SULTAN QABOOS UNIVERSITY – COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FALL- 2010 FINAL EXAMINATION
MATH 3171 – LINEAR ALGEBRA AND MULTIVARIATE CALCULUS FOR ENGINEERS

TIME ALLOWED: 150 MINUTES **TOTAL MARKS: 80**

ANSWER ALL SIX QUESTIONS. SHOW ALL WORK FOR FULL CREDIT

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1. Consider the system of equations $\begin{cases} x_1 \qquad\qquad + x_4 = 0 \\ -x_1 + x_2 + x_3 - x_4 = 1 \\ -3x_1 \qquad + x_3 - x_4 = 0 \\ -2x_1 \qquad + x_3 + x_4 = 4. \end{cases}$

(a) **[5 Marks]** (i) Define rank of a matrix.
(ii) Find the ranks of the coefficient and augmented matrices of the given system.

(b) **[3 Marks]** (i) Determine whether the above system is consistent or not. **Justify your answer.**
(ii) Find the solution of the given system if it is consistent.

(c) **[2 Marks]** Determine whether the coefficient matrix of the given system is invertible or not. **Justify your answer.**

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2. (a) **[2 Marks]** Show that an orthogonal transformation preserves the value of the inner product of vectors.

(b) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 9 & -9 \\ 0 & 5 & -3 \\ 0 & 6 & -4 \end{bmatrix}$.

(i) **[6 Marks]** Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}$.

(ii) **[7 Marks]** Find a non-singular matrix $\mathbf{X}$ such that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is a diagonal matrix, where $\mathbf{X}^{-1}$ is the inverse of $\mathbf{X}$.

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3. (a) **[5 Marks]** Let $\mathbf{U} = [1, \ 2, \ 3]$ and $\mathbf{V} = [0, \ -3, \ 5]$. Verify that

$$\mathbf{U} \times \mathbf{V} = (\mathbf{U} \bullet (\mathbf{V} \times \mathbf{i}))\mathbf{i} + (\mathbf{U} \bullet (\mathbf{V} \times \mathbf{j}))\mathbf{j} + (\mathbf{U} \bullet (\mathbf{V} \times \mathbf{k}))\mathbf{k}.$$

(b) **[3 Marks]** Find an equation of the tangent plane to the graph of $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at the point $\mathrm{P}: (1, 1, 5)$.

(c) **[3 Marks]** Let velocity vector of a fluid motion be

$$\mathbf{V} = [e^x \cos y + yz, \ xz - e^x \sin y, xy + z].$$

Determine whether the flow is irrotational or not.

PLEASE TURN OVER

4. (a) **[4 Marks]** Show that in the absence of friction, the work (W) done by the force $\mathbf{F}$ in moving a particle of constant mass m along the smooth curve C described by the vector function $\mathbf{r}(t)$ from the point A at $t=a$ to the point B at $t=b$ is the same as the change in kinetic energy, $W=\frac{1}{2}m|\mathbf{v}(b)|^2-\frac{1}{2}m|\mathbf{v}(a)|^2$; $\mathbf{v}$ is the velocity of the particle.

(b) **[4 Marks]** Find the work done by the force $\mathbf{F}(x,y,z)=yz\,\mathbf{i}+xz\,\mathbf{j}+xy\,\mathbf{k}$ acting along the curve C given by $C:\mathbf{r}(t)=t^3\,\mathbf{i}+t^2\,\mathbf{j}+t\,\mathbf{k}$, from $t=1$ to $t=3$.

(c) **[2 Marks]** Given $f(x,y,z)=x^4-3y^3+2z^2+15$, find the directional derivative of f at P: (1, 0, 2) along the direction of $2\,\mathbf{i}-\mathbf{j}+2\,\mathbf{k}$.

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5. (a) **[3 Marks]** Show that the function $z=\ln(x^2+y^2)$ satisfies the Laplace's equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

(b) **[3 Marks]** Prove that $\operatorname{div}(f^5\,\mathbf{V})=5f^4(\nabla f\bullet\mathbf{V})+f^5\operatorname{div}(\mathbf{V})$.

(c) **[8 Marks]** (i) Show that the expression
$\mathbf{F}\bullet d\mathbf{r}=(y-yz\sin x)dx+(x+z\cos x)dy+(y\cos x)dz$ is exact.

