MATH 2107 – CALCULUS I – SPRING 10 – FINAL EXAM
MODEL SOLUTIONS

1. We need

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^+} h(x) = h(0) \qquad (1)$$

Now, $\lim_{x\to 0^-} h(x) = 2b$

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \frac{e - e^{\cos x}}{x^2 - 2x^3} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \frac{e^{\cos x}\sin x}{2x - 6x^2} \quad \text{(by L'Hôpital Rule)}$$

$$= \lim_{x\to 0^+} \frac{e^{\cos x}}{2(1-3x)} \cdot \frac{\sin x}{x}$$

$$= \left(\frac{e}{2}\right)(1) = \frac{e}{2}$$

$h(0) = 2b$

$\therefore (1) \Rightarrow \quad 2b = \frac{e}{2} = 2b \Rightarrow \boxed{b = \frac{e}{4}}$

2. (i) Since g is continuous at $x = 3$, $g(3) = \lim_{x\to 3} g(x) = -2$

(ii) $\lim_{x\to 3} \left(\frac{x-3}{\sqrt{x} - \sqrt{3}}\right) g(x)$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})\, g(x)}{(x-3)} = \lim_{x\to 3} (\sqrt{x}+\sqrt{3})\, g(x)$$

$$= (\sqrt{3}+\sqrt{3})\, g(3) = -4\sqrt{3}$$

(iii) $\lim_{x\to 3} g(f(x)) = g(\lim_{x\to 3} f(x))$

$$= g(3) = -2$$

③ $y - x^2 - \sin^{-1}(xy) = 2e$

$\therefore \quad \dfrac{dy}{dx} - 2x - \dfrac{1}{\sqrt{1-x^2y^2}}(xy' + y) = 0$

$\dfrac{dy}{dx}\left[1 - \dfrac{x}{\sqrt{1-x^2y^2}}\right] = 2x + \dfrac{y}{\sqrt{1-x^2y^2}}$

$\dfrac{dy}{dx} = \dfrac{2x + \dfrac{y}{\sqrt{1-x^2y^2}}}{1 - \dfrac{x}{\sqrt{1-x^2y^2}}}$

At $x = 0$, $y = 2e$

$m = y'(0, 2e) = \dfrac{0 + \frac{2e}{1}}{1 - 0} = 2e$

Equation of tangent line: $y = 2e(x-0) + 2e$
$= 2ex + 2e$

④ (a) $y = e^2 + (2x + \sinh(3x^2))^{1/2}$

$y' = 0 + \dfrac{1}{2(2x + \sinh(3x^2))^{1/2}} \cdot (2 + \cosh(3x^2) \cdot 6x)$

$= \dfrac{1 + 3x \cosh(3x^2)}{(2x + \sinh(3x^2))^{1/2}}$

(b) $y = \dfrac{1}{3} \ln\left(\dfrac{x^7}{x^4+1}\right)$

$= \dfrac{1}{3}\left[\ln(x^7) - \ln(x^4+1)\right]$

$= \dfrac{1}{3}\left[7 \ln x - \ln(x^4+1)\right]$

$y' = \dfrac{1}{3}\left[\dfrac{7}{x} - \dfrac{4x^3}{x^4+1}\right]$

(5) Let $f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

f is continuous and differentiable for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Applying the Mean Value Theorem, we have

$$\frac{\tan a - \tan 0}{a - 0} = \sec^2 c, \qquad (1)$$

where c is any number between 0 and a.

(1) $\Rightarrow$ $|\cos^2 c|\,|\tan a| = |a|$

$\Rightarrow$ $|\tan a| \geq |a|$ or $|a| \leq |\tan a|$

$\because |\cos^2 c| \leq 1$.

(6) Let $f(x) = 4 - x - 2^x$

We need to show that f has at least one zero in $[1,2]$.

Note that: f is continuous in $[1,2]$,

$f(1) = 4 - 1 - 2 = 1$,

$f(2) = 4 - 2 - 4 = -2$.

Clearly $f(1)\,f(2) < 0$ [or $f(1)$ & $f(2)$ are of opposite signs]

$\therefore$ By the IVT, f has at least one zero in the given interval.

