d}{dx}\left(\sqrt{x^2+1}\sin x\right)$
$= \sqrt{x^2+1}\cos x + x(x^2+1)^{-1/2}\sin x$

Therefore,
$$\int \left(\sqrt{x^2+1}\cos x + \frac{x}{\sqrt{x^2+1}}\sin x\right)dx$$
$$= \sqrt{x^2+1}\sin x + c$$

69. $\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}(x)+c_1$
$$\int \frac{-1}{\sqrt{1-x^2}}dx = -\sin^{-1}(x)+c_2$$

Therefore,
$\cos^{-1}x + c_1 = -\sin^{-1}x + c_2$

Therefore,
$\sin^{-1}x + \cos^{-1}x =$ constant

To find the value of the constant, let x be any convenient value. Suppose $x=0$; then $\sin^{-1}0 = 0$ and $\cos^{-1}0 = \pi/2$, so
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

70. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x\tan x$$

71. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:
$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}\left(-e^{-x}\right) = e^{-x}$$

72. a) $\int \frac{1}{kx}\,dx = \frac{1}{k}\int \frac{1}{x}\,dx$
$= \frac{1}{k}\ln|x| + c_1$

b) $\int \frac{1}{kx}\,dx = \frac{1}{k}\int \frac{k}{kx}\,dx$
$= \frac{1}{k}\ln|kx| + c_2$

Because
$\frac{1}{k}\ln|kx| = \frac{1}{k}(\ln|k| + \ln|x|)$
$= \frac{1}{k}\ln|x| + \frac{1}{k}\ln|k| = \frac{1}{k}\ln|x| + c$

The two antiderivatives are both correct.

4.2 Sums And Sigma Notation

1. $\sum_{i=1}^{50} i^2 = \frac{(50)(51)(101)}{6} = 42,925$

2. $\left(\sum_{i=1}^{50} i\right)^2 = \left(\frac{50(51)}{2}\right)^2 = 1,625,625$

3. $\displaystyle\sum_{i=1}^{10} \sqrt{i}$
$= 1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$
$+ \sqrt{7} + \sqrt{8} + \sqrt{9} + \sqrt{10}$
≈ 22.47

4. $\displaystyle\sqrt{\sum_{i=1}^{10} i} = \sqrt{\frac{10(11)}{2}} = \sqrt{55}$

5. $\displaystyle\sum_{i=1}^{6} 3i^2 = 3 + 12 + 27 + 48 + 75 + 108$
$= 273$

6. $\displaystyle\sum_{i=3}^{7} i^2 + i = 10 + 20 + 30 + 42 + 56$
$= 158$

7. $\displaystyle\sum_{i=6}^{10} (4i + 2)$
$= (4(6) + 2) + (4(7) + 2) + (4(8) + 2)$
$+ (4(9) + 2) + (4(10) + 2)$
$= 26 + 30 + 34 + 38 + 42$
$= 170$

8. $\displaystyle\sum_{i=6}^{8} (i^2 + 2)$
$= (6^2 + 2) + (7^2 + 2) + (8^2 + 2)$
$= 38 + 51 + 66$
$= 155$

9. $\displaystyle\sum_{i=1}^{70} (3i - 1)$
$\displaystyle = 3 \cdot \sum_{i=1}^{70} i - 70$
$\displaystyle = 3 \cdot \frac{70(71)}{2} - 70$
$= 7,385$

10. $\displaystyle\sum_{i=1}^{45} (3i - 4)$
$\displaystyle = 3 \sum_{i=1}^{45} i - 4 \sum_{i=1}^{45} 1$
$\displaystyle = 3 \left(\frac{45(46)}{2} \right) - 4(45)$
$= 2925$

11. $\displaystyle\sum_{i=1}^{40} (4 - i^2)$
$\displaystyle = 160 - \sum_{i=1}^{40} i^2$
$\displaystyle = 160 - \frac{(40)(41)(81)}{6}$
$= 160 - 22,140$
$= -21,980$

12. $\displaystyle\sum_{i=1}^{50} (8 - i)$
$\displaystyle = 8 \sum_{i=1}^{50} 1 - \sum_{i=1}^{50} i$
$\displaystyle = 8(50) - \frac{50(51)}{2}$
$= -875$

13. $\displaystyle\sum_{i=1}^{100} (i^2 - 3i + 2)$
$\displaystyle = \sum_{i=1}^{100} i^2 - 3 \cdot \sum_{i=1}^{100} i + 200$
$\displaystyle = \frac{(100)(101)(201)}{6} - 3 \cdot \frac{100(101)}{2} + 200$
$= 338,350 - 15,150 + 200$
$= 323,400$

14. $\displaystyle\sum_{i=1}^{140} i^2 + 2i - 4$
$\displaystyle = \sum_{i=1}^{140} i^2 + 2 \sum_{i=1}^{140} i - 4 \sum_{i=1}^{140} 1$
$\displaystyle = \frac{140(141)(281)}{6} + 2 \left(\frac{140(141)}{2} \right)$
$- 4(140)$
$= 943670$

15. $\displaystyle\sum_{i=1}^{200} (4 - 3i - i^2)$
$\displaystyle = 800 - 3 \cdot \sum_{i=1}^{200} i - \sum_{i=1}^{200} i^2$

$$= 800 - 3 \cdot \frac{200(201)}{2} - \frac{(200)(201)(401)}{6}$$
$$= -2,746,200$$

16. $\sum_{i=1}^{250}(i^2+8)$

$$= \sum_{i=1}^{250} i^2 + 8 \cdot 250$$
$$= \frac{(250)(251)(501)}{6} + 2,000$$
$$= 5,241,625$$

17. $\sum_{i=0}^{n}(i^2-3)$

$$= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n}(-3)$$
$$= 0 + \sum_{i=1}^{n} i^2 + (n+1)(-3)$$
$$= \frac{n(n+1)(2n+1)}{6} - 3(n+1)$$
$$= \frac{(n+1)(2n^2+n-18)}{6}$$

18. $\sum_{i=0}^{n}(i^2+5)$

$$= \sum_{i=1}^{n} i^2 + 5n + 5$$
$$= \frac{n(n+1)(2n+1)}{6} + 5n + 5$$

19. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} + 2\sum_{i=1}^{n} \frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} i\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$$
$$= \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2}\right]$$
$$= \frac{2}{6} + 1 = \frac{4}{3}$$

20. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} - 5\sum_{i=1}^{n} \frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 - \frac{5}{n}\sum_{i=1}^{n} i\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{5}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} - \frac{5n(n+1)}{2n^2}$$
$$= \frac{-13n^2 - 12n + 1}{6n^2}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$$
$$= \lim_{n\to\infty} \frac{-13n^2 - 12n + 1}{6n^2}$$
$$= \lim_{n\to\infty} -\frac{13}{6} - \frac{12}{6n} + \frac{1}{6n^2}$$
$$= -\frac{13}{6}$$

21. $\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$

$$= \frac{1}{n}\left[16\sum_{i=1}^{n} \frac{i^2}{n^2} - 2\sum_{i=1}^{n} \frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right.$$
$$\left. -\frac{2}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}$$
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$$
$$= \lim_{n\to\infty}\left[\frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}\right]$$
$$= \frac{16}{3} - 1 = \frac{13}{3}$$

22. $\displaystyle\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\frac{4i^2}{n^2} + 4\sum_{i=1}^{n}\frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n}i^2 + \frac{4}{n}\sum_{i=1}^{n}i\right]$$
$$= \frac{1}{n}\left[\frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right.$$
$$\left. +\frac{4}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{4n(n+1)(2n+1)}{6n^3} + \frac{4n(n+1)}{2n^2}$$
$$= \frac{10n^2+12n+2}{3n^2}$$
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$$
$$= \lim_{n\to\infty}\frac{10n^2+12n+2}{3n^2}$$
$$= \lim_{n\to\infty}\frac{10}{3} + \frac{12}{3n} + \frac{2}{3n^2}$$
$$= \frac{10}{3}$$

23. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5}(x_i^2 + 4x_i)\cdot 0.2$$
$$= (0.2^2 + 4(0.2))(0.2) + \ldots$$
$$+ (1^2+4)(0.2)$$
$$= (0.84)(0.2) + (1.76)(0.2)$$
$$+ (2.76)(0.2) + (3.84)(0.2)$$
$$+ (5)(0.2)$$
$$= 2.84$$

24. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5}(3x_i + 5)\cdot 0.4$$
$$= (3(0.4)+5)(0.4) + \ldots$$
$$+ (3(2)+5)(0.4)$$
$$= (6.2)(0.4) + (7.4)(0.4)$$
$$+ (8.6)(0.4) + (9.8)(0.4)$$
$$+ (11)(0.4)$$
$$= 17.2$$

25. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10}(4x_i^2 - 2)\cdot 0.1$$
$$= (4(2.1)^2 - 2)(0.1) + \ldots$$
$$+ (4(3)^2 - 2)(0.1)$$
$$= (15.64)(0.1) + (17.36)(0.1)$$
$$+ (19.16)(0.1) + (21.04)(0.1)$$
$$+ (23)(0.1) + (25.04)(0.1)$$
$$+ (27.16)(0.1) + (29.36)(0.1)$$
$$+ (31.64)(0.1) + (34)(0.1)$$
$$= 24.34$$

26. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10}(x^3 + 4)\cdot 0.1$$
$$= ((2.05)^3 + 4)(0.1) + \ldots$$
$$+ ((2.95)^3 + 4)(0.1)$$
$$= (202.4375)(0.1)$$
$$= 20.24375$$

27. Distance

$= 50(2) + 60(1) + 70(1/2) + 60(3)$
$= 375$ miles.

28. Distance
$= 50(1) + 40(1) + 60(1/2) + 55(3)$
$= 285$ miles.

29. Remember to convert minutes into hours.
Distance
$= 15\left(\frac{1}{3}\right) + 18\left(\frac{1}{2}\right) + 16\left(\frac{1}{6}\right) + 12\left(\frac{2}{3}\right)$
$= 24\frac{2}{3}$ miles.

30. Remember to convert minutes into hours.
Distance
$= 12\left(\frac{1}{3}\right) + 14\left(\frac{1}{2}\right) + 18\left(\frac{1}{6}\right) + 15\left(\frac{2}{3}\right)$
$= 24$ miles.

31. On the time interval $[0, 0.25]$, the estimated velocity is the average velocity $\frac{120+116}{2} = 118$ feet per second. We estimate the distance traveled during the time interval $[0, 0.25]$ to be $(118)(0.25 - 0) = 29.5$ feet.

Altogether, the distance traveled is estimated as
$= (236/2)(0.25) + (229/2)(0.25)$
$+ (223/2)(0.25) + (218/2)(0.25)$
$+ (214/2)(0.25) + (210/2)(0.25)$
$+ (207/2)(0.25) + (205/2)(0.25)$
$= 217.75$ feet.

32. On the time interval $[0, 0.5]$, the estimated velocity is the average velocity $\frac{10+14.9}{2} = 12.45$ meters per second. We estimate the distance fallen during the time interval $[0, 0.5]$ to be $(12.45)(0.5 - 0) = 6.225$ meters.

Altogether, the distance fallen (estimated)
$= (12.45)(0.5) + (17.35)(0.5)$
$+ (22.25)(0.5) + (27.15)(0.5)$
$+ (32.05)(0.5) + (36.95)(0.5)$
$+ (41.85)(0.5) + (46.75)(0.5)$
$= 118.4$ meters.

33. Want to prove that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

is true for all integers $n \geq 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2}{4},$$

as desired. So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4},$$

for some integer $k \geq 1$.

In this case, we have by the induction assumption that for $n = k + 1$,

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$
$$= \frac{n^2(n+1)^2}{4}$$

as desired.

34. Want to prove that

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

is true for all integers $n \geq 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2(2+2-1)}{12},$$

as desired. So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12},$$

for some integer $k \geq 1$.

In this case, we have by the induction assumption that for $n = k+1$,

$$\sum_{i=1}^{n} i^5 = \sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} i^5 + (k+1)^5$$
$$= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + (k+1)^5$$
$$= \frac{k^2(k+1)^2(2k^2+2k-1) + 12(k+1)^5}{12}$$
$$= \frac{(k+1)^2[k^2(2k^2+2k-1) + 12(k+1)^3]}{12}$$
$$= \frac{(k+1)^2[2k^4+14k^3+35k^2+36k+12]}{12}$$
$$= \frac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12}$$
$$= \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

as desired.

35. $\sum_{i=1}^{10}(i^3 - 3i + 1)$

$$= \sum_{i=1}^{10} i^3 - 3\sum_{i=1}^{10} i + 10$$
$$= \frac{100(11)^2}{4} - 3\frac{10(11)}{2} + 10$$
$$= 2,870$$

36. $\sum_{i=1}^{20}(i^3 + 2i)$

$$= \sum_{i=1}^{20} i^3 + 2\sum_{i=1}^{20} i$$
$$= \frac{400(21)^2}{4} + 2\frac{20(21)}{2}$$
$$= 44,520$$

37. $\sum_{i=1}^{100}(i^5 - 2i^2)$

$$= \sum_{i=1}^{100} i^5 - 2\sum_{i=1}^{100} i^2$$
$$= \frac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$$
$$- 2\frac{20(21)(41)}{6}$$
$$= 171,707,655,800$$

38. $\sum_{i=1}^{100}(2i^5 + 2i + 1)$

$$= 2\sum_{i=1}^{100} i^3 + 2\sum_{i=1}^{100} i + 100$$
$$= 2\frac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$$
$$+ 2 \cdot \frac{100(101)}{2} + 100$$
$$= 171,708,342,700$$

39. $\sum_{i=1}^{n}(ca_i + db_i) = \sum_{i=1}^{n} ca_i + \sum_{i=1}^{n} db_i$

$$= c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$$

40. When $n = 0$, $a = \frac{a - ar}{1 - r}$.

Assume the formula holds for $n = k - 1$, which gives

$$a + ar + \cdots ar^{k-1} = \frac{a - ar^k}{1 - r}.$$

Then for $n = k$, we have

$$
\begin{aligned}
&a + ar + \cdots ar^k \\
&= a + ar + \cdots ar^{k-1} + ar^k \\
&= \frac{a - ar^k}{1-r} + ar^k \\
&= \frac{a - ar^k + ar^k(1-r)}{1-r} \\
&= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r} \\
&= \frac{a - ar^{k+1}}{1-r} \\
&= \frac{a - ar^{n+1}}{1-r}
\end{aligned}
$$

as desired.

41. $\displaystyle\sum_{i=1}^{n} e^{6i/n}\left(\frac{6}{n}\right)$

$$
\begin{aligned}
&= \frac{6}{n}\sum_{i=1}^{n} e^{6i/n} \\
&= \frac{6}{n}\left(\frac{e^{6/n} - e^6}{1 - e^{6/n}}\right) \\
&= \frac{6}{n}\left(\frac{1 - e^6}{1 - e^{6/n}} - 1\right) \\
&= \frac{6}{n}\frac{1 - e^6}{1 - e^{6/n}} - \frac{6}{n}
\end{aligned}
$$

Now $\displaystyle\lim_{n\to\infty}\frac{6}{n} = 0$, and

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{6}{n}\frac{1 - e^6}{1 - e^{6/n}} \\
&= 6(1 - e^6)\lim_{n\to\infty}\frac{1/n}{1 - e^{6/n}} \\
&= 6(1 - e^6)\lim_{n\to\infty}\frac{1}{-6e^{6/n}} \\
&= e^6 - 1,
\end{aligned}
$$

Thus $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} e^{6i/n}\frac{6}{n} = e^6 - 1.$

42. $\displaystyle\sum_{i=1}^{n} e^{(2i)/n}\frac{2}{n}$

$$
\begin{aligned}
&= \frac{2}{n}\left(\frac{e^{2/n} - e^2}{1 - e^{2/n}}\right) \\
&= \frac{2}{n}\left(\frac{1 - e^2}{1 - e^{2/n}} - 1\right) \\
&= \frac{2}{n}\frac{1 - e^2}{1 - e^{2/n}} - \frac{2}{n}
\end{aligned}
$$

Now $\displaystyle\lim_{n\to\infty}\frac{2}{n} = 0$, and

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{2}{n}\frac{1 - e^2}{1 - e^{2/n}} \\
&= 2(1 - e^2)\lim_{n\to\infty}\frac{1/n}{1 - e^{2/n}} \\
&= 2(1 - e^2)\lim_{n\to\infty}\frac{1}{-2e^{2/n}} \\
&= e^6 2 - 1,
\end{aligned}
$$

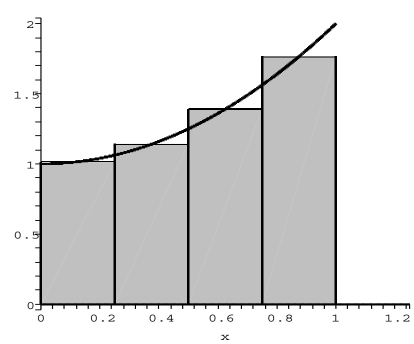
Thus

$$\lim_{n\to\infty}\sum_{i=1}^{n} e^{2i/n}\frac{2}{n} = e^2 - 1.$$

4.3 Area

1. a) Evaluation points:
0.125, 0.375, 0.625, 0.875.
Notice that $\Delta x = 0.25$.

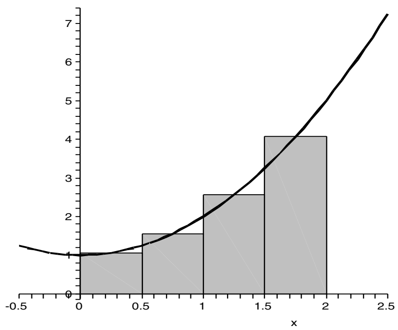
$$
\begin{aligned}
A_4 &= [f(0.125) + f(0.375) + f(0.625) \\
&\quad + f(0.875)](0.25) \\
&= [(0.125)^2 + 1 + (0.375)^2 + 1 \\
&\quad + (0.625)^2 + 1 + (0.875)^2 + 1](0.5) \\
&= 1.38125.
\end{aligned}
$$



b) Evaluation points:
0.25, 0.75, 1.25, 1.75.
Notice that $\Delta x = 0.5$.

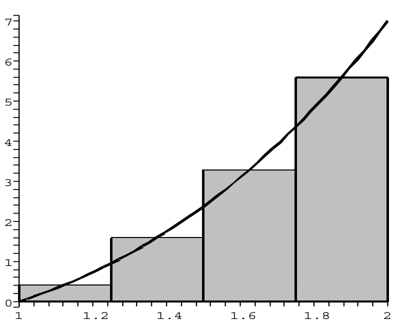
$$
\begin{aligned}
A_4 &= [f(0.25) + f(0.75) + f(1.25) \\
&\quad + f(1.75)](0.5)
\end{aligned}
$$

$= [(0.25)^2+1+(0.75)^2+1+(1.25)^2+1$
$\quad + (1.75)^2 + 1](0.5)$
$= 4.625.$



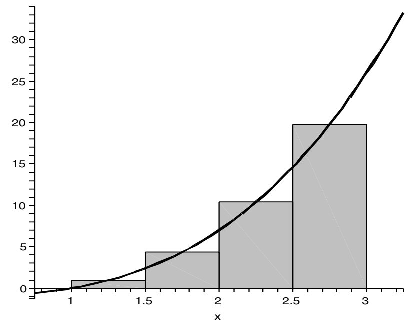
2. a) Evaluation points:
1.125, 1.375, 1.625, 1.875.
Notice that $\Delta x = 0.25$.

$A_4 = [f(1.125) + f(1.375) + f(1.625)$
$\quad + f(1.875)](0.25)$
$= [(1.125)^3 - 1 + (1.375)^3 - 1$
$\quad + (1.625)^3 - 1 + (1.875)^3 - 1](0.25)$
$= 2.7265625.$



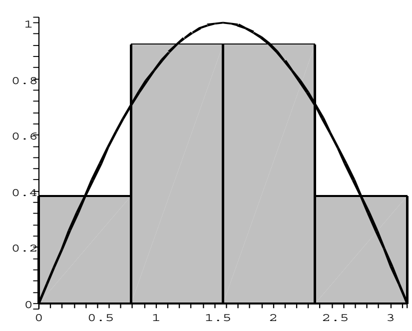
b) Evaluation points:
1.25, 1.75, 2.25, 2.75.
Notice that $\Delta x = 0.5$.

$A_4 = [f(1.25) + f(1.75) + f(2.25)$
$\quad + f(2.75)](0.5)$
$= [(1.25)^3 - 1 + (1.75)^3 - 1$
$\quad + (2.25)^3 - 1 + (2.75)^3 - 1](0.5)$
$= 17.75.$



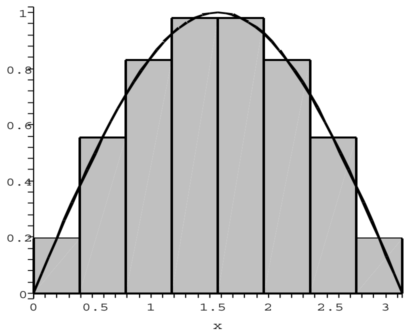
3. a) Evaluation points:
$\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$.
Notice that $\Delta x = \pi/4$.

$A_4 = [f(\pi/8) + f(3\pi/8) + f(5\pi/8)$
$\quad + f(7\pi/8)](\pi/4)$
$= [\sin(\pi/8) + \sin(3\pi/8) + \sin(5\pi/8)$
$\quad + \sin(7\pi/8)](\pi/4)$
$= 2.05234.$



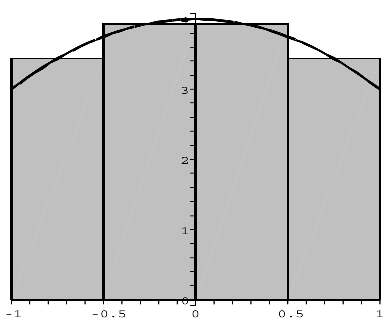
b) Evaluation points:
$\pi/16$, $3\pi/16$, $5\pi/16$, $7\pi/16$, $9\pi/16$,
$11\pi/16$, $13\pi/16$, $15\pi/16$.
Notice that $\Delta x = \pi/8$.

