Sultan Qaboos University-College of Science
Department of Mathematics and Statistics
MATH 3171 - Linear Algebra and Multivariable Calculus for Engineers
Fall Semester 2008 - QUIZ 4 A

Date: Sunday 30 November 2008 Time:20 minutes
NAME: ID. No.

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1. (**5 marks**) Calculate the line integral of the vector field

$$\overrightarrow{F} = (x^2 - 2xy)\overrightarrow{i} + (y^2 - 2xy)\overrightarrow{j}$$

from $(-1,1)$ to $(1,1)$ along the parabola $y = x^2$.

Solution
$C : \overrightarrow{r}(t) = t\,\overrightarrow{i} + t^2\overrightarrow{j}, \quad -1 \le t \le 1,$ (**1 mark**)
$\overrightarrow{r}'(t) = \overrightarrow{i} + 2t\,\overrightarrow{j},$ ($\mathbf{\frac{1}{2}}$ **mark**)
$\overrightarrow{F}(\overrightarrow{r}(t)) = \left(t^2 - 2t^3\right)\overrightarrow{i} + \left(t^4 - 2t^3\right)\overrightarrow{j}$ and (**1 mark**)
$\overrightarrow{F}(\overrightarrow{r}(t))\cdot\overrightarrow{r}'(t) = t^2 - 2t^3 + 2t^5 - 4t^4.$
Therefore,

$$\begin{aligned}
\int_C \overrightarrow{F}\cdot d\overrightarrow{r} &= \int_{-1}^{1} \overrightarrow{F}(\overrightarrow{r}(t))\cdot\overrightarrow{r}'(t)dt \\
&= \int_{-1}^{1} \left(t^2 - 2t^3 + 2t^5 - 4t^4\right)dt \quad (\textbf{1 mark}) \\
&= \frac{1}{3}t^3 - \frac{1}{2}t^4 + \frac{1}{3}t^6 - \frac{4}{5}t^5\Big|_{-1}^{1} \quad (\textbf{1 mark}) \\
&= \frac{-14}{15} \quad (\mathbf{\frac{1}{2}}\ \textbf{mark}).
\end{aligned}$$

2. (**5 marks**) Calculate the curl of the vector field $\overrightarrow{F} = \dfrac{y\,\overrightarrow{i} + x\,\overrightarrow{j}}{\sqrt{x^2+y^2}}$

Solution

The vector $\overrightarrow{F}$ has components $F_1 = \dfrac{y}{\sqrt{x^2+y^2}}$, $F_2 = \dfrac{x}{\sqrt{x^2+y^2}}$, and $F_3 = 0$. (**1 mark**).

Its curl is

$$\begin{aligned}
\text{Curl}\,\overrightarrow{F} &= \overrightarrow{\nabla} \times \overrightarrow{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{k} \quad (\textbf{1 mark}) \\
&= \left(\frac{y^2}{\left(\sqrt{x^2+y^2}\right)^3} - \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^3}\right)\overrightarrow{k} \quad (\textbf{3 marks}) \\
&= \frac{y^2-x^2}{\sqrt{x^2+y^2}}\,\overrightarrow{k}.
\end{aligned}$$

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1. (**5 marks**) Calculate the line integral of the vector field

$$\overrightarrow{F} = (x^2 - 2xy)\overrightarrow{i} + (y^2 - 2xy)\overrightarrow{j}$$

along the line segment from $(-1, 1)$ to $(1, 1)$.