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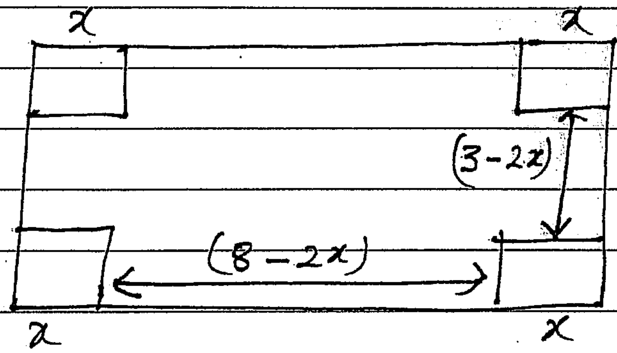


The volume V of the box is given by

$$V(x) = x(8-2x)(3-2x), \qquad 0 \le x \le \frac{3}{2}$$

$$= (3x-2x^2)(8-2x)$$

$$V'(x) = (3-4x)(8-2x) - 2(3x-2x^2)$$

$$= 24-6x-32x+8x^2-6x+4x^2$$

$$= 12x^2-44x+24$$

$$V'(x)=0 \Rightarrow 3x^2-11x+6=0$$

$$\Rightarrow (3x-2)(x-3)=0$$

$$\Rightarrow x = \frac{2}{3}, 3$$

Since $3 \notin [0, \frac{3}{2}]$, the critical number is $x = \frac{2}{3}$

$$V(0) = 0$$

$$V\left(\frac{3}{2}\right) = 0$$

$$V\left(\frac{2}{3}\right) = \frac{2}{3}\left(8-\frac{4}{3}\right)\left(3-\frac{4}{3}\right) = \left(\frac{2}{3}\right)\left(\frac{20}{3}\right)\left(\frac{5}{3}\right) = \frac{200}{27}$$

Thus the maximum volume corresponds to the value $x = \frac{2}{3}$ and is equal to $\frac{200}{27}$ ft^3.

(8) $f(x) = 3x^5 - 5x^3$

$f'(x) = 15x^4 - 15x^2$
$= 15x^2(x^2-1)$, $f''(x) = 60x^3 - 30x$.

$f'(x) = 0 \Rightarrow x = 0, \pm 1$

$f'(x)$: $+$ | $-$ | $-$ | $+$
$-\infty$ $\quad -1 \quad 0 \quad 1 \quad \infty$

$f(x)$: $\nearrow$ $\searrow$ $\searrow$ $\nearrow$

$f(x)$ increases in $(-\infty, -1) \cup (1, \infty)$
$f(x)$ decreases in $(-1, 0) \cup (0, 1)$.

When $x = -1$, $y = 2$
When $x = 1$, $y = -2$
$f(x)$ has a local max. at $x = -1$; Local max pt.: $(-1, 2)$
$f(x)$ has a local min at $x = 1$; Local min. pt.: $(1, -2)$.

Concavity: $f''(x) = 0 \Rightarrow 30x[2x^2 - 1] = 0 \Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}$.

$f''(x)$: $-$ | $+$ | $-$ | $+$
$-\frac{1}{\sqrt{2}}$ (≈ -0.7) $\quad 0 \quad$ $\frac{1}{\sqrt{2}}$ (≈ 0.7)

$f(x)$:

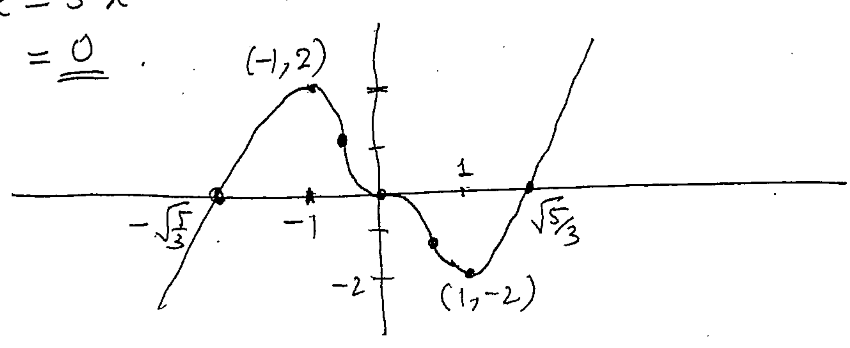
$f(x)$ is concave down ~~in~~ $(-\infty, -\frac{1}{2}) \cup (0, \frac{1}{\sqrt{2}})$

$f(x)$ is concave up in $(-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$.
Inflection points at $x = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ because $f(x)$ changes concavity at these x-values.

x-intercepts: $3x^5 - 5x^3 = 0 \Rightarrow x^3[3x^2 - 5] = 0 \Rightarrow x = 0, x = \pm\sqrt{\frac{5}{3}}$

y-intercept: $f(0) = \underline{\underline{0}}$.