$A_4 = [f(\pi/16) + f(3\pi/16) + f(5\pi/16)$
$\quad + f(7\pi/16) + f(9\pi/16) + f(11\pi/16)$
$\quad + f(13\pi/16) + f(15\pi/16)](\pi/8)$
$= [\sin(\pi/16) + \sin(3\pi/16) + \sin(5\pi/16)$
$\quad + \sin(7\pi/16) + \sin(9\pi/16)$
$\quad + \sin(11\pi/16) + \sin(13\pi/16)$
$\quad + \sin(15\pi/16)](\pi/8)$
$= 2.0129.$



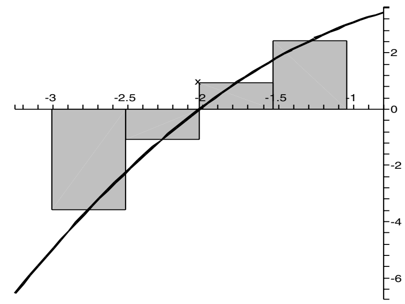
4. a) Evaluation points:
-0.75, -0.25, 0.25, 0.75.
Notice that $\Delta x = 0.5$.

$$
\begin{aligned}
A_4 &= [f(-0.75) + f(-0.25) + f(0.25) \\
&\quad + f(0.75)](0.5) \\
&= [4 - (-0.75)^2 + 4 - (-0.25)^2 + 4 \\
&\quad - (0.25)^2 + 4 - (0.75)^2](0.5) \\
&= 7.375.
\end{aligned}
$$



b) Evaluation points:
-2.75, -2.25, -1.75, -1.25.
Notice that $\Delta x = 0.5$.

$$
\begin{aligned}
A_4 &= [f(-2.75)+f(-2.25)+f(-1.75) \\
&\quad + f(-1.25)](0.5) \\
&= [4 - (-2.75)^2 + 4 - (-2.25)^2 + 4 \\
&\quad - (-1.75)^2 + 4 - (-1.25)^2](0.5) \\
&= -0.625.
\end{aligned}
$$



5. a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$
\begin{aligned}
A_{16} &= \Delta x \sum_{i=0}^{15} f(c_i) \\
&= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} \right)^2 + 1 \right] \approx 1.3027
\end{aligned}
$$

b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \frac{\Delta x}{2}$ where i is from 0 to 15.

$$
\begin{aligned}
A_{16} &= \Delta x \sum_{i=0}^{15} f(c_i) \\
&= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} + \frac{1}{32} \right)^2 + 1 \right] \\
&\approx 1.3330
\end{aligned}
$$

c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$
\begin{aligned}
A_{16} &= \Delta x \sum_{i=0}^{15} f(c_i) \\
&= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} + \frac{1}{16} \right)^2 + 1 \right] \\
&\approx 1.3652
\end{aligned}
$$

6. a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$
A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)
$$

$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8}\right)^2+1\right] \approx 4.4219$$

b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \frac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$

$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8}+\frac{1}{16}\right)^2+1\right] \approx 4.6640$$

c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$

$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8}+\frac{1}{8}\right)^2+1\right] \approx 4.9219$$

7. a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$

$$= \frac{3}{16}\sum_{i=0}^{15}\sqrt{1+\frac{3i}{16}+2} \approx 6.2663$$

b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = 1 + i\Delta x + \Delta x/2$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$

$$= \frac{3}{16}\sum_{i=0}^{15}\sqrt{1+\frac{3i}{16}+\frac{3}{32}+2}$$

$$\approx 6.3340$$

c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$$= \frac{3}{16}\sum_{i=1}^{16}\sqrt{1+\frac{3i}{16}+2} \approx 6.4009$$

8. a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = -1 + i\Delta x - \Delta x$
where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{8})} \approx 4.0991$$

b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = -1 + i\Delta x - \Delta x/2$
where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{16})} \approx 3.6174$$

c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = -1 + i\Delta x$ where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8})} \approx 3.1924$$

9. a) There are 50 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 49.

$$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$$

$$= \frac{\pi}{100}\sum_{i=0}^{50}\cos\left(\frac{\pi i}{100}\right) \approx 1.0156.$$

b) There are 50 rectangles and the evaluation points are given by $c_i = \frac{\Delta x}{2} + i\Delta x$ where i is from 0 to 49.

$$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$$
$$= \frac{\pi}{100} \sum_{i=0}^{50} \cos\left(\frac{\pi}{200} + \frac{\pi i}{100}\right)$$
$$\approx 1.00004.$$

c) There are 50 rectangles and the evaluation points are given by $c_i = \Delta x + i\Delta x$ where i is from 0 to 49.

$$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$$
$$= \frac{\pi}{100} \sum_{i=0}^{50} \cos\left(\frac{\pi}{100} + \frac{\pi i}{100}\right)$$
$$\approx 0.9842.$$

10. a) There are 100 rectangles and the evaluation points are left endpoints which are given by $c_i = -1 + i\Delta x - \Delta x$ where i is from 1 to 100.

$$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$$
$$= \frac{2}{100} \sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{2}{100}\right)^3 - 1\right]$$
$$\approx -2.02.$$

b) There are 100 rectangles and the evaluation points are midpoints which are given by $c_i = -1 + i\Delta x - \Delta x/2$ where i is from 1 to 100.

$$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$$
$$= \frac{2}{100} \sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{1}{100}\right)^3 - 1\right]$$
$$= -2.$$

c) There are 100 rectangles and the evaluation points are right endpoints which are given by $c_i = -1 + i\Delta x$ where i is from 1 to 100.

$$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$$
$$= \frac{2}{100} \sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100}\right)^3 - 1\right]$$
$$\approx -1.98.$$

11.

n	Left Endpoint	Midpoint	Right Endpoint
10	10.56	10.56	10.56
50	10.662	10.669	10.662
100	10.6656	10.6672	10.6656
500	10.6666	10.6667	10.6666
1000	10.6667	10.6667	10.6667
5000	10.6667	10.6667	10.6667

12.

n	Left Endpoint	Midpoint	Right Endpoint
10	0.91940	1.00103	1.07648
50	0.98421	1.00004	1.01563
100	0.99213	1.00001	1.00783
500	0.99843	1.00000	1.00157
1000	0.99921	1.00000	1.00079
5000	0.99984	1.00000	1.00016

13.

n	Left Endpoint	Midpoint	Right Endpoint
10	15.48000	17.96000	20.68000
50	17.4832	17.9984	18.5232
100	17.7408	17.9996	18.2608
500	17.9480	17.9999	18.0520
1000	17.9740	17.9999	18.0260
5000	17.9948	17.9999	18.0052

14.

n	Left Endpoint	Midpoint	Right Endpoint
10	−2.20000	−2	−1.80000
50	−2.04000	−2	−1.96000
100	−2.02000	−2	−1.98000
500	−2.00400	−2	−1.99600
1000	−2.00200	−2	−1.99800
5000	−2.00040	−2	−1.99960

15. $\Delta x = \dfrac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \dfrac{i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 + 1 \\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 1 \\
&= \frac{8n^2+3n+1}{6n^2}
\end{aligned}$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \frac{8n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty} \frac{8}{6} + \frac{3}{6n} + \frac{1}{6n^2} \\
&= \frac{4}{3}.
\end{aligned}$$

16. $\Delta x = \dfrac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \dfrac{i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 3\left(\frac{i}{n}\right)\right] \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n^2}\sum_{i=1}^{n} i \\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&\quad + \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) \\
&= \frac{11n^2+12n+1}{6n^2}
\end{aligned}$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \frac{11n^2+12n+1}{6n^2} \\
&= \lim_{n\to\infty} \frac{11}{6} + \frac{12}{6n} + \frac{1}{6n^2} \\
&= \frac{11}{6}.
\end{aligned}$$

17. $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evaluation points, $x_i = 1 + \dfrac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{2}{n}\sum_{i=1}^{n} 2\left(1+\frac{2i}{n}\right)^2 + 1 \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i^2}{n^2} + \frac{8i}{n} + 3\right) \\
&= \frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^2}\sum_{i=1}^{n} i + 6 \\
&= \frac{16}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&\quad + \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) + 6 \\
&= \frac{16n(n+1)(2n+1)}{6n^3} + \frac{16n(n+1)}{2n^2} \\
&\quad + 6.
\end{aligned}$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \left(\frac{16n(n+1)(2n+1)}{6n^3}\right. \\
&\quad \left. + \frac{16n(n+1)}{2n^2} + 6\right) \\
&= \lim_{n\to\infty} \frac{32}{6} + \frac{16}{2} + 6 \\
&= 19\frac{1}{3}.
\end{aligned}$$

18. $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evaluation points, $x_i = 1 + \dfrac{2i}{n}$.

$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$
$= \dfrac{2}{n}\sum_{i=1}^{n} 4\left(1 + \dfrac{2i}{n}\right) + 2$
$= \dfrac{2}{n}\sum_{i=1}^{n} \dfrac{8i}{n} + 6$
$= \dfrac{16}{n^2}\sum_{i=1}^{n} i + \dfrac{12}{n}\sum_{i=1}^{n} 1$
$= \dfrac{16}{n^2}\left(\dfrac{n(n+1)}{2}\right) + \dfrac{12n}{n}$
$= \dfrac{20n+8}{n}$.

Now to compute the exact area, we take the limit as $n \to \infty$:
$A = \lim_{n\to\infty} A_n$
$= \lim_{n\to\infty} \dfrac{20n+8}{n}$
$= \lim_{n\to\infty} 20 + \dfrac{8}{n}$
$= 20$.

19. Using left hand endpoints:
$L_8 = [f(0.0) + f(0.1) + f(0.2) + f(0.3) + f(0.4) + f(0.5) + f(0.6) + f(0.7)](0.1)$
$= (2.0 + 2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4)(0.1) = 1.81$.

Right endpoints:
$R_8 = [f(0.1) + f(0.2) + f(0.3) + f(0.4) + f(0.5) + f(0.6) + f(0.7) + f(0.8)](0.2)$
$= (2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4 + 0.6)(0.1) = 1.67$.

20. Using left hand endpoints:
$L_8 = [f().0) + f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1.0) + f(1.2) + f(1.4)](0.2)$
$= (2.0 + 2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4)(0.2) = 3.08$.

Right endpoints:
$R_8 = [f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1.0) + f(1.2) + f(1.4) + f(1.6)](0.2)$
$= (2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4 + 2.0)(0.2) = 3.08$.

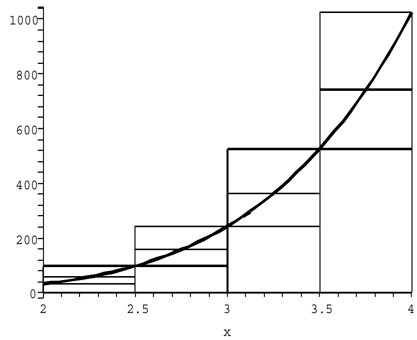
21. Using left hand endpoints:
$L_8 = [f(1.0) + f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5) + f(1.6) + f(1.7)](0.1)$
$= (1.8 + 1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4)(0.1) = 1.182$.

Right endpoints:
$R_8 = [f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5) + f(1.6) + f(1.7) + f(1.8)](0.1)$
$= (1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4 + 2.6)(0.1) = 1.262$.

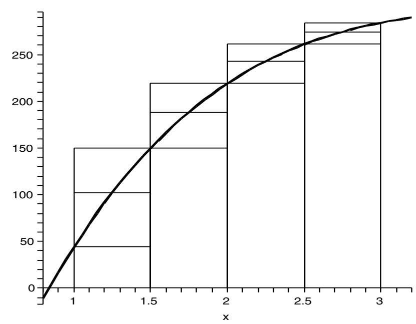
22. Using left hand endpoints:
$L_8 = [f(1.0) + f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2.0) + f(2.2) + f(2.4)](0.2)$
$= (0.0 + 0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4)(0.2) = 1.40$.

Right endpoints:
$R_8 = [f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2.0) + f(2.2) + f(2.4) + f(2.6)](0.2)$
$= (0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4 + 1.0)(0.2) = 1.60$.

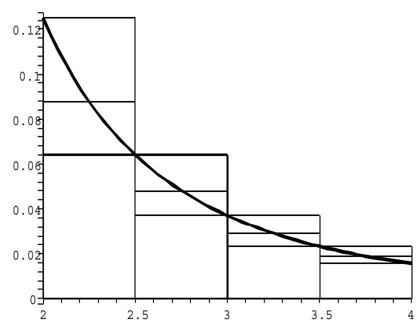
23. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < M < A < R$.



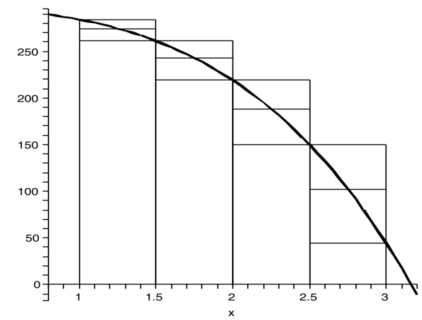
24. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < A < M < R$.



25. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



26. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



27. There are many possible answers here. One possibility is to use $x = 1/6$ on $[0, 0.5]$ and $x = 1/2$ on $[0.5, 1]$.

28. There are many possible answers here. One possibility is to use $x = 1/4$ on $[0, 0.5]$ and $x = 25/36$ on $[0.5, 1]$.

29. We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at $a + (b - a)/n$ (the first right endpoint), then each step of distance Δx takes you to a new right endpoint. To arrive at the i-th right endpoint, you have to take $(i - 1)$ steps to the right of distance Δx. Therefore,
$c_i = a+(b-a)/n+(i-1)\Delta x = a+i\Delta x$.

30. We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at a (the first left endpoint), then each step of distance Δx takes you to a new left endpoint. To arrive at the i-th left endpoint, you have to take $(i - 1)$ steps to the right of distance Δx. Therefore,
$c_i = a + (i - 1)\Delta x$.

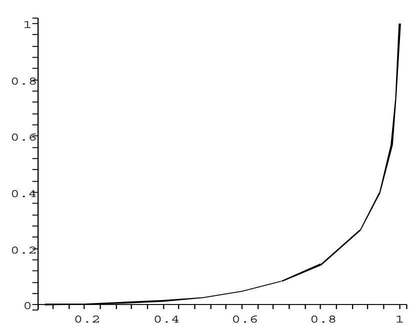
31. We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a+\Delta x/2$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i - 1)$ steps to the

right of distance Δx. Therefore,
$c_i = a + \Delta x/2 + (i-1)\Delta x$
$= a + (i - 1/2)\,\Delta x$, for $i = 1, \ldots, n$.

32. We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a+\Delta x/3$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + \Delta x/3 + (i-1)\Delta x$
$= a + (i - 2/3)\,\Delta x$, for $i = 1, \ldots, n$.

33. $A \approx (0.2-0.1)(0.002) + (0.3-0.2)(0.004)+(0.4-0.3)(0.008)+(0.5-0.4)(0.014)+(0.6-0.5)(0.026)+(0.7-0.6)(0.048)+(0.8-0.7)(0.085)+(0.9-0.8)(0.144) + (0.95-0.9)(0.265) + (0.98-0.95)(0.398) + (0.99-0.98)(0.568)+(1-0.99)(0.736)+1/2\cdot$
$[(0.1-0)(0.002)$
$+ (0.2-0.1)(0.004-0.002) + (0.3-0.2)(0.008-0.004) + (0.4-0.3)(0.014-0.008)+(0.5-0.4)(0.026-0.014) + (0.6-0.5)(0.048-0.026) + (0.7-0.6)(0.085-0.048) + (0.8-0.7)(0.144-0.085)+(0.9-0.8)(0.265-0.144) + (0.95-0.9)(0.398-0.265) + (0.98-0.95)(0.568-0.398) + (0.99-0.98)(0.736-0.568)\ (1-0.99)(1-0.736)]$
≈ 0.092615

The Lorentz curve looks like:



34. Obviously $G = A_1/A_2$ is greater or equal to 0. From the above figure we see that the Lorentz curve is below the diagonal line $y = x$ on the interval $[0, 1]$, hence the area $A_1 \le$ the area A_2.

Furthermore, A_2 = the area of the triangle formed by the points $(0, 0), (1, 0)$ and $(1, 1)$, hence equal to $1/2$.

Now $G = A_1/A_2 = 2A_1$. Using the date in Exercise 33, $G \approx 2 \cdot 0.092615 = 0.185230$.

35. $U_4 = \dfrac{2}{4}\sum_{i=1}^{4}\left(\dfrac{i}{2}\right)^2$
$= \dfrac{1}{8}\sum_{i=1}^{4} i^2 = \dfrac{1}{8}\left[1^2+2^2+3^2+4^2\right]$
$= \dfrac{30}{8} = 3.75$

$L_4 = \dfrac{2}{4}\sum_{i=1}^{4}\left(\dfrac{i-1}{2}\right)^2$
$= \dfrac{1}{8}\sum_{i=1}^{4} i^2 = \dfrac{1}{8}\left[0^2+1^2+2^2+3^2\right]$
$= \dfrac{14}{8} = 1.75$

36. The function $f(x) = x^2$ is symmetric on the two intervals $[-2, 0]$ and $[0, 2]$, so the upper sum U_8 is just double the value of U_4 as calculated in Exercise 35, and the same is for L_8. The answers are
$U_8 = 2\cdot 3.75 = 7.5, L_8 = 2\cdot 1.75 = 3.5$.

37. a) $U_n = \dfrac{2}{n}\sum_{i=1}^{n}\left(\dfrac{2i}{n}\right)^2$
$= \left(\dfrac{2}{n}\right)^3 \sum_{i=1}^{n} i^2$
$= \left(\dfrac{2}{n}\right)^3 \dfrac{n(n+1)(2n+1)}{6}$

$$= \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}$$
$$= \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
$$\lim_{n\to\infty} U_n = \frac{4}{3}(2) = \frac{8}{3}$$

b) $L_n = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2$
$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(i-1)^2$$
$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n-1} i^2$$
$$= \left(\frac{2}{n}\right)^3 \frac{(n-1)(n)(2n-1)}{6}$$
$$= \frac{4}{3}\frac{(n-1)(n)(2n-1)}{n^3}$$
$$= \frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$
$$\lim_{n\to\infty} L_n = \frac{4}{3}(2) = \frac{8}{3}$$

38. a) $U_n = \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{i}{n}\right)^2$
$$= \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n} i^2$$
$$= \left(\frac{1}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{1}{6}\frac{n(n+1)(2n+1)}{n^3}$$
$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
$$\lim_{n\to\infty} U_n = \frac{1}{3}$$

b) $L_n = \frac{1}{n}\sum_{i=1}^{n}\left(-1+\frac{i}{n}\right)^2$
$$= \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n}(i-1)^2$$
$$= \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n-1} i^2$$
$$= \left(\frac{1}{n}\right)^3 \frac{(n-1)(n)(2n-1)}{6}$$
$$= \frac{1}{6}\frac{(n-1)(n)(2n-1)}{n^3}$$
$$= \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$
$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

39. a) $U_n = \frac{2}{n}\sum_{i=1}^{n}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{n^2(n+1)^2}{4}+\frac{2}{n}(n)\right]$$
$$= \frac{4(n+1)^2}{n^2}+2$$
$$= \frac{4(n^2+2n+1)}{n^2}+2$$
$$= 4\left(1+\frac{2}{n}+\frac{1}{n^2}\right)+2$$
$$= 6+\frac{8}{n}+\frac{4}{n^2}$$
$$\lim_{n\to\infty} U_n = 6$$

b) $L_n = \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$
$$= \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=0}^{n-1} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{(n-1)^2n^2}{4}+\frac{2}{n}(n)\right]$$

$$= \frac{4(n-1)^2}{n^2} + 2$$
$$= \frac{4(n^2-2n+1)}{n^2} + 2$$
$$= 4\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) + 2$$
$$= 6 - \frac{8}{n} + \frac{4}{n^2}$$
$$\lim_{n\to\infty} L_n = 6$$

40. a) $U_n = \frac{1}{n}\sum_{i=0}^{n-1}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 - \frac{2}{n^2}\sum_{i=1}^{n-1} i$$
$$= \frac{(n-1)n(2n-1)}{6n^3} - \frac{2(n-1)n}{2n^2}$$
$$= \frac{(n-1)(-4n-1)}{6n^2}$$
$$\lim_{n\to\infty} U_n = -\frac{2}{3}$$

b) $L_n = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{2}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{n(n+1)(2n+1)}{6n^3} - \frac{2(n+1)n}{2n^2}$$
$$= \frac{(n+1)(-4n+1)}{6n^2}$$
$$\lim_{n\to\infty} L_n = -\frac{2}{3}$$

4.4 The Definite Integral

1. $\int_0^3 (x^3 + x)\,dx$

$$= \sum_{i=1}^{n}(c_i^3 + c_i)\Delta x = \sum_{i=1}^{n}(c_i^3 + c_i)\cdot\frac{3}{n},$$
$$c_i = \frac{x_i + x_{i-1}}{2}, x_i = \frac{3i}{n}$$

$n \geq 20 \implies$ Riemann sum ≈ 24.65

2. $\int_0^3 \sqrt{x^2+1}\,dx$

$$= \sum_{i=1}^{n}\sqrt{c_i^2+1}\Delta x = \sum_{i=1}^{n}\sqrt{c_i^2+1}\left(\frac{3}{n}\right),$$
$$c_i = \frac{x_i + x_{i-1}}{2}, x_i = \frac{3i}{n}$$

$n \geq 20 \implies$ Riemann sum ≈ 5.65

3. $\int_0^\pi \sin x^2\,dx$

$$= \sum_{i=1}^{n}\sin c_i^2\Delta x = \sum_{i=1}^{n}\sin c_i^2\left(\frac{\pi}{n}\right),$$
$$c_i = \frac{x_i + x_{i-1}}{2}, x_i = \frac{i\pi}{n}$$

$n \geq 4 \implies$ Riemann sum ≈ 0.80

4. $\int_{-2}^2 e^{-x^2}\,dx$

$$= \sum_{i=1}^{n} e^{-c_i^2}\Delta x = \sum_{i=1}^{n} e^{-c_i^2}\left(\frac{4}{n}\right),$$
$$c_i = \frac{x_i + x_{i-1}}{2}, x_i = -2 + \frac{4i}{n}$$

$n \geq 4 \implies$ Riemann sum ≈ 1.76

5. For n rectangles, $\Delta x = \frac{1}{n}$, $x_i = i\Delta x$.

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n} 2x_i\,\Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(\frac{i}{n}\right)$$
$$= \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{(n+1)}{n}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^2 2x\,dx = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n} = 1$$

6. For n rectangles, $\Delta x = \frac{1}{n}$, $x_i = 1 + i\Delta x$.

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n} 2x_i\,\Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(1+\frac{i}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{2}{n}(n) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)$$
$$= 2 + \frac{(n+1)}{n}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^2 2x\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} 2 + \frac{(n+1)}{n} = 2 + 1 = 3$$

7. For n rectangles, $\Delta x = 2/n$, $x_i = i\Delta x = 2i/n$.

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n}(x_i^2)\,\Delta x = \frac{2}{n}\sum_{i=1}^{n} 2\left(\frac{2i}{n}\right)^2$$
$$= \frac{2}{n}\sum_{i=1}^{n}\frac{4i^2}{n^2} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \frac{4(n+1)(2n+1)}{3n^2}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_0^3 (x^2+1)\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} = \frac{8}{3}$$

8. For n rectangles, $\Delta x = 3/n$, $x_i = i\Delta x = 3i/n$.

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n}(x_i^2+1)\,\Delta x = \frac{3}{n}\sum_{i=1}^{n} 2\left(\frac{3i}{n}\right)^2+1$$
$$= \frac{3}{n}\sum_{i=1}^{n}\frac{18i^2}{n^2}+1$$
$$= \frac{54}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1$$
$$= \frac{54}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{3}{n}\right)n$$
$$= \frac{9(n+1)(2n+1)}{n^2} + 3$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_0^3 (x^2+1)\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{n^2} + 3$$
$$= 9 + 3 = 12$$

