Sultan Qaboos University
Department of Mathematics and Statistics
MATH3171-LINEAR ALGEBRA AND MULTIVARIABLE CALCULUS FOR ENGINEERS
Fall 2008 - Quiz2A - Solutions
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1. $\boxed{\textbf{5 marks}}$ **Find** the spectrum and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution: The eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -\lambda & 3 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0, \qquad \boxed{\textbf{1 mark}}$$

or

$$(-\lambda)(-\lambda)(-1-\lambda) = 0.$$

Thus,

$$\lambda = 0 \ \text{(multiplicity 2)}, \quad \lambda = -1. \qquad \boxed{\textbf{1 mark}}$$

Hence the spectrum of matrix A is the set $\{-1, 0\}$. $\boxed{\textbf{1 mark}}$.

To find the eigenvectors, solve the system of linear, homogeneous algebraic equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ for the different eigenvalues:

(a) For $\lambda = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_2 = x_3 = 0, \quad x_1 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(1, 0, 0)$. $\boxed{\textbf{1 mark}}$

(b) For $\lambda = -1$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 = x_2 = 0, \quad x_3 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(0, 0, 1)$. $\boxed{\textbf{1 mark}}$

2. $\boxed{\textbf{5 marks}}$ Is the matrix $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$ orthogonal? **Explain** your answer.

Solution: Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}. \quad \boxed{\textbf{1 mark}}$$

Multiply to get

$$\begin{aligned} AA^T &= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \boxed{\textbf{2 marks}} \end{aligned}$$

This proves that $A^T = A^{-1}$. $\boxed{\textbf{1 mark}}$ Hence the given matrix is orthogonal. $\boxed{\textbf{1 mark}}$

Note: You may also want to check the two equalities $AA^T = I$ and $A^T A = I$.

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1. **6 marks** **Find** the spectrum and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: The eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \quad \boxed{\textbf{1 mark}}$$

or

$$(2-\lambda)(1-\lambda)(-\lambda) = 0.$$

Thus

$$\lambda = 0, \quad \lambda = 1, \quad \lambda = 2 \quad \boxed{\textbf{1 mark}}.$$

Hence the spectrum of matrix A is the set $\{0, 1, 2\}$. **1 mark**

To find the eigenvectors, solve the system of linear, homogeneous algebraic equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ for the different eigenvalues:

(a) For $\lambda = 0$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 = x_2 = 0, \quad x_3 \text{ takes arbitrary values.}$$

Hence one may take an eigenvector of the form $(0, 0, 1)$. **1 mark**

(b) For $\lambda = 1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 + x_2 = x_3 = 0.$$

Hence one may take an eigenvector of the form $(1,-1,0)$. $\boxed{\textbf{1 mark}}$

(a) For $\lambda = 2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_2 = x_3 = 0, \quad x_1 \text{ takes arbitrary values.}$$

Hence one may take an eigenvector of the form $(1,0,0)$. $\boxed{\textbf{1 mark}}$

2. $\boxed{\textbf{4 marks}}$ Is the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ orthogonal? **Explain** your answer.

Solution: Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}. \quad \boxed{\textbf{1 mark}}$$

Multiply to get

$$\begin{aligned} AA^T &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 1 \end{pmatrix}. \quad \boxed{\textbf{1 mark}} \end{aligned}$$

This proves that $A^T \neq A^{-1}$. $\boxed{\textbf{1 mark}}$ Hence the given matrix is not orthogonal. $\boxed{\textbf{1 mark}}$

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1. $\boxed{\textbf{6 marks}}$ **Find** the spectrum and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution: The eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad \boxed{\textbf{1 mark}}$$

or

$$(2-\lambda)(-4-\lambda)(1-\lambda) = 0.$$

Thus

$$\lambda = 2\ , \quad \lambda = -4, \quad \lambda = 1. \qquad \boxed{\textbf{1 mark}}$$

Hence the spectrum of matrix A is the set $\{-4, 1, 2\}$. $\boxed{\textbf{1 mark}}$

To find the eigenvectors, solve the system of linear, homogeneous algebraic equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ for the different eigenvalues:

(a) For $\lambda = 2$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_2 = 0, \quad x_1 = x_3.$$

Hence one may take an eigenvector of the form $(1, 0, 1)$. $\boxed{\textbf{1 mark}}$

(b) For $\lambda = -4$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 = x_3 = 0, \quad x_2 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(0, 1, 0)$. $\boxed{\textbf{1 mark}}$

(c) For $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 = x_2 = 0, \quad x_3 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(0, 0, 1)$. $\boxed{\textbf{1 mark}}$

2. $\boxed{\textbf{4 marks}}$ Is the matrix $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ orthogonal? **Explain** your answer.