Sketch:



(9) $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

$x_i = 1 + \frac{2i}{n} \quad , \quad i = 0, 1, \cdots n$

$$R_n = \sum_{i=1}^{n} (2x_i^2 + 3x_i)\Delta x$$

$$= \frac{2}{n} \sum_{i=1}^{n} \left[2\left(1+\frac{2i}{n}\right)^2 + 3\left(1+\frac{2i}{n}\right)\right]$$

$$= \frac{2}{n} \sum_{i=1}^{n} \left[2 + \frac{8i}{n} + \frac{8i^2}{n^2} + 3 + \frac{6i}{n}\right]$$

$$= \frac{2}{n} \sum_{i=1}^{n} \left[5 + \frac{14i}{n} + \frac{8i^2}{n^2}\right]$$

$$= \frac{2}{n}\left[\sum_{i=1}^{n} 5 + \frac{14}{n} \sum_{i=1}^{n} i + \frac{8}{n^2} \sum i^2\right]$$

$$= \frac{2}{n}\left[5n + \frac{14\, n(n+1)}{2n} + \frac{8\, n(n+1)(2n+1)}{6n^2}\right]$$

$$= 10 + 14\left(1+\frac{1}{n}\right) + \frac{8}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

The area $A = \lim_{n\to\infty} R_n = 10 + 14 + \frac{16}{3}$

$$= \frac{72+16}{3} = \frac{88}{3} \text{ units.}$$

(10) (a) $\text{L.H.S.} = 1 + 2\sinh^2(2x)$

$$= 1 + 2\left[\frac{e^{2x} - e^{-2x}}{2}\right]^2$$

$$= 1 + \frac{1}{2}\left[e^{4x} + e^{-4x} - 2\right]$$

$$= \frac{1}{2}\left(e^{4x} + e^{-4x}\right)$$

$$= \cosh 4x = \text{R.H.S.}$$

(b) $$\frac{d}{dx}(\sinh^{-1}x) = \frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+1}\right)\right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \cdot \left[1 + \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x\right]$$

$$= \frac{1}{x + \sqrt{x^2+1}}\left[\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}$$

(11) (a) $\displaystyle\int_1^4 \frac{2-x}{x^{3/2}}\,dx$

$= \displaystyle\int_1^4 \left(2x^{-3/2} - x^{-1/2}\right) dx$

$= \left(-\frac{4}{\sqrt{x}} - 2\sqrt{x}\right)\Big|_1^4 = (-2-4)-(-4-2) = -6+6 = 0$

(b) $\displaystyle\int \left(1-x^2+e^{-2x}\right)^{13}\left(x+e^{-2x}\right)dx$

Let $u = 1-x^2+e^{-2x}$ $\therefore du = -2\left(x+e^{-2x}\right)dx$

$I = -\frac{1}{2}\displaystyle\int u^{13}\,du = -\frac{1}{28}u^{14}+C = -\frac{1}{28}\left(1-x^2+e^{-2x}\right)^{14}+C$

(c) $\displaystyle\int_0^{\pi} \frac{\sin x}{3+|\cos x|}\,dx$

$= \displaystyle\int_0^{\pi/2} \frac{\sin x}{3+\cos x}\,dx + \int_{\pi/2}^{\pi} \frac{\sin x}{3-\cos x}\,dx$

$= -\ln(3+\cos x)\Big|_0^{\pi/2} + \ln(3-\cos x)\Big|_{\pi/2}^{\pi}$

$= \left\{-\ln(3)+\ln(4)\right\} + \left\{\ln(4)-\ln(3)\right\}$

$= 2\left(\ln(4)-\ln(3)\right)$

MCQ's / M2107/ FINAL / SP10

VERSION 1		VERSION 2	
12	C	12	B
13	A	13	B
14	A	14	C
15	A	15	D
16	D	16	A
17	B	17	A
18	C	18	D
19	C	19	A
20	C	20	B
21	B	21	B