9. For n rectangles, $\Delta x = 2/n$, $x_i = 1 + i\Delta x = 1 + 2i/n$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n}(x_i^2 - 3)\,\Delta x$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(1+\frac{2i}{n}\right)^2 - 3\right]$$
$$= \sum_{i=1}^{n}\left(\frac{8i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right)$$
$$= \frac{8n(n+1)}{2n^2} + \frac{8n(n+1)(2n+1)}{6n^3} - 4$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^3 (x^2-3)\,dx = \lim_{n\to\infty} R_n$$

$$= \frac{8}{2} + \frac{16}{6} - 4 = \frac{8}{3}$$

10. For n rectangles, $\Delta x = 4/n$, $x_i = -2 + i\Delta x = -2 + 4i/n$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n} (x_i^2 - 1)\,\Delta x$$
$$= \frac{4}{n}\sum_{i=1}^{n} \left(-2 + \frac{4i}{n}\right)^2 - 1$$
$$= \frac{4}{n}\sum_{i=1}^{n} \left(3 - \frac{16i}{n} + \frac{16i^2}{n^2}\right)$$
$$= \frac{12}{n}\sum_{i=1}^{n} 1 - \frac{64}{n^2}\sum_{i=1}^{n} i + \frac{64}{n^3}\sum_{i=1}^{n} i^2$$
$$= \left(\frac{12}{n}\right) n - \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= 12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_{-2}^{2} (x^2 - 1)\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} 12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}$$
$$= 12 - 32 + \frac{64}{3} = \frac{4}{3}$$

11. Notice that the graph of $y = 4 - x^2$ is above the x-axis between $x = -2$ and $x = 2$:

$$\int_{-2}^{2} (4 - x^2)\,dx$$

12. Notice that the graph of $y = 4x - x^2$ is above the x-axis between $x = 0$ and $x = 4$:

$$\int_{0}^{4} (4x - x^2)\,dx$$

13. Notice that the graph of $y = x^2 - 4$ is below the x-axis between $x = -2$ and $x = 2$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_{-2}^{2} -(x^2 - 4)\,dx$$

14. Notice that the graph of $y = x^2 - 4x$ is below the x-axis between $x = 0$ and $x = 4$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_{0}^{4} -(x^2 - 4x)\,dx$$

15. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

16. $\displaystyle -\int_{-\pi/2}^{0} \sin x\,dx + \int_{0}^{\pi/4} \sin x\,dx$

17. $$\left|\int_{0}^{1} (x^3 - 3x^2 + 2x)\,dx\right| + \left|\int_{1}^{2} (x^3 - 3x^2 + 2x)\,dx\right|$$
$$= \int_{0}^{1} (x^3 - 3x^2 + 2x)\,dx - \int_{1}^{2} (x^3 - 3x^2 + 2x)\,dx$$

18. $$\int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx + \int_{2}^{3} (x^3 - 4x)\,dx$$

19. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$

to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\int_0^4 40(1 - e^{-2t})\, dt$. We compute various right hand sums for different values of n:

n	R_n
10	146.9489200
20	143.7394984
50	141.5635684
100	140.7957790
500	140.1662293
1000	140.0865751

It looks like these are converging to about 140. So, the total distance traveled is approximately 140 and the final position is

$$s(b) \approx s(0) + 140 = 0 + 140 = 140.$$

20. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$ to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\int_0^4 30e^{-t/4}\, dt$. We compute various right hand sums for different values of n:

n	R_n
10	72.12494524
20	73.97390774
50	75.09845086
100	75.47582684
500	75.77863788
1000	75.81654616

It looks like these are converging to about 75.8. So, the total distance traveled is approximately 75.8 and the final position is
$s(b) \approx s(0) + 75.8 = -1 + 75.8 = 74.8.$

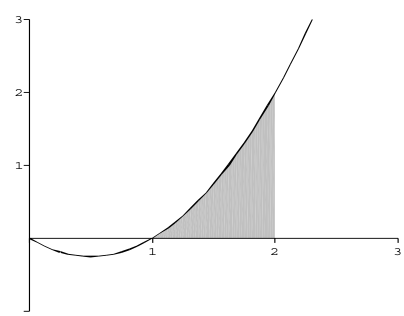
21. $$\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^3 f(x)\,dx$$

22. $$\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx = \int_0^2 f(x)\,dx$$

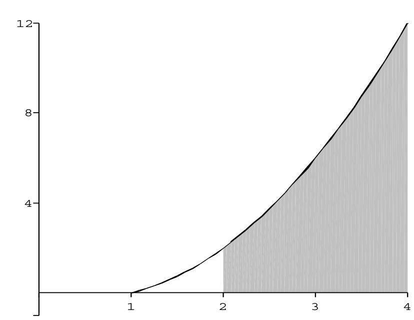
23. $$\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx = \int_0^1 f(x)\,dx$$

24. $$\int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_{-1}^3 f(x)\,dx$$

25.

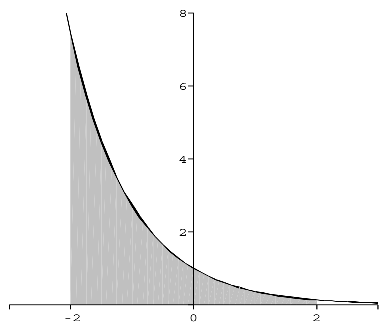


26.



27.

28.



29. The function $f(x) = 3\cos x^2$ is decreasing on $[\pi/3, \pi/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(\pi/3) = 3\cos(\pi^2/9)$. The minimum occurs at the right endpoint and is $f(\pi/2) = 3\cos(\pi^2/4)$.

Using these to estimate the value of the integral gives the following inequality:

$$\frac{\pi}{6} \cdot (3\cos\frac{\pi^2}{4}) \leq \int_{\pi/3}^{\pi/2} 3\cos x^2\, dx \leq \frac{\pi}{6} \cdot (3\cos\frac{\pi^2}{9})$$

$$-1.23 \leq \int_{\pi/3}^{\pi/2} 3\cos x^2\, dx \leq 0.72$$

30. The function $f(x) = e^{-x^2}$ is decreasing on $[0, 1/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(0) = 1$. The minimum occurs at the right endpoint and is $f(1/2) = e^{-1/4}$.

Using these to estimate the value of the integral gives the following inequality:

$$\frac{1}{2}(e^{-1/4}) \leq \int_0^{1/2} e^{-x^2}\, dx \leq \frac{1}{2}(1)$$

$$0.3894 \leq \int_0^{1/2} e^{-x^2}\, dx \leq 0.5$$

31. The function $f(x) = \sqrt{x^2+1}$ is increasing on $[0, 2]$. Therefore, on this interval, the maximum occurs at the right endpoint and is $f(2) = \sqrt{5}$. The minimum occurs at the left endpoint and is $f(0) = 1$.

Using these to estimate the value of the integral gives the following inequality:

$$(2)(1) \leq \int_0^2 \sqrt{x^2+1}\, dx \leq (2)(\sqrt{5})$$

$$2 \leq \int_0^2 \sqrt{x^2+1}\, dx \leq 4.472$$

32. The function $f(x) = \frac{3}{x^3+2}$ is decreasing on $[-1, 1]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(-1) = 3$. The minimum occurs at the right endpoint and is $f(1) = 1$.

Using these to estimate the value of the integral gives the following inequality:

$$(2)(1) \leq \int_{-1}^{1} \frac{3}{x^3+2}\, dx \leq (2)(3)$$

$$2 \leq \int_{-1}^{1} \frac{3}{x^3+2}\, dx \leq 6$$

33. We are looking for a value c, such that

$$f(c) = \frac{1}{2-0}\int_0^2 3x^2\, dx$$

Since $\int_0^2 3x^2\, dx = 8$, we want to find c so that $f(c) = 4$ or, $3c^2 = 4$

Solving this equation using the quadratic formula gives $c = \pm\dfrac{2}{\sqrt{3}}$

We are interested in the value that is in the interval $[0, 2]$, so $c = \dfrac{2}{\sqrt{3}}$.

34. We are looking for a value c, such that

$$f(c) = \frac{1}{1-(-1)} \int_{-1}^{1} (x^2 - 2x)\,dx$$

Since $\displaystyle\int_{-1}^{1} (x^2 - 2x)\,dx = \frac{2}{3}$, we want to find c so that $f(c) = \frac{1}{3}$ or, $c^2 - 2c = \frac{1}{3}$

Solving this equation using the quadratic formula gives $c = \dfrac{3 \pm 2\sqrt{3}}{3}$

We are interested in the value that is in the interval $[-1, 1]$, so $c = \dfrac{3 - 2\sqrt{3}}{3}$.

35. $f_{ave} = \dfrac{1}{4}\displaystyle\int_0^4 (2x+1)\,dx$

$$= \frac{1}{4}\lim_{n\to\infty}\sum_{i=1}^{n} \frac{4}{n}\left(\frac{8i}{n} + 1\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8n(n+1)}{2n^2} + 1\right)$$

$$= 4 + 1 = 5$$

36. $f_{ave} = \dfrac{1}{1}\displaystyle\int_0^1 (x^2 + 2x)\,dx$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i^2}{n^2} + \frac{2i}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6n^3} + \frac{2n(n+1)}{n^2}\right)$$

$$= \frac{2}{6} + 2 = \frac{7}{3}$$

37. $f_{ave} = \dfrac{1}{1-0}\displaystyle\int_0^1 (x^2 - 1)\,dx$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(1 + \frac{2i}{n}\right)^2 - 1\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{4i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{4n(n+1)}{2n^2} + \frac{4n(n+1)(2n+1)}{6n^3}\right)$$

$$= 2 + \frac{4}{3} = \frac{10}{3}$$

38. $f_{ave} = \dfrac{1}{1}\displaystyle\int_0^1 (2x - 2x^2)\,dx$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[2\left(\frac{i}{n}\right) - 2\left(\frac{i}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{2i}{n} + \frac{2i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n(n+1)}{2n^2} + \frac{2n(n+1)(2n+1)}{6n^3}\right)$$

$$= 1 + \frac{1}{3} = \frac{4}{3}$$

39. This is just a restatement of the Integral Mean Value Theorem.

40. Let $c = \dfrac{a+b}{2}$. By definition,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x.$$

We can choose n to be always even, so that $n = 2m$, and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{m\to\infty}\sum_{i=1}^{m} f(c_i)\Delta x + \lim_{m\to\infty}\sum_{i=m+1}^{n} f(c_i)\Delta x$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

41. Between $x = 0$ and $x = 2$, the area below the x-axis is much less than the area above the x-axis. Therefore

$$\int_0^2 f(x)\,dx > 0$$

42. Between $x = 0$ and $x = 2$, the area above the x-axis is much greater than the area below the x-axis. Therefore

$$\int_0^2 f(x)\,dx > 0$$

43. Between $x = 0$ and $x = 2$, the area below the x-axis is slightly greater than the area above the x-axis. Therefore

$$\int_0^2 f(x)\,dx < 0$$

44. Between $x = 0$ and $x = 2$, the area below the x-axis is much greater than the area above the x-axis. Therefore

$$\int_0^2 f(x)\,dx < 0$$

45. Imagine the interval $[0, 2]$ is divided into n subintervals. If n is even, then the point $x = 1$ must be one of the boundary points. If we take the midpoint evaluations to approximate Riemann sums for $\int_0^2 f(x)\,dx$ and $\int_0^2 g(x)\,dx$, all the values $f(c_i)$ and $g(c_i)$ are going to be exactly the same for same index number i, since the only difference between $f(x)$ and $g(x)$ occurs at $x = 1$, and $x = 1$ is never going to be one of the c_i's. Thus the approximated Riemann sums for $\int_0^2 f(x)\,dx$ and $\int_0^2 g(x)\,dx$ are going to be the same.

46. Let $g(x) = |f(x)|$.

Let $h(x) = \begin{cases} f(x) & \text{if } \int_a^b f(x)\,dx > 0 \\ -f(x) & \text{if } \int_a^b f(x)\,dx < 0 \end{cases}$

So $\int_a^b h(x)\,dx = \left| \int_a^b f(x)\,dx \right|$.

Then for each value of x in $[a, b]$, $h(x) \le g(x)$, hence using Theorem 4.3,

$$\int_a^b h(x)\,dx \le \int_a^b g(x)\,dx,$$

which means that

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

47. $\int_0^4 f(x)\,dx$

$= \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx$

$= \int_0^1 2x\,dx + \int_1^4 4\,dx$

$\int_0^1 2x\,dx$ is the area of a triangle with base 1 and height 2 and therefore has area $\frac{1}{2}(1)(2) = 1$.

$\int_1^4 4\,dx$ is the area of a rectangle with base 3 and height 4 and therefore has area $(3)(4) = 12$.

Therefore

$$\int_0^4 f(x)\,dx = 1 + 12 = 13$$

48. $\int_0^4 f(x)\,dx$

$= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$= \int_0^2 2\,dx + \int_2^4 3x\,dx$

$\int_0^2 2\,dx$ is the area of a square with base 2 and height 2 (it is, after all, a square) and therefore has area 4.

$\int_2^4 3x\,dx$ is a trapezoid with height 3 and bases 6 and 12 and therefore has area (using the formula in the front of the text) $\frac{1}{2}(6+12)(2) = 18$.

Therefore

$$\int_0^4 f(x)\,dx = 4 + 18 = 22$$

49. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} b(t)\,dt$.

Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} a(t)\,dt$.

Of course, the net change in population is the number of birth minus the number of deaths:

Population Change
= Births − Deaths

$$= \int_0^{12} b(t)\,dt - \int_0^{12} a(t)\,dt$$

$$= \int_0^{12} [b(t) - a(t)]\,dt.$$

Next we solve the inequality
$410 - 0.3t > 390 + 0.2t$
$20 > 0.5t$ then $t < 40$ months

Therefore $b(t) > a(t)$ when $t < 40$ months. The population is increasing when the birth rate is greater than the death rate, which is during the first 40 month. After 40 months, the population is decreasing. The population would reach a maximum at $t = 40$ months.

50. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} b(t)\,dt$.

Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} a(t)\,dt$.

Of course, the net change in population is the number of birth minus the number of deaths:

Population Change
= Births − Deaths

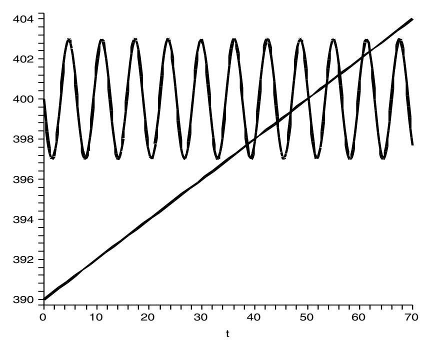
$$= \int_0^{12} b(t)\,dt - \int_0^{12} a(t)\,dt$$

$$= \int_0^{12} [b(t) - a(t)]\,dt.$$

By graphing $b(t)$ and $a(t)$ we see that their graphs intersect 9 times, at
$t \approx 38.5, 40.1, 44.4, 46.9, 50.2, 53.6,$
$56.1, 60.5, 61.9$

This tells us that we have $b(t) > a(t)$ on the intervals
$(0, 38.5), (40.1, 44.4), (46.9, 50.2),$
$(53.6, 56.1), (60.5, 61.9)$

The maximum population will occur when $t = 50.2$.



51. From $PV = 10$ we get $P(V) = 10/V$. By definition,

$$\int_2^4 P(V)\,dV = \int_2^4 \frac{10}{V}\,dV$$

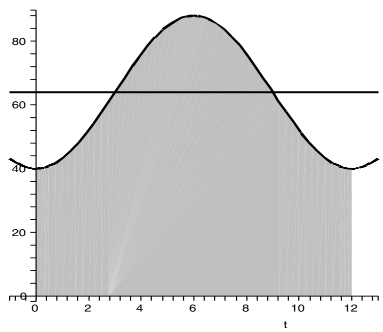
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cdot \frac{10}{2 + \frac{2i}{n}}$$

An estimate of the value of this integral is setting $n = 100$, and then the integral ≈ 6.93

52. The average temperature over the year is

$$\frac{1}{12}\int_0^{12} 64 - 24\cos\left(\frac{\pi}{6}t\right)\ dt$$

If you look at the graphs $T(t)$ and $f(t) = 64$ you should be able to see that the area under $T(t)$ and $f(t)$ between $t = 0$ to $t = 12$ are equal. This means that the average temperature is 64.



53. $\frac{1}{2-0}\int_0^2 f(x)dx = 5$
$\int_0^2 f(x)dx = 10$
and
$\frac{1}{6-2}\int_2^6 f(x)dx = 11$
$\int_2^6 f(x)dx = 44$

The average value of f over $[0, 6]$ is
$\frac{1}{6-0}\int_0^6 f(x)dx$
$= \frac{1}{6}\left(\int_0^2 f(x)dx + \int_2^6 f(x)dx\right)$
$= \frac{1}{6}(10 + 44) = 9$

54. $\frac{1}{b-a}\int_a^b f(x)dx = v$
$\int_a^b f(x)dx = v(b-a)$
and
$\frac{1}{c-b}\int_b^c f(x)dx = w$
$\int_b^c f(x)dx = w(c-b)$

The average value of f over $[a, c]$ is
$\frac{1}{c-a}\int_a^c f(x)dx$
$= \frac{1}{c-a}\left[\int_a^b f(x)dx + \int_b^c f(x)dx\right]$
$= \frac{1}{c-a}[v(b-a) + w(c-b)]$
$= \frac{v(b-a) + w(c-b)}{c-a}$

55. $\int_0^2 3xdx = \frac{1}{2}bh = \frac{1}{2}(2)(6) = 6$

56. $\int_1^4 2xdx = \frac{1}{2}(a+b)h = \frac{1}{2}(2+8)(3)$
$= 15$

57. $\int_0^2 \sqrt{4-x^2} = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi\left(2^2\right) = \pi$

58. $\int_{-3}^0 \sqrt{9-x^2}dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi 3^2$
$= \frac{9\pi}{4}$

59. (a) Average temperature
$= \frac{1}{24}[3(44) + 3(52) + 3(70) + 3(82)$
$+3(86)+3(80)+3(72)+3(56)\ ]$
$= \frac{3}{24}[44 + 52 + 70 + 82 + 86 + 80$
$+ 72 + 56\ ]$
$= \frac{542}{8} = 67.75$

(b) average temperature
$= \frac{1}{24}[3(46) + 3(44) + 3(52) + 3(70)$
$+3(82)+3(86)+3(80)+3(72)\ ]$
$= \frac{3}{24}[46 + 44 + 52 + 70 + 82 + 86$
$+ 80 + 72\ ]$
$= \frac{1}{8}[532] = 66.5$

60. In Exercise 59, the estimate in part (a) is the average temperature of the time interval $[3:00, 12:00]$, recorded every 3 hours. The estimate in part (b) is the average temperature of the time interval $[12:00, 9:00]$, recorded every 3 hours.

61. Since r is the rate at which items are shipped, rt is the number of items shipped between time 0 and time t. Therefore, $Q - rt$ is the number of items remaining in inventory at time t. Since $Q - rt = 0$ when $t = Q/r$, the formula is valid for $0 \le t \le Q/r$. The average value of $f(t) = Q - rt$ on the time interval $[0, Q/r]$ is $\dfrac{1}{Q/r - 0}\displaystyle\int_0^{Q/r} f(t)dt$

$$= \frac{r}{Q}\int_0^{Q/r}(Q - rt)dt$$

$$= \frac{r}{Q}\left[Qt - \frac{1}{2}rt^2\right]_0^{Q/r}$$

$$= \frac{r}{Q}\left[\frac{Q^2}{r} - \frac{r}{2}\frac{Q^2}{r^2}\right]$$

$$= \frac{r}{Q}\left[\frac{Q^2}{2r}\right] = \frac{Q}{2}$$

62. $f(Q) = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{2}$

$f'(Q) = -\dfrac{c_0 D}{Q^2} + \dfrac{c_c}{2}$

Setting $f'(Q) = 0$ gives

$$\frac{c_0 D}{Q^2} = \frac{c_c}{2}$$

$$Q = \sqrt{\frac{2c_0 D}{c_c}}$$

This is the right answer of Q minimizing the total cost $f(Q)$, since when the value of Q is very small, the value of D/Q will get very big, and when the value of Q is very small, the value of $Q/2$ will get very big. This means that the function $f(Q)$ is decreasing on the interval $[0, \sqrt{2c_0D/c_c}]$ and increasing on the interval $[\sqrt{2c_0D/c_c}, \infty]$.

When $Q = \sqrt{2c_0D/c_c}$,

$$c_0\frac{D}{Q} = \frac{c_0 D}{\sqrt{\frac{2c_0D}{c_c}}} = c_c\frac{\sqrt{\frac{2c_0D}{c_c}}}{2} = c_c\frac{Q}{2}$$

63. Delivery is completed in time Q/p, and since in that time Qr/p items are shipped, the inventory when delivery is completed is

$$Q - \frac{Qr}{p} = Q\left(1 - \frac{r}{p}\right)$$

The inventory at any time is given by

$$g(t) = \begin{cases} (p-r)t & \text{for } \ t \in \left[0, \frac{Q}{p}\right] \\ Q - rt & \text{for } \ t \in \left[\frac{Q}{p}, \frac{Q}{r}\right] \end{cases}$$

The graph of g has two linear pieces. The average value of g over the interval $[0, Q/r]$ is the area under the graph (which is the area of a triangle of base Q/r and height $Q(1 - r/p)$) divided by the length of the interval (which is the base of the triangle). Thus the average value of the function is $(1/2)bh$ divided by b, which is

$$(1/2)h = (1/2)Q(1 - r/p)$$

This time the total cost

$$f(Q) = c_0\frac{D}{Q} + c_c\frac{Q}{2}(1 - \frac{r}{p})$$

$$f'(Q) = -\frac{c_0 D}{Q^2} + \frac{c_c(1 - \frac{r}{p})}{2}$$

$$f'(Q) = 0 \text{ gives } \frac{c_0 D}{Q^2} = \frac{c_c}{2}(1 - \frac{r}{p})$$

$$Q = \sqrt{\frac{2c_0 D}{c_c(1 - r/p)}}$$

The order size to minimize the total cost is

$$Q = \sqrt{\frac{2c_0 D}{c_c(1 - r/p)}}$$

64. Use the result from Exercise 62,

$$Q = \sqrt{\frac{2c_0 D}{c_c}}$$
$$= \sqrt{\frac{2(50,000)(4000)}{3800}} \approx 324.44.$$

Since this quantity already takes advantage of largest possible discount, the order size that minimizes the total cost is about 324.44 items.

65. The maximum of

$$F(t) = 9 - 10^8(t - 0.0003)^2$$

occurs when $10^8(t - 0.0003)^2$ reaches its minimum, that is, when $t = 0.0003$. At that time

$F(0.0003) = 9$ thousand pounds.

We estimate the value of

$$\int_0^{0.0006} [9 - 10^8(t - 0.0003)^2]\, dt$$

using midpoint sum and $n = 20$, and get $m\Delta v \approx 0.00360$ thousand pound-seconds, so $\Delta v \approx 360$ ft per second.

66. The impulse-momentum equation of Problem 65 gives

$$5\Delta v$$
$$= \int_0^{0.4} (1000 - 25,000(t - 0.2)^2)\, dt$$
$$= \int_0^{0.4} (-25000t^2 + 10000t)\, dt$$

Using a midpoint sum and $n = 20$ gives an approximation for this integral of 267.0.. This means $5\Delta v \approx 267$ and $\Delta v \approx 53.4$ m/s

67. Since $f(x) = x^3$ is an odd function, the area under the curve and above the x-axis from 0 to 1 is the same as the area under the x-axis and above the curve for -1 to 0.

You can see that $\int_{-1}^{1} x^3 e^{-x} dx < 0$ by thinking of e^{-x} as a weighting factor. Since the weighting factor is always positive, and is greater for $x < 0$ than it is for $x > 0$, the "negative" area to the left of $x = 0$ counts more than the "positive" area to the right of $x = 0$.

