SULTAN QABOOS UNIVERSITY – COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
SUMMER 2010 – INTERM TEST

MATH 3171 – LINEAR ALGEBRA AND MULTIVARIATE CALCULUS FOR ENGINEERS

TIME ALLOWED: 70 MINUTES

SECTION: **ID. NO.** **NAME:**

INSTRUCTIONS

- SHOW ALL WORK FOR FULL CREDIT.
- THE TEST CONSISTS OF 4 QUESTIONS, EACH ON A SEPARATE SHEET.
- PROVIDE YOUR ANSWERS ON THE SHEET IN THE SPACE DIRECTLY BELOW EACH QUESTION.
- PLEASE DO NOT SEPARATE THE PAGES OF THIS BOOKLET

WRITE YOUR ID AT THE TOP OF EACH SHEET

QUESTION NO.	MAXIMUM MARKS	MARKS SCORED
1	15	
2	15	
3	15	
4	15	
TOTAL	60	

1. (a) [10 Marks] Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

(i) Show that A ***satisfies*** **the expression** $A^3 - 2A^2 + A - I = \mathbf{0}$.

(ii) Show that A **is *invertible*. Find the inverse of A *using the above expression*.**

(b) [5 Marks] Show that any square matrix can be written as the sum of a *symmetric* matrix and a *skew- symmetric* matrix.

Solution 1.

(i) $A^2 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 2 & 2 & -3 \\ 1 & 1 & -2 \\ -2 & -1 & 2 \end{bmatrix}$

$$A^3 - 2A^2 + A - I = \begin{bmatrix} 2 & 2 & -3 \\ 1 & 1 & -2 \\ -2 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & -4 \\ 2 & 0 & -2 \\ -2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0}.$$

Q.E.D

(ii) $|A| = 1 \neq 0$ $\therefore$ A is invertible.

NOW $A^3 - 2A^2 + A - I = 0 \Rightarrow I = A^3 - 2A^2 + A$

$$\therefore A^{-1} = A^2 - 2A + I = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & 0 \\ 0 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad \boxed{\text{Ans}}$$

(b) let $A_{n \times n} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = R + S$.

$R^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = R$

$\therefore$ R is a symmetric matrix.

$S^T = \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -S$

$\therefore$ S is a skew-symmetric matrix.

Q.E.D

2. Consider the non-homogeneous linear system

$$\begin{aligned} w \quad -x \quad +y \quad +2z &= 2 \\ 3w \quad +x \quad +4y \quad -z &= 4 \\ 11w \quad +x \quad +14y \quad +z &= \lambda. \end{aligned}$$

(a) [4 Marks] Reduce the *augmented* matrix to the echelon form.

(b) [7 Marks] Determine the value(s) of λ for which this system has
(i) no solution,
(ii) at least one solution. For case (ii) write the *general solution* in matrix form.

(c) [2 Mark] Using part (a) find the rank of the *augmented* matrix.

(d) [2 Mark] Using part (a) determine whether the row vectors $\begin{bmatrix}1 & -1 & 1 & 2\end{bmatrix}$, $\begin{bmatrix}3 & 1 & 4 & -1\end{bmatrix}$ and $\begin{bmatrix}11 & 1 & 14 & 1\end{bmatrix}$ are linearly *dependent* or *independent*.

Solution 2.

(a) $\tilde{A} = \begin{bmatrix} 1 & -1 & 1 & 2 & 2 \\ 3 & 1 & 4 & -1 & 4 \\ 11 & 1 & 14 & 1 & \lambda \end{bmatrix} \underset{R_3-11R_1}{\overset{R_2-3R_1}{\sim}} \begin{bmatrix} 1 & -1 & 1 & 2 & 2 \\ 0 & 4 & 1 & -7 & -2 \\ 0 & 12 & 3 & -21 & \lambda-22 \end{bmatrix} \overset{R_3-3R_2}{\sim}$

$\begin{bmatrix} 1 & -1 & 1 & 2 & 2 \\ 0 & 4 & 1 & -7 & -2 \\ 0 & 0 & 0 & 0 & \lambda-16 \end{bmatrix}$ which is in echelon form.