Solution

$$\begin{aligned} C: \quad \overrightarrow{r}(t) &= (1-t)(-1,1) + t(1,1), \quad (0 \le t \le 1) \\ &= (2t-1)\overrightarrow{i} + \overrightarrow{j}, \quad (\textbf{1 mark}) \end{aligned}$$

$\overrightarrow{r}'(t) = 2\overrightarrow{i}, \quad (\tfrac{\mathbf{1}}{\mathbf{2}}\ \textbf{mark})$

$\overrightarrow{F}(\overrightarrow{r}(t)) = \left[(2t-1)^2 - 2(2t-1)\right]\overrightarrow{i} + [1 - 2(2t-1)]\overrightarrow{j}$ and (**1 mark**)

$\overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) = 8t^2 - 16t + 6.$

Hence

$$\begin{aligned} \int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^1 \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) dt \\ &= \int_0^1 \left(8t^2 - 16t + 6\right) dt \quad (\textbf{1 mark}) \\ &= \frac{8}{3}t^3 - 8t^2 + 6t\Big|_0^1 \quad (\textbf{1 mark}) \\ &= \frac{-2}{3}. \quad (\tfrac{\mathbf{1}}{\mathbf{2}}\ \textbf{mark}) \end{aligned}$$

2. (**5 marks**) Calculate the curl of the vector field $\overrightarrow{F} = \dfrac{y\,\overrightarrow{i} - x\,\overrightarrow{j}}{\sqrt{x^2+y^2}}$

Solution
The vector field $\overrightarrow{F}$ has components $F_1 = \dfrac{y}{\sqrt{x^2+y^2}}$, $F_2 = -\dfrac{x}{\sqrt{x^2+y^2}}$, and $F_3 = 0$. (**1 mark**)
Its curl is

$$\begin{aligned}
\text{Curl } \overrightarrow{F} &= \overrightarrow{\nabla} \times \overrightarrow{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{k} \quad (\textbf{1 mark}) \\
&= \left(-\frac{y^2}{\left(\sqrt{x^2+y^2}\right)^3} - \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^3}\right)\overrightarrow{k} \quad (\textbf{3 marks}) \\
&= -\frac{y^2+x^2}{\left(\sqrt{x^2+y^2}\right)^3}\overrightarrow{k} = -\frac{1}{\sqrt{x^2+y^2}}\,\overrightarrow{k}.
\end{aligned}$$

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1. (**5 marks**) Given that the differential form

$$(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy})dx + (x^3\mathbf{e}^{xy} + 2y)dy$$

is exact, evaluate the integral

$$I = \int_{(1,0)}^{(1,1)} (2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy})dx + (x^3\mathbf{e}^{xy} + 2y)dy.$$

Solution
We find the potential function f from $\frac{\partial f}{\partial y} = x^3\mathbf{e}^{xy} + 2y$.

$$\begin{aligned}
&\Longrightarrow \quad f(x,y) = x^2\mathbf{e}^{xy} + y^2 + h(x) \quad (\mathbf{1\ mark}) \\
&\Longrightarrow \quad \frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} + h'(x) = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \quad (\mathbf{1\ mark}) \\
&\Longrightarrow \quad h'(x) = 0 \Longrightarrow h(x) = k \quad (\mathbf{1\ mark}) \\
&\Longrightarrow \quad f(x,y) = x^2\mathbf{e}^{xy} + y^2 + k \quad (\mathbf{1\ mark}) \\
&\Longrightarrow \quad I = f(1,1) - f(1,0) = \mathbf{e}. \quad (\mathbf{1\ mark})
\end{aligned}$$

2. (**5 marks**) Calculate the divergence of the vector field $\overrightarrow{F} = \dfrac{x\,\overrightarrow{i} + y\,\overrightarrow{j}}{\sqrt{x^2+y^2}}$

Solution
The vector field $\overrightarrow{F}$ has components $F_1 = \dfrac{x}{\sqrt{x^2+y^2}}$ and $F_2 = \dfrac{y}{\sqrt{x^2+y^2}}$.
(**1 mark**)
Its divergence is

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad (\textbf{1 mark}) \\
&= \frac{y^2}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^3} \quad (\textbf{3 marks}) \\
&= \frac{y^2+x^2}{\left(\sqrt{x^2+y^2}\right)^3} = \frac{1}{\sqrt{x^2+y^2}}.
\end{aligned}$$