68. The Riemann sum for n rectangles is $R_n = \sum_{i=1}^{n} f(c_i)\, \Delta x$, where c_i is in the interval $[x_{i-1}, x_i]$ and $\Delta x = x_i - x_{i-1}$.

On the subinterval $[x_{i-1}, x_i]$, we apply the Integral Mean Value Theorem and choose a point c_i such that

$$f(c_i) = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} f(x)\, dx$$
$$= \frac{1}{\Delta x} \int_{x_{i-1}}^{x_i} f(x)\, dx$$

With this choice for each c_i, we have

$$f(c_i)\, \Delta x = \int_{x_{i-1}}^{x_i} f(x)\, dx$$

and therefore

$$R_n = \sum_{i=1}^{n} f(c_i)\, \Delta x = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx$$

Now notice that the endpoints of these integrals are all adjacent and we can apply part (iv) of Theorem 2.2 to combine all the integrals into one integral.

$$R_n = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx$$
$$= \int_{x_0}^{x_n} f(x)\, dx = \int_a^b f(x)\, dx$$

4.5 The Fundamental Theorem Of Calculus

1. $\int_0^2 (2x-3)\,dx$
$= \left(x^2-3x\right)\Big|_0^2 = -2$

2. $\int_0^3 (x^2-2)\,dx$
$= \left(\frac{x^3}{3}-2x\right)\Big|_0^3 = 3$

3. $\int_{-1}^1 (x^3+2x)\,dx$
$= \left(\frac{x^4}{4}+x^2\right)\Big|_{-1}^1 = 0$

4. $\int_0^2 (x^3+3x-1)\,dx$
$= \left(\frac{x^4}{4}-\frac{3x^2}{2}-x\right)\Big|_0^2 = 12$

5. $\int_0^4 (\sqrt{x}+3x)\,dx$
$= \left(\frac{2}{3}x^{\frac{3}{2}}+\frac{3x^2}{2}\right)\Big|_0^4 = \frac{88}{3}$

6. $\int_1^2 (4x-2/x^2)\,dx$
$= \left(2x^2+\frac{2}{x}\right)\Big|_1^2 = 5$

7. $\int_0^1 (x\sqrt{x}+x^{-\frac{1}{2}})\,dx$
$= \left(\frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{1}{2}}\right)\Big|_0^1 = \frac{12}{5}$

8. $\int_0^8 (\sqrt[3]{x}-x^{2/3})\,dx$
$= \left(\frac{3}{4}x^{4/3}-\frac{3}{5}x^{5/3}\right)\Big|_0^8 = -\frac{36}{5}$

9. $\int_0^{\frac{\pi}{4}} (\sec x\tan x)\,dx$
$= \sec x\Big|_0^{\frac{\pi}{4}} = \sqrt{2}-1$

10. $\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = 1$

11. $\int_{\pi/2}^{\pi} (2\sin x-\cos x)\,dx$
$= (-2\cos x-\sin x)\Big|_{\pi/2}^{\pi} = 3$

12. $\int_0^1 (e^x-e^{-x})\,dx$
$= \left(e^x+e^{-x}\right)\Big|_0^1$
$= e+e^{-1}-2$

13. $\int_0^{1/2} \frac{3}{\sqrt{1-x^2}}\,dx$
$= 3\sin^{-1}x\Big|_0^{1/2}$
$= 3\left(\frac{\pi}{6}-0\right) = \frac{\pi}{2}$

14. $\int_{-1}^1 \frac{4}{1+x^2}\,dx$
$= 4\arctan x\Big|_{-1}^1 = 2\pi$

15. $\int_1^4 \frac{x-3}{x}\,dx$
$= \int_1^4 \left(1-3x^{-1}\right)\,dx$
$= (x-3\ln|x|)\Big|_1^4 = 3-3\ln 4$

16. $\int_1^2 \frac{x^2-3x+4}{x^2}\,dx$
$= \int_1^2 1-\frac{3}{x}+\frac{4}{x^2}\,dx$
$= \left(x-3\ln|x|-\frac{4}{x}\right)\Big|_1^2$
$= -3\ln 2+3$

17. $\displaystyle\int_0^4 x(x-2)\,dx$

$\displaystyle= \left(\frac{x^3}{3} - x^2\right)\Bigg|_0^4 = \frac{16}{3}$

18. $\displaystyle\int_0^{\pi/3} 3\sec^2 x\,dx$

$\displaystyle= (3\tan x)\ \Big|_0^{\pi/3} = 3\sqrt{3}$

19. $\displaystyle\int_0^{\ln 2} (e^{x/2})^2\,dx$

$\displaystyle= (e^x)\ \Big|_0^{\ln 2} = 2 - 1 = 1$

20. $\displaystyle\int_0^{\pi} (\sin^2 x + \cos^2 x)\,dx$

$\displaystyle= \int_0^{\pi} 1\,dx = (x)\ \Big|_0^{\pi} = \pi$

21. $\displaystyle\int_0^2 \sqrt{x^2+1}\,dx$

$\displaystyle= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{\left(\frac{2i}{n} + 1\right)}$

Estimating using $n = 20$, we get the Riemann sum ≈ 2.96

22. $\displaystyle\int_0^2 (\sqrt{x}+1)^2\,dx$

$\displaystyle= \int_0^2 (x + 2\sqrt{x} + 1)\,dx$

$\displaystyle= \left(\frac{x^2}{2} + \frac{4}{3}x^{3/2} + x\right)\ \Bigg|_0^2$

$\displaystyle= 4 + \frac{8\sqrt{2}}{3}$

23. $\displaystyle\int_1^4 \frac{x^2}{x^2+4}\,dx$

$\displaystyle= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\frac{(1+(3i/n)^2)}{(1+3i/n)^2+4}$

Estimating using $n = 20$, we get the Riemann sum ≈ 1.71

24. $\displaystyle\int_1^4 \frac{x^2+4}{x^2}\,dx$

$\displaystyle= \int_1^4 1 + \frac{4}{x^2}\,dx$

$\displaystyle= \left(x - 4x^{-1}\right)\ \Big|_1^4 = 6$

25. $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

$\displaystyle= \int_0^{\pi/4} \tan x \sec x\,dx$

$\displaystyle= \sec x\ \Big|_0^{\pi/4} = \sqrt{2} - 1$

26. $\displaystyle\int_0^{\pi/4} \frac{\tan x}{\sec^2 x}\,dx$

$\displaystyle= \int_0^{\pi/4} \sin x \cos x\,dx$

$\displaystyle= \int_0^{\pi/4} \frac{1}{2}\sin 2x\,dx$

$\displaystyle= \left(-\frac{1}{4}\cos 2x\right)\ \Bigg|_0^{\pi/4} = \frac{1}{4}$

27. $f'(x) = x^2 - 3x + 2$

28. $f'(x) = x^2 - 3x - 4$

29. $\displaystyle f'(x) = \left(e^{-(x^2)^2} + 1\right)\frac{d}{dx}(x^2)$

$= \left(e^{-x^4} + 1\right)(2x)$

30. $f'(x) = \sin(x^2+1)(2x)$

31. $f'(x) = -\ln(x^2+1)$

32. $f'(x) = -\sec x.$

33. $y'(x) = \sin\sqrt{x^2+\pi^2}$
At the point in question, $y(0) = 0$ and $y'(0) = \sin\pi = 0$

Therefore, the tangent line has slope 0 and passes through the point $(0,0)$. The equation of this line is $y = 0$.

34. $y'(x) = \ln(x^2 + 2x + 2)$
At the point in question, $y(-1) = 0$ and
$y'(-1) = \ln 1 = 0$

Therefore, the tangent line has slope 0 and passes through the point $(-1,0)$. The equation of this line is $y=0$.

35. $y'(x)=\cos(\pi x^3)$
At the point in question, $y(2)=0$ and $y'(2)=\cos 8\pi=1$

Therefore, the tangent line has slope 1 and passes through the point $(2,0)$. The equation of this line is $y=x-2$.

36. $y'(x)=e^{-x^2+1}$
At the point in question, $y(0)=0$ and $y'(0)=e$

Therefore, the tangent line has slope e and passes through the point $(0,0)$. The equation of this line is $y=ex$.

37. $f'(x)=x^2-3x+2$

Setting $f'(x)=0$ we get

$(x-1)(x-2)=0, x=1,2.$

$$f'(x)\begin{cases} >0 & \text{when } t<1 \text{ or } t>2 \\ <0 & \text{when } 1<t<2 \end{cases}$$

$$f(1)=\int_0^1 (t^2-3t+2)\,dt$$
$$=(t^3/3-3t^2/2+2t)\Big|_0^1=\frac{5}{6}$$
$$f(2)=\int_0^2 (t^2-3t+2)\,dt$$
$$=(t^3/3-3t^2/2+2t)\Big|_0^2=\frac{2}{3}$$

Hence $f(x)$ has a local maximum at the point $(1,5/6)$ and a local minimum at the point $(2,2/3)$.

38. $$\int_0^x [f(t)-g(t)]\,dt$$
$$=\int_0^x [55+10\cos t-(50+2t)]\,dt$$
$$=\int_0^x (5+10\cos t-2t)\,dt$$
$$=5t+\sin t-t^2\Big|_0^x$$
$$=5x+\sin x-x^2$$

Since we are integrating the difference in speeds, the integral represents the distance that Katie is ahead at time x. Of course, if this value is negative, is means that Michael is really ahead.

39. The graph of $y=4-x^2$ is above the x-axis over the interval $[-2,2]$.

$$\int_{-2}^2 (4-x^2)\,dx$$
$$=(4x-\frac{x^3}{3})\Big|_{-2}^2$$
$$=\frac{32}{3}$$

40. The graph of $y=x^2-4x$ is below the x-axis over the interval $[0,4]$.

$$\int_0^4 -(x^2-4x)\,dx$$
$$=\left(-\frac{x^3}{3}+2x^2\right)\Big|_0^4=\frac{32}{3}.$$

41. The graph of $y=x^2$ is above the x-axis over the interval $[0,2]$.

$$\int_0^2 x^2\,dx=\frac{x^3}{3}\Big|_0^2=\frac{8}{3}$$

42. The graph of $y=x^3$ is above the x-axis over the interval $[0,3]$.

$$\int_0^3 x^3\,dx=\left(\frac{x^4}{4}\right)\Big|_0^3=\frac{81}{4}$$

43. The graph of $y=\sin x$ is above the x-axis over the interval $[0,\pi]$.

$$\int_0^\pi \sin x\,dx=-\cos x\Big|_0^\pi=2$$

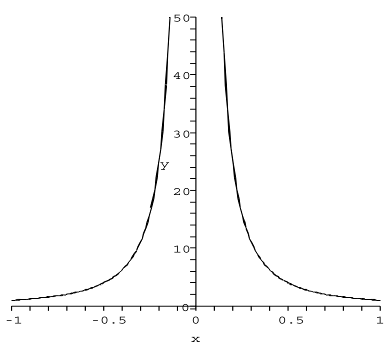
44. The graph of $y=\sin x$ is below the x-axis over the interval $[-\pi/2,0]$ and above the x-axis over the interval $[0,\pi/4]$. Hence we need to compute two separate integrals and add them together:

$$\int_{-\pi/2}^0 -\sin x\,dx+\int_0^{\pi/4} \sin x\,dx$$

$$= (\cos x)\Big|_{-\pi/2}^{0} + (-\cos x)\Big|_{0}^{\pi/4}$$
$$= 1 + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \frac{1}{\sqrt{2}}.$$

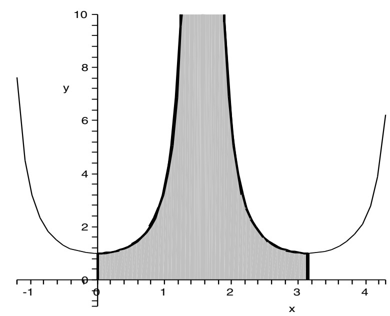
45. If you look at the graph of $1/x^2$, it is obvious that there is positive area between the curve and the x-axis over the interval $[-1, 1]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem.

The problem is that $1/x^2$ is not continuous on $[-1, 1]$ (the discontinuity occurs at $x = 0$) and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I (Theorem 4.1).



46. If you look at the graph of $\sec^2 x$, it is obvious that there is positive area between the curve and the x-axis over the interval $[0, \pi]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem.

The problem is that $\sec^2 x$ is not continuous on $[0, \pi]$ and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I (Theorem 4.1).



47. $s(t) = 40t + \cos t + c$,
$s(0) = 0 + \cos 0 + c = 2$ so therefore $c = 1$ and
$s(t) = 40t + \cos t + 1$.

48. $s(t) = 10e^t + c$,
$s(0) = 10 + c = 2$ so therefore $c = -8$ and
$s(t) = 10e^{-t} - 8$.

49. $v(t) = 4t - \dfrac{t^2}{2} + c_1$,
$v(0) = c_1 = 8$ so therefore $c_1 = 8$ and
$v(t) = 4t - \dfrac{t^2}{2} + 8$.

$s(t) = 2t^2 - \dfrac{t^3}{6} + 8t + c_2$,
$s(0) = c_2 = 0$ so therefore $c_2 = 0$ and
$s(t) = 2t^2 - \dfrac{t^3}{6} + 8t$.

50. $v(t) = 16t - \dfrac{t^3}{3} + c_1$,
$v(0) = c_1 = 0$ so therefore $c_1 = 0$ and
$v(t) = 16t - \dfrac{t^3}{3}$.

$s(t) = 8t^2 - \dfrac{t^4}{12} + c_2$,
$s(0) = c_2 = 30$ so therefore $c_2 = 30$ and
$s(t) = 8t^2 - \dfrac{t^4}{12} + 30$.

51. $\omega(t) = 10t + c_1$ and $\omega(0) = 0$ gives that $c_1 = 0$ and hence $\omega(t) = 10t$.
$\omega(0.8) = 8$ rad/s.
$v(0.8) = 3(8) = 24$ ft/s.

$\theta(t) = 5t^2 + c_2$ and $\theta(0) = 0$, so $c_2 = 0$ and $\theta(t) = 5t^2$
$\theta(0.8) = 5(0.8^2) = 3.2$ rad.

52. The angular velocity of the club is the antiderivative of the angular acceleration:
$\omega(t) = \alpha t + c_1$.
The initial angular velocity is $\omega(0) = 0$ (the club starts at rest) and therefore $c_1 = 0$ and $\omega(t) = \alpha t$.

The angular position of the club is the antiderivative of angular velocity:
$\theta(t) = \dfrac{\alpha t^2}{2} + c_2$.
The initial position of the club is $\theta(0) = 0$ and therefore $c_2 = 0$ and $\theta(t) = \dfrac{\alpha t^2}{2}$.
The club strikes the ball when $\theta(t) = \dfrac{3\pi}{2}$, so we solve
$\dfrac{\alpha t^2}{2} = \dfrac{3\pi}{2}$
and see that this occurs when $t = \sqrt{\dfrac{3\pi}{\alpha}}$.
Next, the angular velocity of the club at this time is
$\omega\left(\sqrt{\dfrac{3\pi}{\alpha}}\right) = \alpha\sqrt{\dfrac{3\pi}{\alpha}} = \sqrt{3\pi\alpha}$.
The linear velocity is 4 times this amount, and we want the linear velocity to be equal to 100 miles per hour, or $\dfrac{440}{3}$ feet per second. Therefore, we need to solve the following equation for α:
$4\sqrt{3\pi\alpha} = \dfrac{440}{3}$.
Solving gives $\alpha = \dfrac{12100}{27\pi} \approx 142.65$ radians per second squared.

53. $f_{ave} = \dfrac{1}{3-1}\displaystyle\int_1^3 (x^2 - 1)\,dx$
$= \dfrac{1}{2}\left(\dfrac{x^3}{3} - x\right)\Big|_1^3 = \dfrac{10}{3}$.

54. $f_{ave} = \dfrac{1}{1-0}\displaystyle\int_0^1 (x^2 + 2x)\,dx$
$= \left(\dfrac{x^3}{3} + x^2\right)\Big|_0^1 = \dfrac{4}{3}$.

55. $f_{ave} = \dfrac{1}{1-0}\displaystyle\int_0^1 (2x - 2x^2)\,dx$
$= \left(x^2 - \dfrac{2x^3}{3}\right)\Big|_0^1 = \dfrac{1}{3}$

56. $f_{ave} = \dfrac{1}{2-1}\displaystyle\int_1^2 (x^3 - 3x^2 + 2x)\,dx$
$= \left(\dfrac{x^4}{4} - x^3 + x^2\right)\Big|_1^2 = -\dfrac{1}{4}$

57. $f_{ave} = \dfrac{1}{\pi/2 - 0}\displaystyle\int_0^{\pi/2} \cos x\,dx$
$= \dfrac{2}{\pi}(\sin x)\Big|_0^{\pi/2} = \dfrac{2}{\pi}$

58. $f_{ave} = \dfrac{1}{\pi/2 - 0}\displaystyle\int_0^{\pi/2} \sin x\,dx$
$= \dfrac{2}{\pi}(-\cos x)\Big|_0^{\pi/2} = \dfrac{2}{\pi}$

59. $\displaystyle\int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx < \int_0^1 f(x)\,dx$

60. $\displaystyle\int_0^1 f(x)\,dx < \int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx$

61. Using the Fundamental Theorem of Calculus, it follows that an antiderivative of e^{-x^2} is $\int_a^x e^{-t^2}\,dt$ where a is a constant.

62. Using the Fundamental Theorem of Calculus, it follows that an antiderivative of $\sin\sqrt{x^2+1}$ is $\int_a^x \sin\sqrt{t^2+1}\,dt$ where a is a constant.

63. $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q (150 - 2q - 3q^2)\,dq - PQ$
$= (150q - q^2 - q^3)\Big|_0^Q - PQ$
$= 150Q - Q^2 - Q^3$
$\quad - (150 - 2Q - 3Q^2)Q$
$= Q^2 + 2Q^3$

When $Q = 4$,
$CS = 16 + 2(64) = 144$ dollars.

When $Q = 6$,
$CS = 36 + 2(216) = 468$ dollars.
The consumer surplus is higher for $Q = 6$ than that for $Q = 4$.

64. $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q 40e^{-0.05q}\,dq - PQ$
$= (-800e^{-0.05q})\Big|_0^Q - PQ$
$= -800e^{-0.05Q} + 800 - 40e^{-0.05Q}$
$= -840e^{-0.05Q} + 800$

When $Q = 10$,
$CS = -840e^{-0.5} + 800 \approx 290.5$ dollars.

When $Q = 20$,
$CS = -840e^{-1} + 800 \approx 491.0$ dollars.
The consumer surplus is higher for $Q = 20$ than that for $Q = 10$.

65. The next shipment must arrive when the inventory is zero. This occurs at time T: $f(t) = Q - r\sqrt{t}$
$f(T) = 0 = Q - r\sqrt{T}$
$r\sqrt{T} = Q$
$T = \dfrac{Q^2}{r^2}$

The average value of f on $[0, T]$ is:
$\dfrac{1}{T}\int_0^T f(t)\,dt$
$= \dfrac{1}{T}\int_0^T (Q - rt^{1/2})\,dt$
$= \dfrac{1}{T}\left[Qt - \dfrac{2}{3}rt^{3/2}\right]_0^T$
$= \dfrac{1}{T}\left[QT - \dfrac{2}{3}rT^{3/2}\right]$
$= Q - \dfrac{2}{3}r\sqrt{T}$
$= Q - \dfrac{2}{3}r\dfrac{Q}{r}$
$= \dfrac{Q}{3}$

66. The total annual cost
$f(Q) = c_0\dfrac{D}{Q} + c_cA = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{3}$.
$f'(Q) = -c_0\frac{D}{Q^2} + c_c\frac{1}{3}$
$f'(Q) = 0$ gives that $Q = \sqrt{\frac{3c_0D}{c_c}}$

This value of Q minimizes the total cost, since
$$f'(Q)\begin{cases} > 0 & \text{when } Q < \sqrt{\frac{3c_0D}{c_c}} \\ < 0 & \text{when } Q > \sqrt{\frac{3c_0D}{c_c}} \end{cases}.$$
When $Q = \sqrt{\frac{3c_0D}{c_c}}$,
$c_0\dfrac{D}{Q} = c_0\dfrac{D}{\sqrt{3c_0D/c_c}} = c_c\dfrac{Q}{3} = c_cA.$

67. When $a < 2$ or $a > 2$, f is continuous. Using the Fundamental Theorem of Calculus,
$\left[\lim_{x\to a} F(x)\right] - F(a)$
$= \lim_{x\to a}\,[F(x) - F(a)]$
$= \lim_{x\to a}\left[\int_0^x f(t)\,dt - \int_0^a f(t)\,dt\right]$
$= \lim_{x\to a}\left[\int_a^x f(t)\,dt\right]$
$= 0$

When $a = 2$,

$\lim_{x\to a^-}\left[\int_a^x f(t)\,dt\right]$

$= \lim_{x\to 2^-} \left[\int_2^x t\,dt\right]$
$= \lim_{x\to 2^-} \left[\frac{t^2}{2}\right]_0^x$
$= \lim_{x\to 2^-} \left[\frac{x^2}{2} - \frac{2^2}{2}\right]$
$= 0$ and
$\lim_{x\to a^+} \left[\int_a^x f(t)\,dt\right]$
$= \lim_{x\to 2^+} \left[\int_2^x (t+1)\,dt\right]$
$= \lim_{x\to 2^+} \left[\frac{t^2}{2} + t\right]_0^x$
$= \lim_{x\to 2^+} \left[\frac{x^2}{2} + x - \frac{2^2}{2} - 2\right]$
$= 0$

Thus, for all values of a,

$\left[\lim_{x\to a} F(x)\right] - F(a) = 0$
$\lim_{x\to a} F(x) = F(a)$

Thus, F is continuous for all x. However, $F'(2)$ does not exist, which is shown as follows:

$F'(2) = \lim_{h\to 0} \frac{F(2+h) - F(2)}{h}$
$= \lim_{h\to 0} \frac{1}{h} \left[\int_0^{2+h} f(t)\,dt - \int_0^2 f(t)\,dt\right]$
$= \lim_{h\to 0} \frac{1}{h} \int_2^{2+h} f(t)\,dt$

We'll show that this limit does not exist by showing that the left and right limits are different. The right limit is

$\lim_{h\to 0^+} \frac{1}{h} \int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h} \int_2^{2+h} (t+1)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h} \left[\frac{t^2}{2} + t\right]_2^{2+h}$
$= \lim_{h\to 0^+} \frac{1}{h} \left[\frac{(2+h)^2}{2} + 2 + h - \frac{2^2}{2} - 2\right]$
$= \lim_{h\to 0^+} \frac{1}{h} \left[\frac{h^2+4h+4}{2} + 2 + h - 4\right]$
$= \lim_{h\to 0^+} \frac{1}{h} \left[\frac{h^2}{2} + 3h\right]$
$= \lim_{h\to 0^+} \frac{1}{h} \left[\frac{h}{2} + 3\right]$
$= 3$

The left limit is $\lim_{h\to 0^-} \frac{1}{h} \int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^-} \frac{1}{h} \int_2^{2+h} t\,dt$
$= \lim_{h\to 0^-} \frac{1}{h} \left[\frac{t^2}{2}\right]_2^{2+h}$
$= \lim_{h\to 0^-} \frac{1}{h} \left[\frac{(2+h)^2}{2} - \frac{2^2}{2}\right]$
$= \lim_{h\to 0^-} \frac{1}{h} \left[\frac{h^2+4h+4}{2} - 2\right]$
$= \lim_{h\to 0^-} \frac{1}{h} \left[\frac{h}{2} + 2\right]$
$= 2$

Thus, $F'(2)$ does not exist. This result does not contradict the Fundamental Theorem of Calculus, because in this situation, $f(x)$ is not continuous, and thus The Fundamental Theorem of Calculus does not apply.