(b) i. For $\lambda \neq 16$, the last row of echelon matrix implies $0 =$ some nonzero number; which is not possible. Therefore, the system has no solution for $\lambda \neq 16$.

ii. For $\lambda = 16$; we have $w - x + y + 2z = 2$ & $4x + y - 7z = -2$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5/4\, y - \frac{1}{4} z + 3/2 \\ -\frac{1}{4} y + \frac{7}{4} z - \frac{1}{2} \\ y \\ z \end{bmatrix}.$$

(c) For $\lambda = 16$; $\text{Rank}(\tilde{A}) = 2$. For $\lambda \neq 16$, $\text{Rank}(\tilde{A}) = 3$.

(d) $\text{Rank}(A) = 2 <$ no. of rows of A.
$\therefore$ The given row vectors of A are linearly dependent.

3. (a) [7 Marks] Solve the following system by Cramer's rule *if it is applicable*

$$\begin{aligned} x \quad +y \qquad &= 2 \\ y \quad +z &= 6 \\ x \qquad\quad +z &= 4. \end{aligned}$$

(b) [4 Marks] If A, B and C are nonsingular square matrices of any order, then *prove* that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

(c) [4 Marks] Prove that the product of any two unitary matrices (of the same size) is unitary.

Solution 3. (a) $D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0$; Cramer's rule is applicable.

$D_1 = \begin{vmatrix} 2 & 1 & 0 \\ 6 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0$; $D_2 = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 6 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 4$; $D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 1 & 0 & 4 \end{vmatrix} = 8$

$x = \frac{D_1}{D} = 0$; $y = \frac{D_2}{D} = 2$; $z = \frac{D_3}{D} = 4$. $\boxed{\text{Ans}}$

(b) $ABC\,(ABC)^{-1} = I$

Multiplying both sides by A^{-1}

$BC\,(ABC)^{-1} = A^{-1}$ $\quad \because A^{-1}A = I$

Multiplying both sides by B^{-1}

$C\,(ABC)^{-1} = B^{-1}A^{-1}$ $\quad \because B^{-1}B = I$

Multiplying both sides by C^{-1}

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ $\quad \because C^{-1}C = I$

(proved)

(c) Let $A_{n\times n}$ and $B_{n\times n}$ are Unitary Matrices. Then $\bar{A}^T = A^{-1}$ and $\bar{B}^T = B^{-1}$. To prove $(\overline{AB})^T = (AB)^{-1}$.

Now $(\overline{AB})^T = (\bar{A}\,\bar{B})^T = \bar{B}^T\bar{A}^T = B^{-1}A^{-1} = (AB)^{-1}$.

Therefore, AB is also unitary matrix.

(proved)

4. Consider $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) [7 Marks] Find the *eigenvalues* and the corresponding *eigenvectors* of A.

(b) [3 Marks] Find algebraic multiplicity, *geometric* multiplicity and *defect* of the corresponding eigenvalues of the matrix A.

(c) [5 Marks] With proper explanation determine whether A is diagonalizable nor not. If possible diagonalize the matrix A.

Solution 4 (a) characteristic equation $|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)(2-\lambda) = 0 \therefore \lambda = 1, 2, 2.$

Eigenvalues are 1, 2 (repeated).

Let $(A-\lambda I)\underline{X} = \underline{0} \Rightarrow \begin{bmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$: $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} x_2 - x_3 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}\right\} x_2 = x_3 = 0.$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an Eigenvector for $\lambda = 1$.

For $\lambda = 2$: $\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - x_2 + x_3 = 0 \quad \therefore x_1 = x_2 - x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are two Eigenvectors for $\lambda = 2$.

(b) $M_1 = 1,\ M_2 = 2,\ m_1 = 1,\ m_2 = 2,\ \Delta_1 = 1-1 = 0,\ \Delta_2 = 2-2 = 0.$

(c) since there are no defects in λ i.e $\Delta_1 = 0, \Delta_2 = 0$, so A is diagonalizable.

Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $|X| = 1 \neq 0 \therefore X^{-1}$ exists.

$XD = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = AX \quad \therefore D = X^{-1}AX.$