Solution: Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \boxed{\textbf{1 mark}}$$

Multiply to get

$$\begin{aligned} AA^T &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \boxed{\textbf{1 mark}} \end{aligned}$$

This proves that $A^T \neq A^{-1}$. $\boxed{\textbf{1 mark}}$ Hence the given matrix is not orthogonal. $\boxed{\textbf{1 mark}}$

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1. $\boxed{\textbf{5 marks}}$ **Find** the spectrum and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution: The eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -\lambda & 3 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \quad \boxed{\textbf{1 mark}}$$

or

$$(-\lambda)\,(-\lambda)\,(-1-\lambda) = 0.$$

Thus

$$\lambda = 0 \ \ \text{(multiplicity 2)}, \quad \lambda = -1. \quad \boxed{\textbf{1 mark}}$$

Hence the spectrum of matrix A is the set $\{-1, 0\}$. $\boxed{\textbf{1 mark}}$

To find the eigenvectors, solve the system of linear, homogeneous algebraic equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ for the different eigenvalues:

(a) For $\lambda = -1$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 = x_2 = 0, \quad x_3 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(0, 0, 1)$. $\boxed{\textbf{1 mark}}$

(b) For $\lambda = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_2 = x_3 = 0, \quad x_1 \text{ takes arbitrary real values.}$$

Hence one may take an eigenvector of the form $(1, 0, 0)$. $\boxed{\textbf{1 mark}}$

2. $\boxed{\textbf{5 marks}}$ Is the matrix $\begin{pmatrix} 1 & i & 1 \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}$ unitary? **Explain** your answer.

Solution: Let

$$A = \begin{pmatrix} 1 & i & 1 \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}.$$

Then

$$\overline{A}^T = \begin{pmatrix} 1 & 0 & i \\ -i & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}. \quad \boxed{\textbf{1 mark}}$$

Multiply to get

$$\begin{aligned} A\overline{A}^T &= \begin{pmatrix} 1 & i & 1 \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ -i & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & i & 2+i \\ -i & 1 & 0 \\ 2-i & 0 & 5 \end{pmatrix}. \quad \boxed{\textbf{2 marks}} \end{aligned}$$

This proves that $A^T \neq A^{-1}$. $\boxed{\textbf{1 mark}}$ Hence the given matrix is not unitary. $\boxed{\textbf{1 mark}}$

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1. $\boxed{\textbf{5 marks}}$ **Find** the spectrum and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: The eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \quad \boxed{\textbf{1 mark}}$$

or

$$(1-\lambda)(1-\lambda)(-\lambda) - (-\lambda) = 0.$$

Thus

$$\lambda = 0 \ \text{ (multiplicity 2)}, \quad \lambda = 2. \quad \boxed{\textbf{1 mark}}$$

Hence the spectrum of matrix A is the set $\{0, 2\}$. $\boxed{\textbf{1 mark}}$

To find the eigenvectors, solve the system of linear, homogeneous algebraic equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ for the different eigenvalues:

(a) For $\lambda = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 + x_2 = x_3 = 0.$$

Hence one may take an eigenvector of the form $(1, -1, 0)$. $\boxed{\textbf{1 mark}}$

(b) For $\lambda = 2$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One finds

$$x_1 - x_2 = x_3 = 0.$$

Hence one may take an eigenvector of the form $(1, 1, 0)$. $\boxed{\textbf{1 mark}}$

2. $\boxed{\textbf{5 marks}}$ Is the matrix $\begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ -i & 0 & i \end{pmatrix}$ unitary? **Explain** your answer.

Solution: Let

$$A = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ -i & 0 & i \end{pmatrix}.$$

Then

$$\overline{A}^T = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & -i \end{pmatrix}. \quad \boxed{\textbf{1 mark}}$$

Multiply to get

$$\begin{aligned} A\overline{A}^T &= \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ -i & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & -1+i \\ 0 & 1 & 0 \\ -1-i & 0 & 2 \end{pmatrix}. \quad \boxed{\textbf{2 marks}} \end{aligned}$$

This proves that $\overline{A}^T \neq A^{-1}$. $\boxed{\textbf{1 mark}}$ Hence the given matrix is not unitary. $\boxed{\textbf{1 mark}}$