68. Let $h_1(x) = \int_0^{x+k} g(t)\,dt$
and $h_2(x) = \int_0^x g(t)\,dt$.
Then $h_1'(x) = g(x+k)$ and
$h_2'(x) = g(x)$.
Since $f(x) = \frac{1}{k}(h_1(x) - h_2(x))$,

$f'(x) = \frac{1}{k}(h_1'(x+k) - h_2'(x))$
$= \frac{g(x+k) - g(x)}{k}$

In other words, the derivative of $f(x)$

is the slope of the secant line between the two points $(x, g(x))$ and $(x+k, g(x+k))$.

69. $g(x) = \int_0^x \left[\int_0^u f(t)\,dt \right] du$
$g'(x) = \int_0^x f(t)\,dt$
$g''(x) = f(x)$
A zero of f corresponds to a zero of the second derivative of g (possibly an inflection point of g).

70. When $x = 0$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(0) = f(0)$.
When $0 < x < 1$, $\lim_{n\to x^n} = 0$, and then
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= f(\lim_{n\to infty} x^n) = f(0)$.
When $x = 1$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(1) = f(1)$.
The integral $\int_0^1 g_n(x)\,dx$ represents the net area between the graph of $f(x^n)$ and the x-axis. As n approaches ∞,
$f(x^n) \to \begin{cases} f(0) & \text{when } 0 \le x < 1 \\ f(1) & \text{when } x = 1 \end{cases}$
Thus the integral $\int_0^1 g_n(x)\,dx$ approaches the area of the shape of a rectangle with length 1 and width $f(0)$ (possibly negative), which means
$\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = f(0)$

71. The integrals in parts (a) and (c) are improper, because the integrands have asymptotes at one of the limits of integration. The Fundamental Theorem of Calculus applies to the integral in part (b).

72. The Fundamental Theorem of Calculus applies to the integral in part (a) and (b). The integral in part (c) is improper since the point $x = \pi/2$ lies in the interval $[0, 2]$, and $\sec x$ is not defined at $x = \pi/2$.

73. (a) $\lim_{n\to\infty} \frac{1}{n} \left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\pi \right]$
$= \int_0^1 \sin(\pi x)\,dx$
$= -\frac{1}{\pi}\cos(\pi x)\Big|_0^1$
$= -\frac{1}{\pi}(\cos\pi - \cos 0)$
$= -\frac{1}{\pi}(-1-1)$
$= \frac{2}{\pi}$

(b) $\lim_{n\to\infty} \frac{2}{n} \left[\frac{1}{1+2/n} + \frac{1}{1+4/n} + \cdots + \frac{1}{3} \right]$
$= \int_0^2 \frac{1}{1+x}\,dx$
$= \ln|1+x|\Big|_0^2$
$= \ln 3 - \ln 1$
$= \ln 3$

74. (a) $\lim_{n\to\infty} \frac{1}{n} \left[e^{4/n} + e^{8/n} + \cdots + e^4 \right]$
$= \int_0^1 e^{4x}\,dx = \frac{1}{4} e^{4x}\Big|_0^1 = \frac{e^4 - 1}{4}$

(b) $\lim_{n\to\infty} \frac{4}{n} \left[\frac{2}{\sqrt{n}} + \frac{2\sqrt{2}}{\sqrt{n}} + \cdot \frac{2}{1} \right]$
$= \int_0^4 \sqrt{x}\,dx$
$= \frac{2}{3} x^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}$

75. Let $F(x) = \int_{a(x)}^{b(x)} f(t)\,dt$,
$G(x) = \int_0^{b(x)} f(t)\,dt$,
$H(x) = \int_0^{a(x)} f(t)\,dt$,

Then
$F(x) = G(x) - H(x)$.
$G'(x) = f(b(x))b'(x)$
$H'(x) = f(a(x))a'(x)$.

$F'(x) = G'(x) - H'(x)$
$= f(b(x))b'(x) - f(a(x))a'(x)$.

4.6 Integration by Substitution

1. Let $u = x^3 + 2$ and then $du = 3x^2\,dx$ and
$$\int x^2\sqrt{x^3+2}\,dx = \frac{1}{3}\int u^{-1/2}\,du$$
$$= \frac{2}{9}u^{3/2} + c = \frac{2}{9}(x^3+2)^{3/2} + c$$

2. Let $u = x^4 + 1$ and then $du = 4x^3\,dx$ and
$$\int x^3(x^4+1)^{-2/3}\,dx = \frac{1}{4}\int u^{-2/3}\,du$$
$$= \frac{3}{4}u^{1/3} + c = \frac{3}{4}(x^4+1)^{1/3} + c$$

3. Let $u = \sqrt{x} + 2$ and then $du = \frac{1}{2}x^{-1/2}\,dx$ and
$$\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\,dx = 2\int u^3\,du$$
$$= \frac{2}{4}u^4 + c = \frac{1}{2}(\sqrt{x}+2)^4 + c$$

4. Let $u = \sin x$ and then $du = \cos\,dx$ and
$$\int \sin x\cos x\,dx = \int u\,du$$
$$= \frac{u^2}{2} + c = \frac{\sin^2 x}{2} + c$$

5. Let $u = x^4 + 3$ and then $du = 4x^3\,dx$ and
$$\int x^3\sqrt{x^4+3}\,dx = \frac{1}{4}\int u^{1/2}\,du$$
$$= \frac{1}{6}u^{3/2} + c = \frac{1}{6}(x^4+3)^{3/2} + c$$

6. Let $u = \tan x$ and then $du = \sec^2 x\,dx$ and
$$\int \sec^2 x\sqrt{\tan x}\,dx = \int u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(\tan x)^{3/2} + c$$

7. Let $u = \cos x$ and then $du = -\sin x\,dx$ and
$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int \frac{du}{\sqrt{u}}$$
$$= -2\sqrt{u} + c = -2\sqrt{\cos x} + c$$

8. Let $u = \sin x$ and then $du = \cos x\,dx$ and
$$\int \sin^3 x\cos x\,dx = \int u^3\,du$$
$$= \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$$

9. Let $u = x^3$ and then $du = 3x^2\,dx$ and
$$\int x^2\cos x^3\,dx = \frac{1}{3}\int \cos u\,du$$
$$= \frac{1}{3}\sin u + c = \frac{1}{3}\sin x^3 + c$$

10. Let $u = \cos x + 3$ and then $du = -\sin x\,dx$ and
$$\int \sin x(\cos x+3)^{3/4}\,dx = -\int u^{3/4}\,du$$
$$= -\frac{4}{7}u^{7/4} + c = -\frac{4}{7}(\cos x+3)^{7/4} + c$$

11. Let $u = x^2 + 1$ and then $du = 2x\,dx$ and
$$\int xe^{x^2+1}\,dx = \frac{1}{2}\int e^u\,du$$
$$= \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2+1} + c$$

12. Let $u = e^x + 4$ and then $du = e^x\,dx$ and
$$\int e^x\sqrt{e^x+4}\,dx = \int \sqrt{u}\,du$$
$$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(e^x+4)^{3/2} + c$$

13. Let $u = \sqrt{x}$ and then $du = \frac{1}{2\sqrt{x}}\,dx$ and

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du$$
$$= 2e^u + c = 2e^{\sqrt{x}} + c$$

14. Let $u = x^2 + 2x - 1$ and then $du = 2(x+1)\,dx$ and
$$\int \frac{x+1}{(x^2+2x-1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du$$
$$= -\frac{1}{2}u^{-1} + c = -\frac{1}{2(x^2+2x-1)} + c$$

15. Let $u = \ln x$, and then $du = \frac{1}{x}\,dx$ and
$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du$$
$$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(\ln x)^{3/2} + c$$

16. Let $u = \frac{1}{x}$ and then $du = -\frac{1}{x^2}\,dx$ and
$$\int \frac{\cos\frac{1}{x}}{x^2}\,dx = -\int \cos u\,du$$
$$= -\sin u + c = -\sin\frac{1}{x} + c$$

17. Let $u = \sqrt{x} + 1$ and then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and
$$\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx = 2\int u^{-2}\,du$$
$$= -2u^{-1} + c = -2\left(\sqrt{x}+1\right)^{-1} + c$$

18. Let $u = x^2 + 4$ and then $du = 2x\,dx$ and
$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+4| + c$$
$$= \frac{1}{2}\ln(x^2+4) + c$$

19. Let $u = \ln x + 1$ and then $du = \dfrac{1}{x}\,dx$ and
$$\int \frac{4}{x(\ln x+1)^2}\,dx = 4\int u^{-2}\,du$$
$$= -4u^{-1} + c = -4(\ln x + 1)^{-1} + c$$

20. Let $u = \cos 2x$ and then $du = -2\sin 2x\,dx$ and
$$\int \tan 2x\,dx = -\frac{1}{2}\int \frac{1}{u}\,du$$
$$= -\frac{1}{2}\ln|u| + c = -\frac{1}{2}\ln|\cos 2x| + c$$

21. Let $u = \sin^{-1} x$ and then $du = \frac{1}{\sqrt{1-x^2}}\,dx$ and
$$\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx = \int u^3\,du$$
$$= \frac{u^4}{4} + c = \frac{(\sin^{-1}x)^4}{4} + c$$

22. Let $u = x^3$ and then $du = 3x^2\,dx$ and
$$\int x^2\sec^2 x^3\,dx = \frac{1}{3}\int \sec^2 u\,du$$
$$= \frac{1}{3}\tan u + c = \frac{1}{3}\tan x^3 + c$$

23. Let $u = x^2$ and then $du = 2xdx$ and
$$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$$
$$= \frac{1}{2}\sin^{-1}u + c = \frac{1}{2}\sin^{-1}x^2 + c$$

24. Let $u = 1 - x^4$ and then $du = -4x^3\,dx$ and
$$\int \frac{x^3}{(1-x^4)^{1/2}}\,dx = -\frac{1}{4}\int u^{-1/2}\,du$$
$$= -\frac{1}{2}u^{1/2} + c = -\frac{1}{2}(1-x^4)^{1/2} + c$$

25. Let $u = x^3$ and then $du = 3x^2dx$ and
$$\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{1}{1+u^2}du$$
$$= \frac{1}{3}\tan^{-1}u + c = \frac{1}{3}\tan^{-1}x^3 + c$$

26. Let $u = 1 + x^6$ and then $du = 6x^5\,dx$ and
$$\int \frac{x^5}{1+x^6}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$$
$$= \frac{1}{6}\ln|u| + c = \frac{1}{6}\ln|1+x^6| + c$$

27. Let $u = x + 7$ and then $du = dx, x = u - 7$ and

$$\int \frac{2x+3}{x+7}\,dx = \int \frac{2(u-7)+3}{u}\,du$$
$$= \int \left(2 - \frac{11}{u}\right) du = 2u - 11\ln|u| + c$$
$$= 2(x+7) - 11\ln|x+7| + c$$

28. Let $u = x+3$ and then $du = dx$ and
$$\int \frac{x^2}{(x+3)^{1/3}}\,dx = \int \frac{(u-3)^2}{u^{1/3}}\,du$$
$$= \int (u^{5/3} - 6u^{2/3} + 9u^{-1/3})\,du$$
$$= \frac{3}{8}u^{8/3} - \frac{18}{5}u^{5/3} + \frac{18}{2}u^{2/3} + c$$
$$= \frac{3}{8}(x+3)^{8/3} - \frac{18}{5}(x+3)^{5/3}$$
$$+ \frac{18}{2}(x+3)^{2/3} + c$$

29. Let $u = \sqrt{1+\sqrt{x}}$ and then
$(u^2-1)^2 = x$
$2(u^2-1)(2u)du = dx$ and
$$\int \frac{1}{\sqrt{1+\sqrt{x}}}dx$$
$$= \int \frac{4u(u^2-1)}{u}\,du$$
$$= 4\int (u^2-1)\,du$$
$$= 4\left(\frac{u^3}{3} - u\right) + c$$
$$= \frac{4}{3}(1+\sqrt{x})^{3/2} - 4(1+\sqrt{x})^{1/2} + c$$

30. Let $u = x^2$ and then $du = 2x\,dx$ and
$$\int \frac{dx}{x\sqrt{x^4-1}}$$
$$= \int \frac{du/2}{u\sqrt{u^2-1}}$$
$$= \frac{1}{2}\sec^{-1} u + c$$
$$= \frac{1}{2}\sec^{-1} x^2 + c$$

31. Let $u = x^2+1$ and then
$du = 2x\,dx, u(0) = 1, u(2) = 5$
$$\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^5 \sqrt{u}\,du$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{1}{3}\left(\sqrt{125}-1\right)$$
$$= \frac{5}{3}\sqrt{5} - \frac{1}{3}$$

32. Let $u = \pi x^2$ and then $du = 2\pi x\,dx$ and
$$\int_1^3 x\sin(\pi x^2)\,dx = \frac{1}{2\pi}\int_\pi^{9\pi} \sin u\,du$$
$$= (\sin u)\big|_\pi^{9\pi} = 0$$

33. Let $u = x^2+1$ and then
$du = 2x\,dx, u(-1) = 2 = u(1)$ and
$$\int_{-1}^1 \frac{x}{(x^2+1)^{1/2}}\,dx$$
$$= \frac{1}{2}\int_2^2 u^{-1/2}\,du = 0$$

34. Let $u = x^3$ and then
$du = 3x^2\,dx, u(0) = 0, u(2) = 8$ and
$$\int_0^2 x^2e^{x^3}\,dx = \frac{1}{3}\int_0^8 e^u\,du$$
$$= \frac{1}{3}e^u\Big|_0^8 = \frac{1}{3}(e^8-1)$$

35. Let $u = e^x$ and then
$du = e^x dx, u(0) = 1, u(2) = e^2$ and
$$\int_0^2 \frac{e^x}{1+e^{2x}}dx = \int_1^{e^2} \frac{1}{1+u^2}du$$
$$= \tan^{-1} u\Big|_1^{e^2} = \tan^{-1} e^2 - \tan^{-1} 1$$
$$= \tan^{-1} e^2 - \frac{\pi}{4}$$

36. Let $u = \sqrt{x}$ and then $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$,
$u(0) = \cos 0 = 1, u(\pi^2) = \cos\pi = -1$
and
$$\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int_1^{-1} \cos u\,du$$
$$= \sin u\Big|_1^{-1} = \sin(-1) - \sin(1)$$

37. Let $u = \sin x$ and then $du = \cos x\,dx$
$u(\pi/4) = 1/\sqrt{2}, u(\pi/2) = 1$ and
$$\int_{\pi/4}^{\pi/2} \cot x\,dx = \int_{1/\sqrt{2}}^1 \frac{1}{u}\,du$$
$$= \ln|u|\big|_{1/\sqrt{2}}^1 = \ln\sqrt{2}$$

38. Let $u = \ln x$ and then $du = \frac{1}{x}\,dx$
$u(1) = 0, u(e) = 1$ and

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

39. $$\int_1^4 \frac{x-1}{\sqrt{x}}\,dx = \int_1^4 \left(x^{1/2} - x^{-1/2}\right)dx$$
$$= \left(\frac{2}{3}x^{3/2} - 2x^{1/2}\right)\Big|_1^4$$
$$= \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{8}{3}$$

40. Let $u = x^2 + 1$ and then $du = 2x\,dx$ and

$$\int_0^1 \frac{x}{(x^2+1)^{1/2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}\,du$$
$$= \left(u^{1/2}\right)\Big|_1^2 = \sqrt{2} - 1$$

41. (a) $\int_0^\pi \sin x^2\,dx \approx .77$ using midpoint evaluation with $n \geq 40$

(b) Let $u = x^2$ and then $du = 2x\,dx$ and

$$\int_0^\pi x\sin x^2\,dx = \frac{1}{2}\int_0^{\pi^2} \sin u\,du$$
$$= \frac{1}{2}\left(-\cos u\right)\Big|_0^{\pi^2}$$
$$= -\frac{1}{2}\cos\pi^2 + \frac{1}{2}$$
$$\approx 0.95134$$

42. (a) Let $u = x^2$ and then $du = 2x\,dx, u(-1) = 1, u(1) = 1$

$$\int_{-1}^1 xe^{-x^2}\,dx = \frac{1}{2}\int_1^1 d^{-u}\,du = 0$$

(b) $\int_{-1}^1 e^{-x^2}\,dx \approx 1.4937$ using midpoint evaluation with $n \geq 50$.

43. (a) $\int_0^2 \frac{4x^2}{(x^2+1)^2}\,dx \approx 1.414$ using right endpoint evaluation with $n \geq 50$.

(b) Let $u = x^2 + 1$ and then $du = 2x\,dx, x^2 = u - 1$ and

$$\int_0^2 \frac{4x^3}{(x^2+1)^2}\,dx = \int_1^5 2\frac{u-1}{u^2}\,du$$
$$= \int_1^5 \left(2u^{-1} - 2u^{-2}\right)du$$
$$= \left(2\ln|u| + 2u^{-1}\right)\Big|_1^5$$
$$= 2\ln 5 - \frac{8}{5}$$

44. (a) $\int_0^{\pi/4} \sec x\,dx \approx .88$ using midpoint evaluation with $n \geq 10$.

(b) $$\int_0^{\pi/4} \sec^2 x\,dx$$
$$= (\tan x)\Big|_0^{\pi/4} = 1$$

45. $\frac{1}{2}\int_0^4 f(u)\,du$

46. $\frac{1}{3}\int_1^8 f(u)\,du$

47. $\int_0^1 f(u)\,du$

48. $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_0^2 f(u)\,du$

49. $$\int_{-a}^a f(x)dx$$
$$= \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

Let $u = -x$ and $du = -dx$ in the first integral. Then,

$$\int_{-a}^a f(x)\,dx$$
$$= -\int_a^0 f(-u)\,du + \int_0^a f(x)\,dx$$
$$= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

If f is even, then $f(-u) = f(u)$, and so

$\int_{-a}^{a} f(x)dx$
$= \int_0^a f(u)\,du + \int_0^a f(x)\,dx$
$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$
$= 2\int_0^a f(x)\,dx$

If f is odd, then $f(-u) = -f(u)$, and so

$\int_{-a}^{a} f(x)dx$
$= -\int_0^a f(u)du + \int_0^a f(x)dx$
$= -\int_0^a f(x)dx + \int_0^a f(x)dx$
$= 0$

50. First, let $u = x - T$, then for any a,

$\int_T^{a+T} f(x)\,dx = \int_0^a f(u+T)\,du$
$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx$

If we let $a = T$, then we get

$\int_0^T f(x)\,dx = \int_T^{2T} f(x)\,dx.$

If we let $a = 2T$, then we get

$\int_0^{2T} f(x)\,dx = \int_T^{3T} f(x)\,dx$ and then

$\int_0^T f(x)\,dx = \int_T^{2T} f(x)\,dx$
$= \int_0^{2T} f(x)\,dx - \int_0^T f(x)\,dx$
$= \int_T^{3T} f(x)\,dx - \int_T^{2T} f(x)\,dx$
$= \int_{2T}^{3T} f(x)\,dx$

It is straightforward to see that for any integer i,

$\int_0^T f(x)\,dx = \int_{iT}^{(i+1)T} f(x)\,dx$

Now suppose $0 \leq a \leq T$, then

$\int_0^T f(x)\,dx - \int_a^{a+T} dx$
$= \int_0^a f(x)\,dx - \int_T^{a+T} f(x)\,dx$
$= 0$

So $\int_0^T f(x)\,dx = \int_a^{a+T} dx$

Next suppose a is any number. Then a must lie in some interval $[iT, (i+1)T]$ for some integer i. Use the similar method as in above, we shall get

$\int_{iT}^{(i+1)T} f(x)\,dx = \int_a^{a+T} f(x)\,dx$

And since

$\int_{iT}^{(i+1)T} f(x)\,dx = \int_0^T f(x)\,dx$

we get $\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$

51. Let $u = 10 - x$, so that $du = -\,dx$. Then,

$I = \int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$
$= -\int_{x=0}^{x=10} \frac{\sqrt{10-u}}{\sqrt{10-u}+\sqrt{u}}\,du$
$= -\int_{u=10}^{u=0} \frac{\sqrt{10-u}}{\sqrt{10-u}+\sqrt{u}}\,du$
$= \int_{u=0}^{u=10} \frac{\sqrt{10-u}}{\sqrt{10-u}+\sqrt{u}}\,du$
$I = \int_{x=0}^{x=10} \frac{\sqrt{10-x}}{\sqrt{10-x}+\sqrt{x}}\,dx$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$\frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}$
$= \frac{\sqrt{x}+\sqrt{10-x}-\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}$
$= 1 - \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}$

Thus,

$$\int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$$
$$= \int_0^{10} \left[1 - \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\right]\,dx$$
$$= \int_0^{10} 1\,dx - \int_0^{10} \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$$
$$I = \int_0^{10} 1\,dx - I$$
$$2I = 10$$
$$I = 5$$

52. Let $u = a - x$, so that $du = -\,dx$. Then,

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$$
$$= -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du$$
$$= \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du$$
$$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx$$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$$\frac{f(x)}{f(x)+f(a-x)}$$
$$= \frac{f(x)+f(a-x)-f(a-x)}{f(x)+f(a-x)}$$
$$= 1 - \frac{f(a-x)}{f(a-x)+f(x)}$$

Thus,

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$$
$$= \int_0^a \left[1 - \frac{f(a-x)}{f(a-x)+f(x)}\right]\,dx$$
$$= \int_0^a 1\,dx - \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx$$
$$I = \int_0^a 1\,dx - I$$
$$2I = a$$
$$I = a/2$$

53. Let $u = 6 - x$, so that $du = -\,dx$.

Then,

$$I = \int_2^4 \frac{\sin^2(9-x)}{\sin^2(9-x)+\sin^2(x+3)}\,dx$$
$$= -\int_4^2 \frac{\sin^2(u+3)}{\sin^2(u+3)+\sin^2(9-u)}du$$
$$= \int_2^4 \frac{\sin^2(u+3)}{\sin^2(u+3)+\sin^2(9-u)}du$$
$$= \int_2^4 \frac{\sin^2(x+3)}{\sin^2(x+3)+\sin^2(9-x)}\,dx$$
$$= \int_2^4 \left[1 - \frac{\sin^2(9-x)}{\sin^2(x+3)+\sin^2(9-x)}\right]dx$$
$$I = \int_2^4 1\,dx - I$$
$$2I = 2$$
$$I = 1$$

54. Let $u = 6 - x$, so that $du = -\,dx$. Then,

$$I = \int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}\,dx$$
$$= -\int_4^2 \frac{f(u+3)}{f(u+3)+f(9-u)}du$$
$$= \int_2^4 \frac{f(u+3)}{f(u+3)+f(9-u)}du$$
$$= \int_2^4 \frac{f(x+3)}{f(x+3)+f(9-x)}\,dx$$
$$= \int_2^4 \left[1 - \frac{f(9-x)}{f(x+3)+f(9-x)}\right]dx$$
$$I = \int_2^4 1\,dx - I$$
$$2I = 2$$
$$I = 1$$

55. Let $6-u = x+4$; that is, let $u = 2-x$, so that $du = -dx$.

Then,

$$
\begin{aligned}
I &= \int_0^2 \frac{f(x+4)}{f(x+4)+f(6-x)}dx \\
&= -\int_2^0 \frac{f(6-u)}{f(6-u)+f(u+4)}du \\
&= \int_0^2 \frac{f(6-u)}{f(6-u)+f(u+4)}du \\
&= \int_0^2 \frac{f(6-x)}{f(6-x)+f(x+4)}dx \\
&= \int_0^2 \frac{f(6-x)+f(x+4)-f(x+4)}{f(6-x)+f(x+4)}dx \\
&= \int_0^2 \left[1 - \frac{f(x+4)}{f(6-x)+f(x+4)}\right] dx \\
I &= \int_0^2 1dx - I \\
2I &= 2 \\
I &= 1
\end{aligned}
$$

56. Let $u = x^{1/6}$, so that $du = \frac{1}{6}x^{-5/6}\,dx$.

Then,

$$
\begin{aligned}
I &= \int \frac{1}{x^{5/6}+x^{2/3}}\,dx \\
&= \int \frac{x^{-5/6}\,dx}{1+x^{-1/6}} \\
&= \int \frac{6\,du}{1+\frac{1}{u}} \\
&= \int \frac{6u}{u+1}\,du
\end{aligned}
$$

Let $v = u+1$, then $dv = du$ and $u = v-1$. Then,

$$
\begin{aligned}
I &= \int \frac{6u}{u+1}\,du \\
&= \int \frac{6(v-1)}{v}\,dv \\
&= \int \left(6 - \frac{6}{v}\right) dv \\
&= 6v - 6\ln|v| + c \\
&= 6(u+1) - 6\ln|u+1| + c \\
&= 6(x^{1/6}+1) - 6\ln|x^{1/6}+1| + c
\end{aligned}
$$

57. Let $u = x^{1/6}$, so that $du = (1/6)x^{-5/6}dx$, which means $6u^5du = dx$.

Thus,

$$
\begin{aligned}
&\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx \\
&= 6\int \frac{u^5}{u^3+u^2}du \\
&= 6\int \frac{u^3}{u+1}du \\
&= 6\int \left[u^2 - u + 1 - \frac{1}{u+1}\right] du \\
&= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + c \\
&= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} \\
&\quad - 6\ln|x^{1/6}+1| + c
\end{aligned}
$$

58. Let $u = x^{1/q}$, then $q\,du = x^{(1-q)/q}dx$, and

$$
\begin{aligned}
I &= \int \frac{1}{x^{(p+1)/q}+x^{p/q}}\,dx \\
&= \int \frac{x^{(1-q)/q}dx}{x^{(p+2-q)/q}+x^{(p+1-q)/q}}\,dx \\
&= q\int \frac{1}{u^{p+2-q}+u^{p+1-q}}\,du \\
&= q\int \frac{u^{q-1-p}}{u+1}\,du
\end{aligned}
$$

The rest of the calculation will depend on the values of p and q.