(ii) Evaluate the integral $\displaystyle\int_{(0,0,0)}^{(\pi/2,1,1)}\mathbf{F}\bullet d\mathbf{r}$.

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6. (a) **[5 Marks]** Describe the region of integration and evaluate $\displaystyle\int_0^1\int_0^{2x}e^{x^2}\,dy\,dx$.

(b) **[6 Marks]** Let $f(x,y)=x$ be the density of mass in the region $R:0\le y\le\sqrt{4-x^2}$, $0\le x\le 2$. Find the moment of inertia about the y-axis of the mass in R.

(c) **[9 Marks]** State Green's theorem in the plane. Hence verify it for the function $\mathbf{F}=[y,-x]$ around the boundary C of the region $R:\{(x,y)\,|\,x^2+y^2\le 4\}$ considering anticlockwise rotation.

TOTAL 80 MARKS *WISH YOU GOOD LUCK*

Solutions of Final Exam. Math 3171, Fall 2010.

1. (a) (i) Rank of a matrix: The maximum number of linearly independent row (or, column) vectors of a matrix $A = [a_{ij}]$ is called the rank of A and is denoted by rank A.

(ii) $$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ -3 & 0 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 & 4 \end{bmatrix}$$ using Gaussian elimination.

$$\xrightarrow[R_4+2R_1]{R_2+R_1,\ R_3+3R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

which is in echelon form.

$\text{Rank}(A) = 4$, $\text{Rank}(\tilde{A}) = 4$ as we have four pivot elements in the echelon matrix.

(b) (i) Since $\text{Rank}(A) = \text{Rank}(\tilde{A})$, so, the given system is consistent.

(ii) Now from the echelon matrix:

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_2 + x_3 &= 1 \\ x_3 + 2x_4 &= 0 \\ x_4 &= 4 \end{aligned}$$

By backward substitution we get $x_1 = -4$, $x_2 = 9$, $x_3 = -8$, $x_4 = 4$; which is the required soln of the given system.

(c) Since $\text{rank}(A) = 4$, the order of the matrix (or $\det(A) \neq 0$), the coefficient matrix of the given system is invertible.

2 (a) Let A be an orthogonal matrix. Then $A^TA = AA^T = I$.
Also, let $\underline{a}, \underline{b} \in R^n$. Then $\underline{a}\cdot\underline{b} = \underline{a}^T\underline{b}$.
Suppose $\underline{u} = A\underline{a}$ and $\underline{v} = A\underline{b}$. Then

$$\underline{u}\cdot\underline{v} = \underline{u}^T\underline{v} = (A\underline{a})^T(A\underline{b}) = \underline{a}^TA^TA\,\underline{b} = \underline{a}^TI\underline{b} = \underline{a}^T\underline{b}$$

$\therefore \underline{u}\cdot\underline{v} = \underline{a}\cdot\underline{b}$. $\boxed{Q.E.D}$

(b) Characteristic equation: $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 9 & -9 \\ 0 & 5-\lambda & -3 \\ 0 & 6 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(\lambda^2-\lambda-2) = 0$$

$$\Rightarrow \lambda = 2, \quad (\lambda-2)(\lambda+1) = 0$$

$$\therefore \lambda = 2, \lambda = -1.$$

Thus, Eigenvalues are $\lambda_1 = -1$, $\lambda_{2,3} = 2$.

Let $\underline{x}$ be the Eigenvector of A corresponding to the Eigenvalue λ. Then $(A-\lambda I)\underline{x} = \underline{0}$.

Now,
$$\begin{bmatrix} 2-\lambda & 9 & -9 \\ 0 & 5-\lambda & -3 \\ 0 & 6 & -4-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

For $\lambda = -1$: Eqn. (1) implies

$$\begin{bmatrix} 3 & 9 & -9 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 9 & -9 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 - 3x_3 = 0$
$2x_2 - x_3 = 0$

The system contains two equations with three unknowns. Hence there is one free variable say x_3.

put $x_3 = 2$, then $x_2 = 1$, $x_1 = 3$.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$ is an Eigenvector of A for $\lambda = -1$.

For $\lambda = 2$: Eqn (1) implies

$$\begin{bmatrix} 0 & 9 & -9 \\ 0 & 3 & -3 \\ 0 & 6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, $\begin{bmatrix} 0 & 9 & -9 \\ 0 & 3 & -3 \\ 0 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore x_2 - x_3 = 0$

$\Rightarrow x_2 = x_3$,

x_1 & x_3 are free variables.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are two Eigenvectors of A for $\lambda = 2$.