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1. (**5 marks**) Given that the differential form

$$(2x^3y^4 + x)dx + (2x^4y^3 + y)dy$$

is exact, evaluate the integral

$$I = \int_{(1,0)}^{(1,1)} (2x^3y^4 + x)dx + (2x^4y^3 + y)dy.$$

Solution
We find the potential function f from $\frac{\partial f}{\partial x} = 2x^3y^4 + x$.

$$\Longrightarrow \quad f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y) \quad (\textbf{1 mark})$$
$$\Longrightarrow \quad \frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 3x^4y^3 + y \quad (\textbf{1 mark})$$
$$\Longrightarrow \quad h'(y) = y \Longrightarrow h(y) = \frac{1}{2}y^2 + k \quad (\textbf{1 mark})$$
$$\Longrightarrow \quad f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}\left(x^2 + y^2\right) + k \quad (\textbf{1 mark})$$
$$\Longrightarrow \quad I = f(1,1) - f(1,0) = 1. \quad (\textbf{1 mark})$$

2. (**5 marks**) Calculate the divergence of the vector field $\overrightarrow{F} = \dfrac{x\,\overrightarrow{i} - y\,\overrightarrow{j}}{\sqrt{x^2+y^2}}$

Solution
The vector field $\overrightarrow{F}$ has components $F_1 = \dfrac{x}{\sqrt{x^2+y^2}}$ and $F_2 = -\dfrac{y}{\sqrt{x^2+y^2}}$.
(**1 mark**)
Its divergence is

$$
\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad (\textbf{1 mark}) \\
&= \frac{y^2}{\left(\sqrt{x^2+y^2}\right)^3} - \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^3} \quad (\textbf{3 marks}) \\
&= \frac{y^2 - x^2}{\left(\sqrt{x^2+y^2}\right)^3}.
\end{aligned}
$$

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1. (**5 marks**) Evaluate the integral

$$\int_C f(\overrightarrow{r})ds$$

where $f(x, y, z) = xyz$, $C : \overrightarrow{r}(t) = (\cos t, \sin t, 3t),\ 0 \leq t \leq 4\pi$, and s is the arc length.

Solution

$$\begin{aligned}
\int_C f(\overrightarrow{r})ds &= \int_0^{4\pi} 3t\cos t\sin t\sqrt{\sin^2 t + \cos^2 t + 9}dt \quad (\textbf{1 mark}) \\
&= \sqrt{10}\int_0^{4\pi} 3t\left(\frac{1}{2}\sin 2t\right)dt \quad (\textbf{1 mark}) \\
&= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin 2t dt \\
&= \frac{3\sqrt{10}}{2}\left[\frac{1}{4}\sin 2t - \frac{t}{2}\cos 2t\right]_0^{4\pi} \quad (\textbf{2 marks}) \\
&= -3\pi\sqrt{10}. \quad (\textbf{1 mark})
\end{aligned}$$

2. (**5 marks**) Calculate the divergence of the vector field $F(x, y, z)$ given by

$$\overrightarrow{F}(x, y, z) = \mathbf{e}^{xyz}\,\overrightarrow{i} + \frac{x}{y^2}\mathbf{e}^y\,\overrightarrow{j} + \ln(yz)\,\overrightarrow{k}.$$

Solution
The vector $\overrightarrow{F}$ has components $F_1 = \mathbf{e}^{xyz}$, $F_2 = \dfrac{x}{y^2}\mathbf{e}^y$, and $F_3 = \ln(yz)$.
(**1 mark**)
Its divergence is

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad (\textbf{1 mark}) \\
&= \frac{\partial}{\partial x}\left(\mathbf{e}^{xyz}\right) + \frac{\partial}{\partial y}\left(\frac{x}{y^2}\mathbf{e}^y\right) + \frac{\partial}{\partial z}\left(\ln(yz)\right) \\
&= yz\mathbf{e}^{xyz} + \frac{x}{y^2}\mathbf{e}^y - \frac{2x}{y^3}\mathbf{e}^y + \frac{1}{z}. \quad (\textbf{3 marks})
\end{aligned}$$