59. First let $u = \ln\sqrt{x}$, so that $du = x^{-1/2}(1/2)x^{-1/2}dx$, so that $2du = \dfrac{1}{x}dx$. Then,

$$
\begin{aligned}
&\int \frac{1}{x\ln\sqrt{x}}dx = 2\int \frac{1}{u}du \\
&= 2\ln|u| + c \\
&= 2\ln|\ln\sqrt{x}| + c
\end{aligned}
$$

Now use the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$. Then,

$$\int \frac{1}{x\ln\sqrt{x}}dx = \int \frac{1}{x\ln(x^{1/2})}dx$$

$$= \int \frac{1}{x\left(\frac{1}{2}\right)\ln x} dx$$
$$= 2\int \frac{1}{u} du$$
$$= 2\ln|u| + c_1$$
$$= 2\ln|\ln x| + c_1$$

The two results differ by a constant, and so are equivalent, as can be seen as follows:

$$2\ln|\ln\sqrt{x}| = 2\ln|\ln(x^{1/2})|$$
$$= 2\ln\left|\frac{1}{2}\ln x\right|$$
$$= 2\left[\ln\frac{1}{2} + \ln|\ln x|\right]$$
$$= 2\ln\frac{1}{2} + 2\ln|\ln x|$$
$$= 2\ln|\ln x| + \text{ constant}$$

60. Let $u = \ln x^2$, then $du = (1/x^2)2x\,dx = (2/x)dx$, and

$$\int \frac{1}{x\ln x^2}\,dx = \int \frac{du}{2u}$$
$$= \frac{1}{2}\ln|u| + c$$
$$= \frac{1}{2}\ln|\ln x^2| + c$$

Let $u = \ln x$, then $du = (1/x)dx$, and

$$\int \frac{1}{x\ln x^2}\,dx = \int \frac{1}{x(2\ln x)}\,dx$$
$$= \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\ln|\ln x| + c_1$$

The above two answers are equivalent, because

$$\frac{1}{2}\ln|\ln x^2| = \frac{1}{2}\ln|2\ln x|$$
$$= \frac{1}{2}(\ln 2 + \ln|\ln x|)$$
$$= \frac{1}{2}\ln|\ln x| + \frac{\ln 2}{2}$$

61. The point is that if we let $u = x^4$, then we get $x = \pm u^{1/4}$, and so we need to pay attention to the sign of u and x. A safe way is to solve the original indefinite integral in terms of x, and then solve the definite integral using boundary points in terms of x.

$$\int_{-2}^{1} 4x^4dx = \int_{x=-2}^{x=1} u^{1/4}du$$
$$= \frac{4}{5}u^{5/4}\Big|_{x=-2}^{x=1}$$
$$= \frac{4}{5}x^5\Big|_{x=-2}^{x=1}$$
$$= \frac{4}{5}(1^5 - (-2)^5)$$
$$= \frac{4}{5}(1 - (-32))$$
$$= \frac{4(33)}{5} = \frac{132}{5}$$

62. The problem is that it is not true on the entire interval $[0,\pi]$ that $\cos x = \sqrt{1-\sin^2 x}$. This is only true on the interval $[0,\pi/2]$. To make this substitution correctly, one must break up the integral:

$$\int_0^{\pi} \cos x(\cos x)\,dx$$
$$= \int_0^{\pi/2} \cos x(\cos x)\,dx + \int_{\pi/2}^{\pi} \cos x(\cos x)\,dx$$
$$= \int_{x=0}^{x=\pi/2} \sqrt{1-u^2}\,du - \int_{x=\pi/2}^{x=\pi} \sqrt{1-u^2}\,du$$
$$= \left(\frac{u}{2} + \frac{\sin^{-1} u}{2}\right)\Big|_{x=0}^{x=\pi/2} - \left(\frac{u}{2} + \frac{\sin^{-1} u}{2}\right)\Big|_{x=\pi/2}^{x=\pi}$$
$$= \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2}\right)\Big|_{x=0}^{x=\pi/2} - \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2}\right)\Big|_{x=\pi/2}^{x=\pi}$$

$$= \left(\frac{1}{2} + \frac{\pi}{4}\right) - 0 - 0 + \left(\frac{1}{2} + \frac{\pi}{4}\right)$$
$$= 1 + \frac{\pi}{2}$$

63. Let $u = 1/x$, so that $du = -1/x^2 dx$, which means that
$-1/u^2 du = dx$. Then,

$$\int_a^1 \frac{1}{x^2+1} dx = -\int_{1/a}^1 \frac{1/u^2}{1/u^2+1} du$$
$$= \int_1^{1/a} \frac{1}{1+u^2} du$$
$$= \int_1^{1/a} \frac{1}{1+x^2} dx$$

The last equation follows from the previous one because u and x are dummy variables of integration. Thus,

$$\tan^{-1} x \Big|_a^1 = \tan^{-1} x \Big|_1^{1/a}$$
$$\tan^{-1} 1 - \tan^{-1} a = \tan^{-1} \frac{1}{a} - \tan^{-1} 1$$
$$\tan^{-1} a + \tan^{-1} \frac{1}{a} = 2 \tan^{-1} 1$$
$$\tan^{-1} a + \tan^{-1} \frac{1}{a} = \frac{\pi}{2}$$

64. If $u = 1/x$, then $du = -dx/x^2$ and

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx$$
$$= \int \frac{1}{x^2\sqrt{1-1/x^2}} dx$$
$$= -\int \frac{1}{\sqrt{1-u^2}} du$$
$$= -\sin^{-1} u + c$$
$$= -\sin^{-1} 1/x + c$$

On the other hand,

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c_1$$

So $-\sin^{-1} 1/x = \sec^{-1} x + c_2$

Let $x = 1$, we get
$\sin^{-1} 1 = \sec^{-1} 1 + c_2$

$$\frac{\pi}{2} = 0 + c_2$$
$$c_2 = \frac{\pi}{2}$$

Hence $-\sin^{-1} 1/x = \sec^{-1} x + \dfrac{\pi}{2}$

65. $\bar{x} = \dfrac{\int_{-2}^2 x\sqrt{4-x^2}\, dx}{\int_{-2}^2 \sqrt{4-x^2}\, dx}$

Examine the denominator of $\bar{x}$, the graph of $\sqrt{4-x^2}$, which is indeed a semicircle, is symmetric over the two intervals $[-2, 0]$ and $[0, 2]$, while multiplying by x changes the symmetry into anti-symmetry. In other words,

$$\int_{-2}^0 x\sqrt{4-x^2}\, dx = -\int_0^2 x\sqrt{4-x^2}\, dx$$

so that

$$\int_{-2}^2 x\sqrt{4-x^2}\, dx$$
$$= \int_{-2}^0 x\sqrt{4-x^2}\, dx + \int_0^2 x\sqrt{4-x^2}\, dx$$
$$= 0$$

Hence $\bar{x} = 0$.

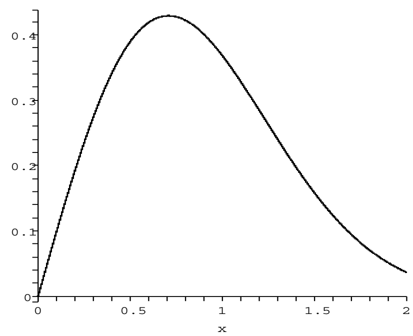
Now the integral $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\, dx$ is the area of a semicircle with radius 2, thus its value $= (1/2)\pi 2^2 = 2\pi$.

Then

$$\bar{y} = \frac{\int_{-2}^2 (\sqrt{4-x^2})^2\, dx}{2\int_{-2}^2 \sqrt{4-x^2}\, dx}$$
$$= \frac{\int_{-2}^2 (4-x^2)\, dx}{2 \cdot 2\pi}$$
$$= \frac{\int_{-2}^0 (4-x^2)\, dx + \int_0^2 (4-x^2)\, dx}{4\pi}$$
$$= \frac{2\int_0^2 (4-x^2)\, dx}{4\pi}$$
$$= \frac{\int_0^2 (4-x^2)\, dx}{2\pi}$$
$$= \frac{1}{2\pi}\left(4x - \frac{x^3}{3}\right) \Big|_0^2$$

$= \dfrac{8}{3\pi}$

66. These animals are likely to be found 0.7 miles from the pond.



Let $u = -x^2$, then
$du = -2x\,dx,\ u(0) = 0, u(2) = -4$ and

$$\int_0^2 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^{-4} e^u\,du$$
$$= -\frac{1}{2}(e^{-4} - 1) = \frac{1 - e^{-4}}{2}$$

67. $V(t) = V_p \sin(2\pi f t) V^2(t)$

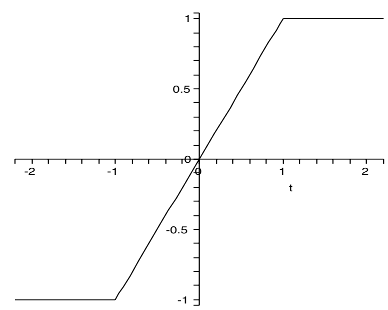
$$= V_p^2 \sin^2(2\pi f t)$$
$$= V_p^2 \left(\frac{1}{2} - \frac{1}{2}\cos(4\pi f t)\right)$$
$$= \frac{V_p^2}{2}(1 - \cos(4\pi f t))$$

$$\text{rms} = \sqrt{f\int_0^{1/f} V^2(t)\,dt}$$
$$= \sqrt{f\int_0^{1/f} \frac{V_p^2}{2}(1 - \cos(4\pi f t))\,dt}$$
$$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\int_0^{1/f}(1 - \cos(4\pi f t))\,dt}$$
$$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\left(t - \frac{\sin(4\pi f t}{4\pi f}\right)\Big|_0^{1/f}}$$
$$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\frac{1}{f}} = \frac{V_p}{\sqrt{2}}$$

68. $\displaystyle\int_{-2}^{2} f^2(t)\,dt$

$$= \int_{-2}^{-1} 1\,dt + \int_{-1}^{1} t^2\,dt + \int_1^2 1\,dt$$
$$= 1 + \frac{2}{3} + 1 = \frac{8}{3}$$

$$\text{rms} = \sqrt{\frac{1}{4}\int_{-2}^{2} f^2(t)\,dt}$$
$$= \sqrt{\frac{1}{4}\left(\frac{8}{3}\right)} = \sqrt{\frac{2}{3}}$$



4.7 Numerical Integration

1. Midpoint Rule:

$$\int_0^1 \left(x^2 + 1\right)\,dx$$
$$\approx \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$
$$= \frac{85}{64}$$

Trapezoidal Rule:

$$\int_0^1 \left(x^2 + 1\right)\,dx$$
$$\approx \frac{1 - 0}{2(4)}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right]$$

$= \dfrac{43}{32}$

Simpson's Rule:

$\displaystyle\int_0^1 \left(x^2+1\right)\,dx$

$= \dfrac{1-0}{3(4)}\left[f(0)+4f\left(\dfrac{1}{4}\right)+2f\left(\dfrac{1}{2}\right)\right.$

$\left.+4f\left(\dfrac{3}{4}\right)+f(1)\right]$

$= \dfrac{4}{3}$

2. Midpoint Rule:

$\displaystyle\int_0^2 (x^2+1)\,dx$

$\approx \dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+f\left(\dfrac{5}{4}\right)\right.$

$\left.+f\left(\dfrac{7}{4}\right)\right]$

$= \dfrac{1}{2}\left(\dfrac{17}{16}+\dfrac{25}{16}+\dfrac{41}{16}+\dfrac{65}{16}\right) = \dfrac{37}{8}$

Trapezoidal Rule:

$\displaystyle\int_0^2 (x^2+1)\,dx$

$\approx \dfrac{1}{4}\left[f(0)+2f\left(\dfrac{1}{2}\right)+2f(1)\right.$

$\left.+2f\left(\dfrac{3}{2}\right)+f(2)\right]$

$= \dfrac{1}{4}\left(1+\dfrac{5}{2}+4+\dfrac{13}{2}+5\right) = \dfrac{19}{4}$

Simpson's Rule:

$\displaystyle\int_0^2 (x^2+1)\,dx$

$\approx \dfrac{1}{6}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)\right.$

$\left.+4f\left(\dfrac{3}{2}\right)+f(2)\right]$

$= \dfrac{1}{6}(1+5+4+13+5) = \dfrac{14}{3}$

3. Midpoint Rule:

$\displaystyle\int_1^3 \dfrac{1}{x}\,dx$

$\approx \dfrac{3-1}{4}\left[f\left(\dfrac{5}{4}\right)+f\left(\dfrac{7}{4}\right)+f\left(\dfrac{9}{4}\right)\right.$

$\left.+f\left(\dfrac{11}{4}\right)\right]$

$= \dfrac{1}{2}\left(\dfrac{4}{5}+\dfrac{4}{7}+\dfrac{4}{9}+\dfrac{4}{11}\right)$

$= \dfrac{3776}{3465}$

Trapezoidal Rule:

$\displaystyle\int_1^3 \dfrac{1}{x}\,dx$

$\approx \dfrac{3-1}{2(4)}\left[f(1)+2f\left(\dfrac{3}{2}\right)+2f(2)\right.$

$\left.+2f\left(\dfrac{5}{2}\right)+f(3)\right]$

$= \dfrac{1}{4}\left(1+\dfrac{4}{3}+1+\dfrac{4}{5}+\dfrac{1}{3}\right)$

$= \dfrac{67}{60}$

Simpson's Rule:

$\displaystyle\int_1^3 \dfrac{1}{x}\,dx$

$\approx \dfrac{3-1}{3(4)}\left[f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)\right.$

$\left.+4f\left(\dfrac{5}{2}\right)+f(3)\right]$

$= \dfrac{1}{6}\left(1+\dfrac{8}{3}+1+\dfrac{8}{5}+\dfrac{1}{3}\right)$

$= \dfrac{11}{10}$

4. Midpoint Rule:

$\displaystyle\int_{-1}^1 (2x-x^2)\,dx$

$\approx \dfrac{1}{2}\left[f\left(-\dfrac{3}{4}\right)+f\left(-\dfrac{1}{4}\right)+f\left(\dfrac{1}{4}\right)\right.$

$\left.+f\left(\dfrac{3}{4}\right)\right]$

$= \dfrac{1}{2}\left(-\dfrac{33}{16}-\dfrac{9}{16}+\dfrac{7}{16}+\dfrac{15}{16}\right)$

$= \dfrac{-5}{8}$

Trapezoidal Rule:

$$\int_{-1}^{1} (2x - x^2)\, dx$$
$$\approx \frac{1}{4}\left[f(-1) + 2f\left(-\frac{1}{2}\right) + 2f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right]$$
$$= \frac{1}{4}\left(-3 - \frac{5}{2} + 0 + \frac{3}{2} + 1\right)$$
$$= -\frac{3}{4}$$

Simpson's Rule:

$$\int_{-1}^{1} (2x - x^2)\, dx$$
$$\approx \frac{1}{6}\left[f(-1) + 4f\left(-\frac{1}{2}\right) + 2f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$
$$= \frac{1}{6}(-3 - 5 + 0 + 3 + 1)$$
$$= -\frac{2}{3}$$

5. (a) Left Endpoints:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{2-0}{4}[f(0) + f(.5) + f(1) + f(1.5)]$$
$$= \frac{1}{2}(1 + .25 + 0 + .25)$$
$$= .75$$

(b) Midpoint Rule:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{2-0}{4}[f(.25) + f(.75) + f(1.25) + f(1.75)]$$
$$= \frac{1}{2}(.65 + .15 + .15 + .65)$$
$$= .7$$

(c) Trapezoidal Rule:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{2-0}{2(4)}[f(0) + 2f(.5) + 2f(1) + 2f(1.5) + f(2)]$$
$$= \frac{1}{4}(1 + .5 + 0 + .5 + 1)$$
$$= .75$$

6. (a) Left Endpoints:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{1}{2}(f(0) + f(.5) + f(1) + f(1.5))$$
$$= \frac{1}{2}(0.5 + 0.8 + 0.5 + 0.1)$$
$$= 0.95$$

(b) Midpoint Rule:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{1}{2}(0.7 + 0.8 + 0.4 + 0.2)$$
$$= 1.05$$

(c) Trapezoidal Rule:

$$\int_0^2 f(x)\, dx$$
$$\approx \frac{1}{4}[0.5 + 2(0.8) + 2(0.5) + 2(0.1) + 0.5]$$
$$= 0.95$$

7.

n	Midpoint	Trapezoidal	Simpson
10	0.5538	0.5889	0.5660
20	0.5629	0.5713	0.5655
50	0.5652	0.5666	0.5657

8.

n	Midpoint	Trapezoidal	Simpson
10	0.386939	0.385578	0.386476
20	0.386600	0.386259	0.386485
50	0.386504	0.386450	0.386486

9.

n	Midpoint	Trapezoidal	Simpson
10	0.88220	0.88184	0.88207
20	0.88211	0.88202	0.88208
50	0.88209	0.88207	0.88208

10.

n	Midpoint	Trapezoidal	Simpson
10	0.886210	0.886202	0.886207
20	0.886208	0.886206	0.886207
50	0.886207	0.886207	0.886207

11.

n	Midpoint	Trapezoidal	Simpson
10	3.9775	3.9775	3.9775
20	3.9775	3.9775	3.9775
50	3.9775	3.9775	3.9775

12.

n	Midpoint	Trapezoidal	Simpson
10	3.333017	3.336997	3.334337
20	3.334012	3.335007	3.334344
50	3.334291	3.334450	3.334344

13. The exact value of this integral is

$$\int_0^1 5x^4\,dx = x^5\Big|_0^1 = 1 - 0 = 1$$

n	Midpoint	EM_n
10	1.00832	8.3×10^{-3}
20	1.00208	2.1×10^{-3}
40	1.00052	5.2×10^{-4}
80	1.00013	1.3×10^{-4}

n	Trapezoidal	ET_n
10	0.98335	1.6×10^{-2}
20	0.99583	4.1×10^{-3}
40	0.99869	1.0×10^{-3}
80	0.99974	2.6×10^{-4}

n	Simpson	ES_n
10	1.000066	6.6×10^{-5}
20	1.0000041	4.2×10^{-6}
40	1.00000026	2.6×10^{-7}
80	1.00000016	1.6×10^{-8}

14. The exact value of this integral is

$$\int_1^2 \frac{1}{x}\,dx = \ln 2$$

n	Midpoint	EM_n
10	0.692835	3.1×10^{-4}
20	0.693069	7.8×10^{-5}
40	0.693128	2.0×10^{-5}
80	0.693142	4.9×10^{-6}

n	Trapezoidal	ET_n
10	0.693771	6.2×10^{-4}
20	0.693303	1.6×10^{-4}
40	0.693186	3.9×10^{-5}
80	0.693157	9.8×10^{-6}

n	Simpson	ES_n
10	0.693150	3.1×10^{-6}
20	0.693147	1.9×10^{-7}
40	0.693147	1.2×10^{-8}
80	0.693147	8.0×10^{-10}

15. The exact value of this integral is

$$\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi = 0$$

n	Midpoint	EM_n
10	0	0
20	0	0
40	0	0
80	0	0

n	Trapezoidal	ET_n
10	0	0
20	0	0
40	0	0
80	0	0

n	Simpson	ES_n
10	0	0
20	0	0
40	0	0
80	0	0

16. The exact value of this integral is

$$\int_0^{\pi/4} \cos x \, dx = \frac{1}{\sqrt{2}}$$

n	Midpoint	EM_n
10	0.707289	1.8×10^{-4}
20	0.707152	4.5×10^{-5}
40	0.707118	1.1×10^{-5}
80	0.707110	2.8×10^{-6}

n	Trapezoidal	ET_n
10	0.706743	3.6×10^{-4}
20	0.707016	9.1×10^{-5}
40	0.707084	2.3×10^{-5}
80	0.707101	5.7×10^{-6}

n	Simpson	ES_n
10	0.7071087	1.5×10^{-7}
20	0.7071068	9.5×10^{-9}
40	0.7071068	6×10^{-10}
80	0.7071068	6×10^{-10}

17. If you double n, the error in the Midpoint Rule is divided by 4, the error in the Trapezoidal Rule is divided by 4 and the error in the Simpson's Rule is divided by 16.

18. If you halve the interval length $b - a$, the error in the Midpoint Rule is divided by 8, the error in the Trapezoidal Rule is divided by 8 and the error in the Simpson's Rule is divided by 32.