(ii) Let $\underline{X} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. $|\underline{X}| = 1 \neq 0$ $\therefore \underline{X}$ is non-singular and $\underline{X}^{-1}$ exists.

Now,

$$[X : I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -3 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \therefore \underline{X}^{-1} = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Now, $$X^{-1}AX = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 9 & -9 \\ 0 & 5 & -3 \\ 0 & 6 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is a diagonal matrix.

3 (a) $\underline{U} \times \underline{V} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & -3 & 5 \end{vmatrix} = [19, -5, -3].$

$\underline{V} \times \underline{i} = [0, -3, 5] \times [1, 0, 0] = [0, 5, 3]$

$\underline{U} \cdot (\underline{V} \times \underline{i}) = [1, 2, 3] \cdot [0, 5, 3] = 19$

$\underline{V} \times \underline{K} = [0, -3, 5] \times [0, 0, 1] = [-3, 0, 0]$

$\underline{U} \cdot (\underline{V} \times \underline{K}) = [1, 2, 3] \cdot [-3, 0, 0] = -3$

$\underline{V} \times \underline{j} = [0, -3, 5] \times [0, 1, 0] = [-5, 0, 0]$

$\underline{U} \cdot (\underline{V} \times \underline{j}) = [1, 2, 3] \cdot [-5, 0, 0] = -5$

Therefore,

$(\underline{U} \cdot (\underline{V} \times \underline{i}))\underline{i} + (\underline{U} \cdot (\underline{V} \times \underline{j}))\underline{j} + (\underline{U} \cdot (\underline{V} \times \underline{K}))\underline{K} =$

$19\underline{i} - 5\underline{j} - 3\underline{K} = \underline{U} \times \underline{V}.$

(Verified).

(b) Let $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4.$

$\nabla f = [x, y, -1].$

$\nabla f]_P = [1, 1, -1]$ which is a normal vector to the surface of f at P.

Eqn of the tangent plane

$(x-1) + (y-1) - (z-5) = 0$

$\Rightarrow x + y - z + 3 = 0$

$\boxed{\text{Ans}}$.

3(c) $\nabla \times \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y + yz & xz - e^x \sin y & xy + z \end{vmatrix}$

$= [x - x,\ y - y,\ z - e^x \sin y + e^x \sin y - z]$

$= [0, 0, 0]$.

Since $\nabla \times \underline{V} = \underline{0}$, so the fluid motion is irrotational.

4(a) Let $\underline{F}$ be the force, t be the time. Then velocity, $\underline{V} = \frac{d\underline{r}}{dt}$.

Now work done, $W = \int_c \underline{F} \cdot d\underline{r} = \int_a^b \underline{F} \cdot \underline{V}\, dt$

Following Newton's second law of motion

$\underline{F} = m\underline{a} = m \frac{d\underline{V}}{dt} = m\underline{V}'$.

$\therefore\ W = \int_a^b m \underline{V}' \cdot \underline{V}\, dt$

Now, $\frac{d}{dt}(\underline{V} \cdot \underline{V}) = 2\underline{V}' \cdot \underline{V}$

$\therefore\ \underline{V}' \cdot \underline{V} = \frac{1}{2} \frac{d}{dt} |\underline{V}|^2$

So, $W = \int_a^b \frac{1}{2} m \frac{d}{dt} |\underline{V}|^2\, dt = \left[\frac{1}{2} m |\underline{V}|^2\right]_a^b$

$= \frac{1}{2} m |\underline{V}(b)|^2 - \frac{1}{2} m |\underline{V}(a)|^2$

$=$ Gain in kinetic energy.

Showed.