19. Midpoint Rule:
$\ln 4 \approx 1.366162$

Trapezoidal Rule:
$\ln 4 \approx 1.428091$

Simpson's Rule:
$\ln 4 \approx 1.391621$

20. Midpoint Rule:
$\ln 8 \approx 1.987287$

Trapezoidal Rule:
$\ln 8 \approx 2.289628$

Simpson's Rule:
$\ln 8 \approx 2.137327$

21. Midpoint Rule:
$\sin 1 \approx 0.843666$

Trapezoidal Rule:
$\sin 1 \approx 0.837084$

Simpson's Rule:
$\sin 1 \approx 0.841489$

22. Midpoint Rule:
$e^2 \approx 7.322986$

Trapezoidal Rule:
$e^2 \approx 7.521610110$

Simpson's Rule:
$e^2 \approx 7.391210186$

23. $f(x) = \dfrac{1}{x}, f''(x) = \dfrac{2}{x^3}, f^{(4)}(x) = \dfrac{24}{x^5}$
Then $K = 2, L = 24$

Hence according to Theorems 9.1 and 9.2

$$|ET_4| \le 2\frac{(4-1)^3}{12 \cdot 4^2} \approx 0.281$$

$$|EM_4| \le 2\frac{(4-1)^3}{24 \cdot 4^2} \approx 0.141$$

$$|ES_4| \le 24\frac{(4-1)^5}{180 \cdot 4^4} \approx 0.127$$

24. $f(x) = \cos x, f''(x) = -\cos x,$
$f^{(4)}(x) = \cos x$
Then $K = L = 1$

Hence according to Theorems 9.1 and 9.2

$$|ET_4| \le 1\frac{1}{12 \cdot 4^2} \approx 0.005$$

$$|EM_4| \le 1\frac{1}{24 \cdot 4^2} \approx 0.003$$

$$|ES_4| \le 1\frac{1}{180 \cdot 4^4} \approx 2.17 \times 10^{-5}$$

25. Using Theorems 9.1 and 9.2, and the calculation in Example 9.10, we find the following lower bounds for the number of steps needed to guarantee accuracy of 10^{-7} in Exercise 19:

Midpoint: $\sqrt{\dfrac{2 \cdot 3^3}{24 \cdot 10^{-7}}} \approx 4745$

Trapezoidal: $\sqrt{\dfrac{2 \cdot 3^3}{14 \cdot 10^{-7}}} \approx 6709$

Simpson's: $\sqrt[4]{\dfrac{24 \cdot 3^5}{180 \cdot 10^{-7}}} \approx 135$

26. Midpoint: $|E_n| K \dfrac{(b-a)^3}{24n^2} = \dfrac{1}{24n^2}$

We want $\dfrac{1}{24n^2} \leq 10^7$

$24n^2 \geq 10^7$

$n^2 \geq \dfrac{10^7}{24}$

$n \geq \sqrt{\dfrac{10^7}{24}} \approx 645.5$

So need $n \geq 646$.

Trapezoid: $|ET_n| K \dfrac{(b-a)^3}{12n^2} = \dfrac{1}{12n^2}$

We want $n^2 \geq \dfrac{10^7}{12}$

$n \geq \sqrt{\dfrac{10^7}{12}} \approx 912.87$

$n \geq 913$

Simpson: $|ES_n| L \dfrac{(b-a)^5}{180n^4} = \dfrac{1}{180n^4}$

$\dfrac{1}{180n^4} \leq 10^{-7}$

$180n^4 \geq 10^7$

$n^4 \geq \dfrac{10^7}{180}$

$n \geq \sqrt[4]{\dfrac{10^7}{180}} \approx 15.4$

So need $n \geq 16$.

27. We use $K = 60, L = 120$.

n	$\lvert EM_n\rvert$	Error bound
10	8.3×10^{-3}	2.5×10^{-2}

n	$\lvert ET_n\rvert$	Error bound
10	1.6×10^{-2}	5×10^{-2}

n	$\lvert ES_n\rvert$	Error bound
10	7.0×10^{-5}	6.6×10^{-3}

28. We use $K = L = 1$.

n	$\lvert EM_n\rvert$	Error bound
10	0	1.3×10^{-2}

n	$\lvert ET_n\rvert$	Error bound
10	0	2.6×10^{-2}

n	$\lvert ES_n\rvert$	Error bound
10	0	1.7×10^{-4}

29. Trapezoidal Rule:

$$\int_0^2 f(x)\,dx$$
$$\approx \frac{2-0}{2(8)}[f(0) + 2f(0.25) + 2f(0.5) + 2f(.75) + 2f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)]$$
$$= \frac{1}{8}[4.0 + 9.2 + 10.4 + 9.6 + 10 + 9.2 + 8.8 + 7.6 + 4.0]$$
$$= 9.1$$

Simpson's Rule:

$$\int_0^2 f(x)\,dx$$
$$\approx \frac{2-0}{3(8)}[f(0) + 4f(.25) + 2f(.5) + 4f(.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)]$$
$$= \frac{1}{12}(4.0 + 18.4 + 10.4 + 19.2 + 10 + 18.4 + 8.8 + 15.2 + 4.0)$$
$$\approx 9.033$$

30. Trapezoidal Rule:

$$\int_0^2 f(x)\,dx$$
$$\approx \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 2f(1) + 2f(1.25)$$

$+ 2f(1.5) + 2f(1.75) + f(2)]$
$= \frac{0.25}{2}[(1.0)+2(0.6)+2(0.2)+2(-0.2)$
$+ 2(-0.4) + 2(0.4) + 2(0.8) + 2(1.2)$
$+ (2.0)]$
$= 1.025$

Simpson's Rule:
$\int_0^2 f(x)\, dx$
$\approx \frac{0.25}{3}[f(0) + 4f(0.25) + 2f(0.5)$
$+ 4f(0.75) + 2f(1) + 4f(1.25)$
$+ 2f(1.5) + 4f(1.75) + f(2)]$
$= \frac{0.25}{3}[(1.0)+4(0.6)+2(0.2)+4(-0.2)$
$+ 2(-0.4) + 4(0.4) + 2(0.8) + 4(1.2)$
$+ (2.0)]$
≈ 1.016667

31. Simpson's Rule:
$\int_0^{120} f(x)\, dx$
$\approx \frac{120-0}{3(12)}[f(0) + 4f(10) + 2f(20)$
$+4f(30)+2f(40)+4f(50)+2f(60)$
$+ 4f(70) + 2f(80) + 4f(90)$
$+ 2f(100) + 4f(110) + f(120)]$
$= \frac{10}{3}(56+216+116+248+116+232$
$+124+224+104+192+80+128+22)$
≈ 6193

32. Simpson's Rule:
$\int_0^{120} f(x)\, dx$
$\approx \frac{10}{3}[f(0)+4f(10)+2f(20)+4f(30)$
$+2f(40)+4f(50)+2f(60)+4f(70)$
$+ 2f(80) + 4f(90) + 2f(100)$
$+ 4f(110) + f(120)]$
$= \frac{10}{3}[26 + 4(30) + 2(28) + 4(22)$
$+2(28)+4(32)+2(30)+4(33)+2(31)$
$+ 4(28) + 2(30) + 4(32) + (22)]$
$= 3500$

33. Simpson's Rule:
$\int_0^{24} v(t)\, dt$
$\approx \frac{12-0}{3(12)}[f(0) + 4f(1) + 2f(2)$
$+ 4f(3) + 2f(4) + 4f(5) + 2f(6)$
$+ 4f(7) + 2f(8) + 4f(9) + 2f(10)$
$+ 4f(11) + f(12)]$
$= \frac{1}{3}(40 + 168 + 80 + 176 + 96 + 200$
$+ 92 + 184 + 84 + 176 + 80 + 168$
$+ 42)$
$= 529$

34. Simpson's Rule:
$\int_0^{24} v(t)\, dt$
$\approx \frac{2}{3}[f(0) + 4f(2) + 2f(4) + 4f(6)$
$+ 2f(8) + 4f(10) + 2f(12) + 4f(14)$
$+2f(16)+4f(18)+2f(20)+4f(22)$
$+ f(24)]$
$= \frac{2}{3}[(26)+4(30)+2(28)+4(30)+2(28)$
$+4(32)+2(30)+4(33)+2(31)+4(28)$
$+ 2(30) + 4(32) + (32)]$
$= 728$

35. Simpson's Rule:
$\int_0^{2.4} f(x)\, dx$
$\approx \frac{2.4-0}{3(12)}[f(0) + 4f(.2) + 2f(.4)$
$+ 4f(.6) + 2f(.8) + 4f(1) + 2f(1.2)$
$+4f(1.4)+2f(1.6)+4f(1.8)+2f(2)$
$+ 4f(2.2) + f(2.4)]$
$= \frac{1}{15}(0 + .8 + .8 + 4 + 3.2 + 8 + 4.4$
$+ 8 + 3.2 + 4.8 + 1.2 + .8 + 0)$
≈ 2.6

36. Simpson's Rule:
$\int_0^{2.4} f(x)\, dx$
$\approx \frac{0.2}{3}[f(0) + 4f(0.2) + 2f(0.4)$
$+ 4f(0.6) + 2f(0.8) + 4f(1.0)$
$+ 2f(1.2) + 4f(1.4) + 2f(1.6)$
$+ 4f(1.8) + 2f(2) + 4f(2.2)$

$$+ f(2.4)]$$
$$= \frac{0.2}{3}[0 + 4(0.1) + 2(0.4) + 4(0.8) + 2(1.4) + 4(1.8) + 2(2.0) + 4(2.0) + 2(1.6) + 4(1.2) + 2(0.6) + 4(0.2) + 0]$$
$$\approx 2.426667$$

37. a) Midpoint Rule:
$M_n < \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n > \int_a^b f(x)\, dx$

c) Simpson's Rule:
not enough information.

38. a) Midpoint Rule:
$M_n < \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n > \int_a^b f(x)\, dx$

c) Simpson's Rule:
$S_n \geq \int_a^b f(x)\, dx$

39. a) Midpoint Rule:
$M_n > \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n < \int_a^b f(x)\, dx$

c) Simpson's Rule:
not enough information.

40. a) Midpoint Rule:
$M_n > \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n < \int_a^b f(x)\, dx$

c) Simpson's Rule:
$S_n \leq \int_a^b f(x)\, dx$

41. a) Midpoint Rule:
$M_n < \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n > \int_a^b f(x)\, dx$

c) Simpson's Rule:
$S_n = \int_a^b f(x)\, dx$

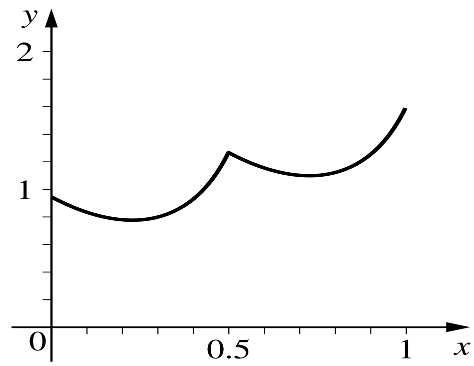
42. a) Midpoint Rule:
$M_n = \int_a^b f(x)\, dx$

b) Trapezoidal Rule:
$T_n = \int_a^b f(x)\, dx$

c) Simpson's Rule:
$S_n = \int_a^b f(x)\, dx$

43. $\frac{1}{2}(R_L + R_R)$
$$= \sum_{i=0}^{n-1} f(x_i) + \sum_{i=1}^{n} f(x_i)$$
$$= f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \sum_{i=1}^{n-1} f(x_i) + f(x_n)$$
$$= f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) = T_n$$

44.



45. $I_1 = \int_0^1 \sqrt{1-x^2}\,dx$ is one fourth of the area of a circle with radius 1, so

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

$$I_2 = \int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^1$$

$$= \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

n	$S_n(\sqrt{1-x^2})$	$S_n(\frac{1}{1+x^2})$
4	0.65652	0.78539
8	0.66307	0.78539

The second integral $\int \frac{1}{1+x^2}\,dx$ provides a better algorithm for estimating π.

46. $$\int_{-h}^{h} (Ax^2 + Bx + c)\,dx$$

$$= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + cx\right)\Big|_{-h}^{h}$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

$$= \frac{h}{3}(2Ah^2 + 6C)$$

$$= \frac{h}{3}[f(-h) + 4f(0) + f(h)]$$

47. (a) $\int_{-1}^{1} x\,dx = 0$

$$\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right) = 0$$

(b) $\int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$$\left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3}$$

(c) $\int_{-1}^{1} x^3\,dx = 0$

$$\left(-\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{3}}\right)^3 = 0$$

48. Simpson's Rule with $n = 2$:

$$\int_{-1}^{1} \pi\cos\left(\frac{\pi x}{2}\right)\,dx$$

$$\approx \frac{2}{6}\,(f(-1) + 4f(-1/3) + f(1))$$

$$= \frac{1}{3}\,(\pi\cos(-\pi/2) + 4\pi\cos(-\pi/6) + \pi\cos(\pi/2))$$

$$= \frac{\pi}{3}\,(0 + 2\sqrt{3} + 0) = \frac{2\pi}{\sqrt{3}}$$

$$\approx 3.6276$$

Gaussian quadrature:

$$\int_{-1}^{1} \pi\cos\left(\frac{\pi x}{2}\right)\,dx$$

$$\approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

$$= \pi\cos\left(-\frac{\pi}{2\sqrt{3}}\right) + \pi\cos\left(\frac{\pi}{2\sqrt{3}}\right)$$

$$\approx 3.87164$$

49. Simpson's Rule is not applicable because $\frac{\sin x}{x}$ is not defined at $x = 0$.

$$L = \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

The two functions $f(x)$ and $\frac{\sin x}{x}$ differ only at one point $x = 0$, so

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{x}\,dx$$

We can now apply Simpson's Rule with $n = 2$:

$$\int_0^{\pi} f(x)\,dx$$

$$\approx \frac{\pi}{6}\left(1 + \frac{4\sin\pi}{\pi/2} + \frac{\sin\pi}{\pi}\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{3} + \frac{8}{3\pi}\right)$$
$$\approx \frac{\pi}{2} \cdot 1.18$$

50. The function $\frac{\sin x}{x}$ is not defined at $x = 0$, and it is symmetric across the y-axis. We define a new function

$$f(x) = \begin{cases} \sin x/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

over the interval $[0, \pi/2]$, and

$$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x}\, dx = 2\int_0^{\pi/2} f(x)\, dx$$

Use Simpson's Rule on $n = 2$:

$$\int_0^{\pi/2} f(x)\, dx$$
$$\approx \frac{\pi}{12}\left(1 + \frac{\frac{\sqrt{2}}{2}}{\pi/4} + \frac{1}{\pi/2}\right)$$
$$\approx \frac{\pi}{2} \cdot 15.22$$

Hence

$$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x}\, dx \approx \frac{\pi}{2} \cdot 30.44$$

51. Let I be the exact integral. Then we have

$$T_n - I \approx -2(M_n - I)$$
$$T_n - I \approx 2I - 2M_n$$
$$T_n + 2M_n \approx 3I$$
$$\frac{T_n}{3} + \frac{2}{3}M_n \approx I$$

52. The text does not say this, but we want to show that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$$

In this case, we have data points: $x_0, x_1, x_2, x_3, \ldots, x_{2n}$.

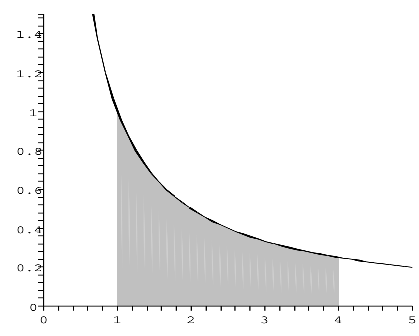
The midpoint rule will use the points: $x_1, x_3, \ldots, x_{2n-1}$

The trapezoidal rule will use the points: $x_0, x_2, \ldots, x_{2n}$

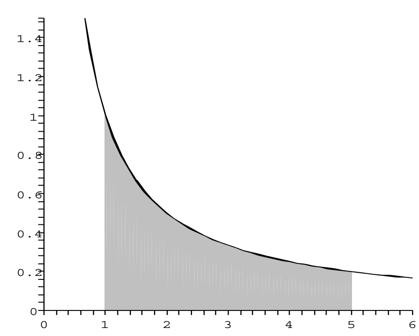
$$\frac{1}{3}T_n + \frac{2}{3}M_n$$
$$= \left(\frac{1}{3}\right)\left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{2n-2}) + f(x_{2n})]$$
$$+ \left(\frac{2}{3}\right)\left(\frac{b-a}{n}\right)[f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{2n-1})]$$
$$= \left(\frac{b-a}{6n}\right)[f(x_0)4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$
$$= S_{2n}$$

4.8 The Natural Logarithm As An Integral

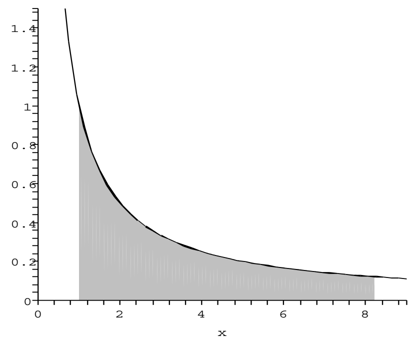
1. $\ln 4 = \ln 4 - \ln 1 = \ln x \Big|_1^4 = \displaystyle\int_1^4 \frac{dx}{x}$



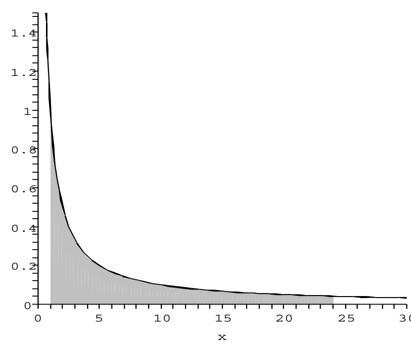
2. $\ln 5 = \displaystyle\int_1^5 \frac{dx}{x}$



3. $\ln 8.2 = \displaystyle\int_1^{8.2} \frac{dx}{x}$



4. $\ln 24 = \displaystyle\int_1^{24} \frac{dx}{x}$



5. $\ln 4 = \displaystyle\int_1^{4} \frac{dx}{x}$

$$\approx \frac{3}{12}\left(\frac{1}{1} + 4\frac{1}{1.75} + 2\frac{1}{1.5} + 4\frac{1}{3.25} + \frac{1}{4}\right)$$

≈ 1.3868

6. $\ln 5 = \displaystyle\int_1^{5} \frac{dx}{x}$

$$\approx \frac{4}{12}\left(\frac{1}{1} + 4\frac{1}{2} + 2\frac{1}{3} + 4\frac{1}{4} + \frac{1}{5}\right)$$

≈ 1.6108

7. (a) Simpson's Rule with $n = 32$:

$$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.386296874$$

(b) Simpson's Rule with $n = 64$:

$$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.386294521$$

8. (a) Simpson's Rule with $n = 32$:

$$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609445754$$

(b) Simpson's Rule with $n = 64$:

$$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609438416$$

9. $\dfrac{7}{2}\ln 2$

10. $\ln 2$

11. $\ln\left(\dfrac{3^2 \cdot \sqrt{3}}{9}\right) = \dfrac{1}{2}\ln 3$

12. $\ln\left(\dfrac{\frac{1}{9}\cdot\frac{1}{9}}{3}\right) = -5\ln 3$

13. $\dfrac{1}{\sqrt{x^2+1}} \cdot \dfrac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x$

14. $\dfrac{5x^4 \sin x \cos x + x^5 \cos^2 x - x^5 \sin x}{x^5 \sin x \cos x}$

15. $\dfrac{x^5+1}{x^4} \cdot \dfrac{4x^3(x^5+1) - x^4(5x^4)}{(x^5+1)^2}$

16. $\sqrt{\dfrac{x^5+1}{x^3}} \cdot \dfrac{1}{2} \cdot \left(\dfrac{x^3}{x^5+1}\right)^{-1/2}$

$\cdot\, \dfrac{3x^2(x^5+1) - x^3(5x^4)}{(x^5+1)^2}$

17. $\displaystyle\int \frac{3x^3}{x^4+5}\,dx = \frac{3}{4}\ln|x^4+5| + c$

$= \dfrac{3}{4}\ln(x^4+5) + c$

18. $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)} = 2\ln|\sqrt{x}+1| + c$

$= 2\ln(\sqrt{x}+1) + c$

19. $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + c$

20. $\displaystyle\int \frac{1}{\sqrt{1-x^2}\sin^{-1} x}\,dx$

$= \ln|\sin^{-1} x| + c$

21. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\ln|1+e^{2x})| + c$

$= \dfrac{1}{2}\ln(1+e^{2x}) + c$

22. Let $u = e^x, du = e^x\,dx$

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2}$$

$= \tan^{-1} u + c = \tan^{-1} e^x + c$

23. Let $u = 2/x, du = (-2/x^2)\,dx$

$$\int \frac{e^{2/x}}{x^2}\,dx = -\frac{1}{2}\int e^u\,du$$

$= -\dfrac{1}{2e^u} + c = -\dfrac{1}{2}e^{2/x} + c$

24. Let $u = \ln x^3, du = (3/x)\,dx$

$$\int \frac{\sin(\ln x^3)}{x}\,dx = \frac{1}{3}\int \sin u\,du$$

$= -\dfrac{1}{3}\cos u + c$

$= -\dfrac{1}{3}\cos(\ln x^3) + c$

25. $\displaystyle\int_0^1 \frac{x^2}{x^3-4}\,dx = \frac{1}{3}\ln|x^3-4|\,\Big|_0^1$

$= \dfrac{1}{3}\ln 3 - \dfrac{1}{3}\ln 4 = \dfrac{1}{3}\ln\dfrac{3}{4}$

26. $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln|e^x + e^{-x}|\,\Big|_0^1$

$= \ln(e + e^{-1}) - \ln 2$

$= \ln\left(\dfrac{e+e^{-1}}{2}\right)$

27. $\displaystyle\int_0^1 \tan x\,dx = \int_0^1 \frac{\sin x}{\cos x}\,dx$

$= -\ln|\cos x|\,\Big|_0^1$

$= -\ln|\cos 1| - \ln|\cos 0|$

$= -\ln(\cos 1)$

28. Let $u = \ln x, du = dx/x$

$$\int \frac{\ln x}{x}\,dx = \int u\,dx = \frac{u^2}{2} + c$$

$= \dfrac{(\ln x)^2}{2} + c$

$$\int_1^2 \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}\,\Big|_1^2$$

$= \dfrac{\ln^2 2}{2} - \dfrac{\ln^2 1}{2} = \dfrac{\ln^2 2}{2}$

29. $\displaystyle\int_0^1 \frac{e^x - 1}{e^{2x}}\,dx = \int_0^1 (e^{-x} - e^{-2x})\,dx$

$= \left(-e^{-x} + \dfrac{1}{2}e^{-2x}\right)\Big|_0^1$

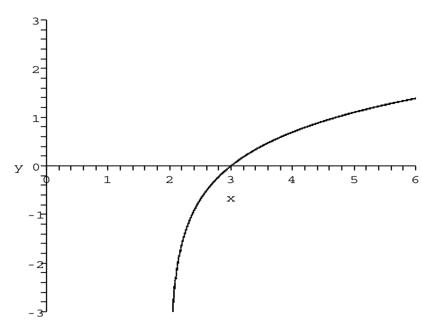
$= -\,e^{-1} + \dfrac{1}{2}\,e^{-2} + \dfrac{1}{2}$

30. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx = \ln|\ln x|\,\Big|_e^{e^2}$

$= \ln|\ln e^2| - \ln|\ln e| = \ln 2 - \ln 1$

$= \ln 2$

31.