4(b) $\underline{F} = yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k}$; $\underline{r}(t) = t^3\underline{i} + t^2\underline{j} + t\underline{k}$

$\underline{F}(\underline{r}(t)) = t^3\underline{i} + t^4\underline{j} + t^5\underline{k}$; $\underline{r}'(t) = 3t^2\underline{i} + 2t\underline{j} + \underline{k}$

$\underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) = 3t^5 + 2t^5 + t^5 = 6t^5$.

work done, $w = \int_C \underline{F} \cdot d\underline{r} = \int_1^3 6t^5\,dt = \left[t^6\right]_1^3$

$= 729 - 1 = 728$. Unit

$\boxed{\text{Ans}}$

4(c) $f = x^4 - 3y^3 + 2z^2 + 15$

$\nabla f = 4x^3\underline{i} - 9y^2\underline{j} + 4z\underline{k}$

$\nabla f]_p = 4\underline{i} + 8\underline{k} = [4, 0, 8]$

Let $\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}$. Then $|\underline{a}| = \sqrt{4+1+4} = 3$

Directional derivative,

$$\left(D_{\underline{a}} f\right)_p = \nabla f]_p \cdot \frac{\underline{a}}{|\underline{a}|}$$

$$= [4, 0, 8] \cdot \left[\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right]$$

$$= \frac{8}{3} + \frac{16}{3}$$

$$= 8$$

$\boxed{\text{Ans}}$

5 (a) $z = \ln(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = 0.$$

Showed

(6) L.H.S $\quad \operatorname{div}(f^5 \underline{V}) = \frac{\partial}{\partial x}(f^5 V_1) + \frac{\partial}{\partial y}(f^5 V_2) + \frac{\partial}{\partial z}(f^5 V_3)$

$$= 5f^4 \frac{\partial f}{\partial x} V_1 + f^5 \frac{\partial V_1}{\partial x} + 5f^4 \frac{\partial f}{\partial y} V_2 + f^5 \frac{\partial V_2}{\partial y} + 5f^4 \frac{\partial f}{\partial z} V_3 + f^5 \frac{\partial V_3}{\partial z}.$$

$$= 5f^4 \left(\frac{\partial f}{\partial x} V_1 + \frac{\partial f}{\partial y} V_2 + \frac{\partial f}{\partial z} V_3\right) + f^5\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$= 5f^4 (\nabla f \cdot \underline{V}) + f^5 (\nabla \cdot \underline{V}).$$

= R.H.S proved

5 (c) $F_1 = y - yz\sin x$, $F_2 = x + z\cos x$, $F_3 = y\cos x$

$\frac{\partial F_1}{\partial y} = 1 - z\sin x = \frac{\partial F_2}{\partial x}$; $\frac{\partial F_2}{\partial z} = \cos x = \frac{\partial F_3}{\partial y}$; $\frac{\partial F_3}{\partial x} = -y\sin x = \frac{\partial F_1}{\partial z}$.

OR $\nabla \times \underline{F} = \underline{0}$.

$\therefore$ $\underline{F}\cdot d\underline{r}$ is exact.

Let $\underline{F} = \text{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$.

$\frac{\partial f}{\partial x} = y - yz\sin x$; $\frac{\partial f}{\partial y} = x + z\cos x$; $\frac{\partial f}{\partial z} = -y\sin x$

$f = yx + yz\cos x + g(y, z)$

$\frac{\partial f}{\partial y} = x + z\cos x + \frac{\partial g}{\partial y}$

$\Rightarrow$ $x + z\cos x = x + z\cos x + \frac{\partial g}{\partial y}$

$\Rightarrow$ $\frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$

$\therefore$ $f = yx + yz\cos x + h(z)$

Differentiating partially w.r. to z

$\frac{\partial f}{\partial z} = y\cos x + h'(z)$

$\Rightarrow y\cos x = y\cos x + h'(z)$

$\Rightarrow$ $h'(z) = 0$

$\therefore$ $h(z) = C$

$\therefore$ potential of $\underline{F}$, $f = xy + yz\cos x + C$.

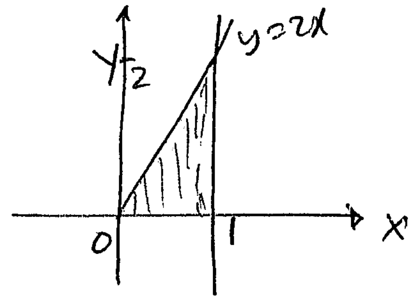
put $c = 0$. Then $f = xy + yz\cos x$.

Since $\underline{F}\cdot d\underline{r}$ is exact, so $\int_C \underline{F}\cdot d\underline{r}$ is independent of the path.

Now $\int_C \underline{F}\cdot d\underline{r} = \int_{(0,0,0)}^{(\frac{\pi}{2},1,1)} \underline{F}\cdot d\underline{r} = \left[xy + yz\cos x\right]_{(0,0,0)}^{(\frac{\pi}{2},1,1)}$

$= \frac{\pi}{2}$ $\boxed{\text{Ans}}$

6 (a) $\int_0^1 \int_0^{2x} e^{x^2} dx$



$= \int_0^1 e^{x^2} \left[y \right]_0^{2x} dx$

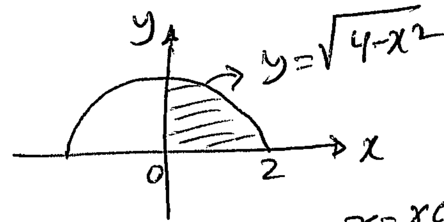
$= \int_0^1 2x e^{x^2} dx$

Let $u = x^2$
$du = 2x\,dx$
$x = 0;\ u = 0$
$x = 1;\ u = 1.$

$= \int_0^1 e^u \, du$

$= \left[e^u \right]_0^1$

$= e - 1.$ $\boxed{\text{Ans}}$



6 (b) $I_Y = \iint_R x^2 f(x,y)\, dA$

$= \int_0^2 \int_0^{\sqrt{4-x^2}} x^3 \, dy\, dx \Rightarrow$

$= \int_0^2 x^3 \sqrt{4-x^2}\, dx$

$= \int_0^2 x^2 \left(\sqrt{4-x^2}\right) x\, dx$

let $u = 4 - x^2$
$du = -2x\,dx$
$x = 0;\ u = 4$
$x = 2;\ u = 0$

$= \int_4^0 (4-u)\sqrt{u} \left(-\tfrac{1}{2} du\right)$

$= +\tfrac{1}{2} \int_0^4 \left(4\sqrt{u} - u^{3/2}\right) du$

$= +\tfrac{1}{2} \left[4 \cdot \tfrac{2}{3} u^{3/2} - \tfrac{2}{5} u^{5/2} \right]_0^4$

$= \tfrac{1}{2} \left[\tfrac{8}{3} \cdot 8 - \tfrac{2}{5} \cdot 32 \right]$

$= \frac{64}{15}$ $\boxed{\text{Ans}}$

OR

$x = r\cos\theta$
$dy\,dx = r\,d\theta\,dr$

$I_Y = \int_{r=0}^{2} \int_{\theta=0}^{\pi/2} r^3 \cos^3\theta \cdot r\,d\theta\,dr$ using polar Co-ordinates.

$= \int_0^2 r^4\, dr \int_0^{\pi/2} \cos^3\theta\, d\theta$

$= \left[\frac{r^5}{5}\right]_0^2 \int_0^{\pi/2} (1 - \sin^2\theta)\cos\theta\, d\theta$

$= \frac{32}{5} \cdot \int_0^1 (1 - u^2)\, du$ put $u = \sin\theta$

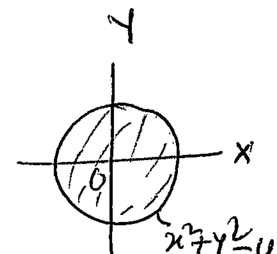
$= \frac{32}{5} \cdot \left[u - \frac{u^3}{3} \right]_0^1$

$= \frac{32}{5} \cdot \frac{2}{3}$

$= \frac{64}{15}.$

6(c) Green's thm: Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R.

Then $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1 dx + F_2 dy$; where R is on the left as we move in the direction of integration.



<u>L.H.S</u> $\quad \underline{F} = [y, -x], \quad F_1 = y, F_2 = -x$

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1 - 1 = -2.$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = -2 \iint_R dA$$

$= -2 \cdot$ area of the circle

$= -2 \cdot \pi (2)^2 = -8\pi.$

<u>R.H.S</u> parameterise the circle $\underline{x^2+y^2=4}$; $0 \le t \le 2\pi$

$x = 2\cos t, \; y = 2\sin t$

$$\oint_C F_1 dx + F_2 dy = \int_0^{2\pi} 2\sin t(-2\sin t\, dt) + (-2\cos t)(2\cos t\, dt)$$

$$= -\int_0^{2\pi} 2 \cdot 2\, dt$$

$$= -4 [t]_0^{2\pi}$$

$$= -8\pi$$

L.H.S = R.H.S

Hence Green's thm is verified. 田.