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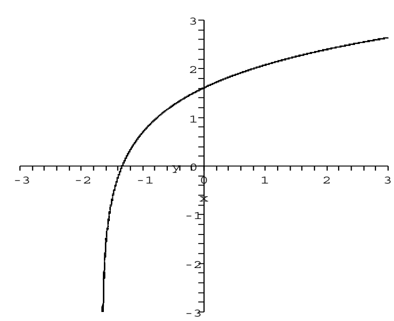

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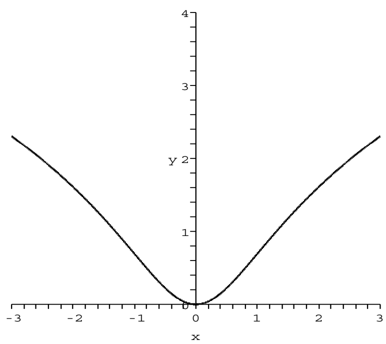

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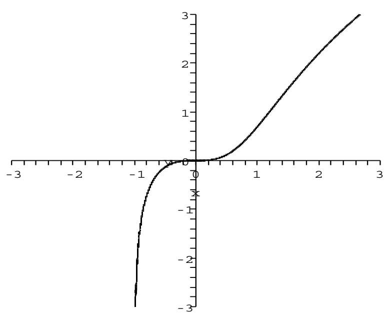

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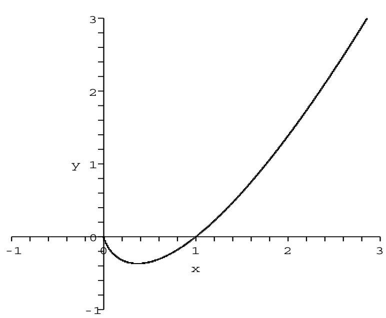

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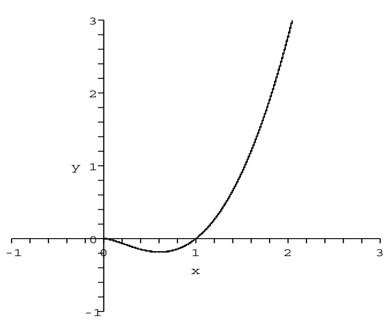


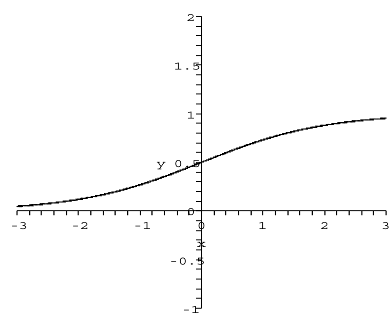
37. $\ln\left(\frac{a}{b}\right) = \ln\left(a \cdot \frac{1}{b}\right) = \ln a + \ln\left(\frac{1}{b}\right)$
$= \ln a - \ln b$

38. Let $y = e^x = \lim_{n\to\infty} x_n$,
where $x_n = (1 + x/n)^n$
Then
$x_n^{1/n} = 1 + x/n$
$n(x_n^{1/n} - 1) = x.$

On the other hand,
$y = e^x$, so
$\ln y = x = n(x_n^{1/n} - 1)$

Take limits on both sides, we get
$\ln y = \lim_{n\to\infty} n(x_n^{1/n} - 1)$
$= \lim_{n\to\infty} n(y^{1/n} - 1)$

39. $f(x) = \dfrac{1}{1 + e^{-x}}$



Using $\lim_{x\to\infty} e^{-x} = 0$ we get
$\lim_{x\to\infty} \dfrac{1}{1+e^{-x}} = 1$
Using $\lim_{x\to-\infty} e^{-x} = \infty$ we get
$\lim_{x\to\infty} \dfrac{1}{1+e^{-x}} = 0$
The function $f(x)$ is increasing over $(-\infty, \infty)$ and when $x = 0$,
$f(0) = \dfrac{1}{1+1} = \dfrac{1}{2}.$
So $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

The threshold value for $g(x)$ to switch is $x = 0$.

One way of modifying the function to move the threshold to $x = 4$ is to let $f(x) = \dfrac{1}{1 + e^{-(x-4)}}$

40. $1 - (9/10)^1 0 \approx 0.65132$
$1 - (19/20)^2 0 \approx 0.64151$
$1 - (9/10)^1 0 > 1 - (19/20)^2 0$

The probability of winning is lower.

When taking the limit as $n \to \infty$,

$$\lim_{n\to\infty}\left[1 - \left(\frac{n-1}{n}\right)^n\right]$$
$$= 1 - \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n$$
$$= 1 - \lim_{n\to\infty}\left(1 + \frac{-1}{n}\right)^n$$
$$= 1 - e^{-1}$$

41. $h = \ln e^h = \displaystyle\int_1^{e^h} \frac{1}{x}\,dx = \frac{e^h - 1}{\bar{x}}$,
for some $\bar{x}$ in $(0, h)$
$\dfrac{e^h - 1}{h} = \bar{x}$
as $h \to 0^+, \bar{x} \to 0,$ then

$$\lim_{h\to 0^+}\frac{e^h - 1}{h} = 0$$

$-h = \ln e^{-h}$
$= \displaystyle\int_1^{e^{-h}} \frac{1}{x}\,dx = \frac{e^{-h} - 1}{\bar{x}}$,
for some $\bar{x}$ in $(-h, 0)$
$\dfrac{e^{-h} - 1}{-h} = \bar{x}$
as $h \to 0^+, -h \to 0^-, \bar{x} \to 0,$ then
$$\lim_{h\to 0^+}\frac{e^{-h} - 1}{-h} = 0$$

42. $f(x) = \ln x,$ then
$f'(x) = \dfrac{1}{x}$ and $f'(1) = 1$.

On the other hand
$$f'(a) = \lim_{x\to a}\frac{\ln x - \ln a}{x - a}$$
$$f'(1) = \lim_{x\to 1}\frac{\ln x - \ln 1}{x - 1} = 1$$
$$\lim_{x\to 1}\frac{\ln x}{x - 1} = 1$$

Thus the reciprocal of $\dfrac{\ln x}{x-1}$ has the same limit,
$$\lim_{x\to 1}\frac{x - 1}{\ln x} = 1$$

Substituting $x = e^h$,
$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

43. $\ln\left[\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\right]$
$$= \lim_{n\to\infty}\ln\left(1 + \frac{1}{n}\right)^n$$
$$= \lim_{n\to\infty} n\ln\left(1 + \frac{1}{n}\right)$$
$$= \lim_{n\to\infty}\frac{\ln(1 + 1/n}{)}1/n$$
$$= \lim_{n\to\infty}\frac{-1/n^2}{-1/n^2(1 + 1/n)}$$
$$= \lim_{n\to\infty}\frac{1}{1 + 1/n}$$
$$= 1$$

44. $f(x) = \ln x - 1$
$f'(x) = \dfrac{1}{x}$

$x_0 = 3$
$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{\ln 3 - 1}{1/3}$
$= 6 - 3\ln 3 \approx 2.704163133$
$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 2.718245098$
$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \approx 2.718281827$

$e \approx 2.718282183$

Three steps are needed to start at $x_0 = 3$ and obtain five digits of accuracy.

45. $s(x) = x^2 \ln(1/x)$
$s'(x) = 2x \ln 1/x + x^2 \cdot x \cdot (-1/x^2)$
$= 2x \ln(1/x) - x = x(2\ln(1/x) - 1)$

$s'(x) = 0$ gives
$x = 0$ (which is impossible) or
$\ln(1/x) = 1/2, x = e^{-1/2}$

Since $s'(x) \begin{cases} < 0 & \text{if } x < e^{-1/2} \\ > 0 & \text{if } x > e^{-1/2} \end{cases}$

The value $x = e^{-1/2}$ maximizes the transmission speed.

Ch. 4 Review Exercises

1. $\displaystyle\int (4x^2 - 3)\,dx = \frac{4}{3}x^3 - 3x + c$

2. $\displaystyle\int (x - 3x^5)\,dx = \frac{x^2}{2} - \frac{1}{2}x^6 + c$

3. $\displaystyle\int \frac{4}{x}\,dx = 4\ln|x| + c$

4. $\displaystyle\int \frac{4}{x^2}\,dx = -\frac{4}{x} + c$

5. $\displaystyle\int 2\sin 4x\,dx = -\frac{1}{2}\cos 4x + c$

6. $\displaystyle\int 3\sec^2 x\,dx = 3\tan x + c$

7. $\displaystyle\int (x - e^{4x})\,dx = \frac{x^2}{2} - \frac{1}{4}e^{4x} + c$

8. $\displaystyle\int 3\sqrt{x}\,dx = 2x^{3/2} + c$

9. $\displaystyle\int \frac{x^2+4}{x}\,dx = \int (x + 4x^{-1})\,dx$
$\displaystyle= \frac{x^2}{2} + 4\ln|x| + c$

10. $\displaystyle\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + c$

11. $\displaystyle\int e^x(1 - e^{-x})\,dx = \int (e^x - 1)\,dx$
$= e^x - x + c$

12. $\displaystyle\int e^x(1+e^x)^2\,dx$
$\displaystyle= \int (e^x + 2e^{2x} + e^{3x})\,dx$
$\displaystyle= e^x + e^{2x} + \frac{1}{3}\,e^{3x} + c$

13. Let $u = x^2 + 4$, then $du = 2x\,dx$ and
$\displaystyle\int x\sqrt{x^2+4}\,dx$
$\displaystyle= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}\,u^{3/2} + c$
$\displaystyle= \frac{1}{3}\,(x^2+4)^{3/2} + c$

14. $\displaystyle\int x(x^2+4)\,dx = \int (x^3 + 4x)\,dx$
$\displaystyle= \frac{x^4}{4} + 2x^2 + c$

15. Let $u = x^3$, $du = 3x^2\,dx$
$\displaystyle\int 6x^2\cos x^3\,dx = 2\int \cos u\,du$
$= 2\sin u + c = 2\sin x^3 + c$

16. Let $u = x^2$, $du = 2x\,dx$
$\displaystyle\int 4x\sec x^2\tan x^2\,dx$
$\displaystyle= 2\int \sec u\tan u\,du$
$= 2\sec u + c = 2\sec x^2 + c$

17. Let $u = 1/x$, $du = -1/x^2\,dx$
$\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du$
$= -e^u + c = -e^{1/x} + c$

18. Let $u = \ln x$, $du = dx/x$
$\displaystyle\int \frac{\ln x}{x}\,dx = \int u\,du$
$\displaystyle= \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

19. $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$
$= -\ln|\cos x| + c$

20. Let $u = 3x+1, du = 3\,dx$

$$\int \sqrt{3x+1}dx = \frac{1}{3}\int u^{1/2}\,du$$
$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}+c = \frac{2}{9}(3x+1)^{3/2}+c$$

21. $f(x) = \int (3x^2+1)\,dx = x^3+x+c$

$f(0) = c = 2$

$f(x) = x^3+x+2$

22. $f(x) = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}+c$

$f(0) = -\frac{1}{2}+c = 3$

$c = \frac{7}{2}$

$f(x) = -\frac{1}{2}e^{-2x}+\frac{7}{2}$

23. $s(t) = \int (-32t+10)\,dt$

$= -16t^2+10t+c$

$s(0) = c = 2$

$s(t) = -16t^2+10t+2$

24. $v(t) = \int 6\,dt = 6t+c_1$

$v(0) = c_1 = 10$

$v(t) = 6t+10$

$s(t) = \int (6t+10)\,dt = 3t^2+10t+c_2$

$s(0) = c_2 = 0$

$s(t) = 3t^2+10t$

25. $\sum_{i=1}^{6}(i^2+3i)$

$= (1^2+3\cdot1)+(2^2+3\cdot2)+(3^2+3\cdot3)$
$+(4^2+3\cdot4)+(5^2+3\cdot5)+(6^2+3\cdot6)$

$= 4+10+18+28+40+54$

$= 154$

26. $\sum_{i=1}^{12} i^2 = 650$

27. $\sum_{i=1}^{100}(i^2-1)$

$= \sum_{i=1}^{100} i^2 - \sum_{i=1}^{100} 1$

$= \frac{100(101)(201)}{6} - 100$

$= 338,250$

28. $\sum_{i=1}^{100}(i^2+2i)$

$= \sum_{i=1}^{100} i^2 + 2\cdot\sum_{i=1}^{100} i$

$= \frac{100(101)(201)}{6} + 100(101)$

$= 348,450$

29. $\frac{1}{n^3}\sum_{i=1}^{n}(i^2-i)$

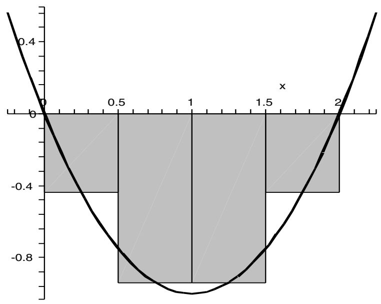
$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - \cdot\sum_{i=1}^{n} i\right)$$
$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)$$
$$= \frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i^2-i)$$
$$= \lim_{n\to\infty}\left(\frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}\right)$$
$$= \frac{2}{6} - 0 = \frac{1}{3}$$

30. Evaluation points: $0.25, 0.75, 1.25, 1.75$

Riemann sum $= \Delta x \sum_{i=1}^{n} f(c_i)$

$$= \frac{2}{4}\sum_{i=1}^{4}(c_i^2 - 2c_i)$$
$$= \frac{1}{2}\big[(0.25^2-2\cdot0.25)+(0.75^2-2\cdot0.75)$$
$$+(1.25^2-2\cdot1.25)+(1.75^2-2\cdot1.75)\big]$$
$$= -2.75$$



31. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 2.65625$

32. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 0.6875$

33. Riemann sum $= \frac{3}{8}\sum_{i=1}^{8} c_i^2 \approx 4.668$

34. Riemann sum $= \frac{1}{8}\sum_{i=1}^{8} c_i^2 \approx 0.6724$

35. **(a)** Left-endpoints:

$$\int_0^{1.6} f(x)\,dx \approx \frac{1.6-0}{8}(f(0)+f(.2)+f(.4) + f(.6)+f(.8)+f(1)+f(1.2) + f(1.4)) = \frac{1}{5}(1+1.4+1.6+2+2.2+2.4 + 2+1.6) = 2.84$$

(b) Right-endpoints:

$$\int_0^{1.6} f(x)\,dx \approx \frac{1.6-0}{8}(f(.2)+f(.4)+f(.6) + f(.8)+f(1)+f(1.2)+f(1.4) + f(1.6)) = \frac{1}{5}(1.4+1.6+2+2.2+2.4 + 2+1.6+1.4) = 2.92$$

(c) Trapezoidal Rule:

$$\int_0^{1.6} f(x)\,dx \approx \frac{1.6-0}{2(8)}[f(0)+2f(.2)+2f(.4) + 2f(.6)+2f(.8)+2f(1) + 2f(1.2)+2f(1.4)+f(1.6)] = 2.88$$

(d) Simpson's Rule:

$$\int_0^{1.6} f(x)\,dx \approx \frac{1.6-0}{3(8)}[f(0)+4f(.2)+2f(.4) + 4f(.6)+2f(.8)+4f(1) + 2f(1.2)+4f(1.4)+f(1.6)] \approx 2.907$$

36. **(a)** Left-endpoints:

$$\int_1^{4.2} f(x)\,dx \approx (0.4)[f(1.0)+f(1.4)+f(1.8) + f(2.2)+f(2.6)+f(3.0) + f(3.4)+f(3.8)] = (0.4)(4.0+3.4+3.6+3.0 + 2.6+2.4+3.0+3.6) = 10.24$$

(b) Right-endpoints:

$$\int_1^{4.2} f(x)\,dx \approx (0.4)[f(1.4)+f(1.8)+f(2.2) + f(2.6)+f(3.0)+f(3.4) + f(3.8)+f(4.2)] = (0.4)(3.4+3.6+3.0+2.6 + 2.4+3.0+3.6+3.4) = 10.00$$

(c) Trapezoidal Rule:

$$\int_1^{4.2} f(x)\,dx \approx \frac{0.4}{2}[f(1.0)+2f(1.4)+2f(1.8) + 2f(2.2)+2f(2.6)+2f(3.0) + 2f(3.4)+2f(3.8)+f(4.2)] = (0.2)[4.0+2(3.4)+2(3.6)$$

$$+ 2(3.0) + 2(2.6) + 2(2.4) + 2(3.0) + 2(3.6) + 3.4]$$
$$= 10.12$$

(d) Simpson's Rule:

$$\int_1^{4.2} f(x)\,dx$$
$$\approx \frac{0.4}{3}[f(1.0)+4f(1.4)+2f(1.8) + 4f(2.2) + 2f(2.6) + 4f(3.0) + 2f(3.4) \quad + 4f(3.8) + f(4.2)]$$
$$= \frac{0.4}{3}[4.0 + 4(3.4) + 2(3.6) + 4(3.0) + 2(2.6) + 4(2.4) + 2(3.0) + 4(3.6) + 3.4]$$
$$\approx 10.05333$$

37. See Example 7.10.

Simpson's Rule is expected to be most accurate.

38. In this situation, the Midpoint Rule will be less than the actual integral. The Trapezoid Rule will be an overestimate.

39. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} 2\cdot\left(\frac{i}{n}\right)^2$$
$$= \frac{2}{n^3}\sum_{i=1}^{n} i^2$$
$$= \left(\frac{2}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{(n+1)(2n+1)}{3n^2}$$

Now, to find the integral, we take the limit:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2}$$
$$= \frac{2}{3}$$

40. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \dfrac{2}{n}$ and $x_i = \dfrac{2i}{n}$

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2 + 1$$
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$
$$= \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{2}{n}\right)n$$
$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$

Now, to find the integral, we take the limit:

$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{3n^2} + 2\right)$$
$$= \frac{8}{3} + 2 = \frac{14}{3}$$

41. Area $= \displaystyle\int_0^3 (3x - x^2)\,dx$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{9}{2}$$

42. Area

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx - \int_1^2 (x^3 - 3x^2 + 2x)\,dx$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

43. The velocity is always positive, so distance traveled is equal to change in position.

$$\text{Dist} = \int_1^2 (40 - 10t)\ dt$$
$$= (40t - 5t^2)\Big|_1^2 = 25$$

44. The velocity is always positive, so distance traveled is equal to change in position.
$$\text{Dist} = \int_0^2 20e^{-t/2}\ dt = (-40e^{-t/2})\Big|_0^2$$
$$= 40(-e^{-1} + 40) \approx 25.2848$$

45. $f_{ave} = \dfrac{1}{2}\displaystyle\int_0^2 e^x\ dx = \dfrac{e^2-1}{2} \approx 3.19$

46. $f_{ave} = \dfrac{1}{4}\displaystyle\int_0^4 (4x - x^2)\ dx = \dfrac{8}{3}$

47. $\displaystyle\int_0^2 (x^2-2)\,dx = \left(\frac{x^3}{3} - 2x\right)\Big|_0^2 = -\frac{4}{3}$

48. $\displaystyle\int_{-1}^1 (x^3 - 2x)\,dx = \left(\frac{x^4}{4} - x^2\right)\Big|_{-1}^1 = 0$

49. $\displaystyle\int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = 1$

50. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = 1$

51. $\displaystyle\int_0^{10} (1 - e^{-t/4})\,dt$
$$= \left(t + 4e^{-t/4}\right)\Big|_0^{10} = 6 + 4e^{-5/2}$$

52. $\displaystyle\int_0^1 te^{-t^2}\ dt$
$$= \left(-\frac{1}{2}e^{-t^2}\right)\Big|_0^1 = -\frac{1}{2}\left(e^{-1} - 1\right)$$

53. $\displaystyle\int_0^2 \frac{x}{x^2+1}\ dx = \frac{1}{2}\ \ln|x^2+1|\Big|_0^2$
$$= \frac{\ln 5}{2}$$

54. $\displaystyle\int_1^2 \frac{\ln x}{x}\ dx = \left(\frac{\ln^2 x}{2}\right)\Big|_1^2 = \frac{\ln^2 2}{2}$

55. $\displaystyle\int_0^2 x\sqrt{x^2+4}\,dx$
$$= \left(\frac{1}{2}\cdot\frac{2}{3}\cdot(x^2+4)^{3/2}\right)\Big|_0^2$$
$$= \frac{16\sqrt{2} - 8}{3}$$

56. $\displaystyle\int_0^2 x(x^2+1)\,dx$
$$= \left(\frac{1}{4}(x^2+1)^2\right)\Big|_0^2 = 6$$

57. $\displaystyle\int_0^1 (e^x - 2)^2\,dx = \int_0^1 (e^{2x - 4e^x + 4})\,dx$
$$= \left(\frac{1}{2}e^{2x} - 4e^x + 4x\right)\Big|_0^2$$
$$= \left(\frac{e^2}{2} - 4e + 4\right) - \left(\frac{1}{2} - 4\right)$$
$$= \frac{e^2}{2} - 4e + \frac{15}{2}$$

58. $\displaystyle\int_{-\pi}^{\pi} \cos(x/2)\,dx$
$$= (2\sin(x/2))\Big|_{-\pi}^{\pi} = 4$$

59. $f'(x) = \sin x^2 - 2$

60. $f'(x) = \sqrt{(x^2)^2 + 1}\cdot 2x$

61. a) Midpoint Rule:
$$\int_0^1 \sqrt{x^2+4}\,dx$$
$$\approx \frac{1-0}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$
$$\approx 2.079$$

b) Trapezoidal Rule:
$$\int_0^1 \sqrt{x^2+4}\,dx$$
$$\approx \frac{1-0}{2(4)}\left[f(0) + 2f\left(\frac{1}{4}\right)\right.$$

$+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)$
$+f(1)]$
≈ 2.083

c) Simpson's Rule:
$$\int_0^1 \sqrt{x^2+4}\,dx$$
$$\approx \frac{1-0}{3(4)}\left[f(0)+4f\left(\frac{1}{4}\right)\right.$$
$$\left.+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]$$
$$\approx 2.080$$

62. a) Midpoint Rule:
$$\int_0^2 e^{-x^2/4}\,dx$$
$$\approx \frac{2}{4}[f(0.25)+f(0.75)$$
$$+f(1.25)+f(1.75)]$$
$$\approx 1.497494$$

b) Trapezoidal Rule:
$$\int_0^2 e^{-x^2/4}\,dx$$
$$\approx \frac{2}{8}[f(0)+2f(.5)+2f(1)$$
$$+2f(1.5)+f(2)]$$
$$\approx 1.485968$$

c) Simpson's Rule:
$$\int_0^2 e^{-x^2/4}\,dx$$
$$\approx \frac{2}{12}[f(0)+4f(.5)+2f(1)$$
$$+4f(1.5)+f(2)]$$
$$\approx 1.493711$$

63.

n	Midpoint	Trapezoid	Simpson's
20	2.08041	2.08055	2.08046
40	2.08045	2.08048	2.08046

64.

n	Midpoint	Trapezoid	Simpson's
20	1.493802	1.493342	1.493648
40	1.493687	1.493572	1.493648